The reaction rate is first order with a rate constant (in $\min^{-1}$)

$$k = 2.4 \times 10^{15} e^{-20{,}000/T} \ (T \text{ in } °R).$$

For the batch case, linearize the model around $T = \overline{T}$.

18.10 A batch reactor converts component A into B, which in turn decomposes into C:

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

where $k_1 = k_{10}e^{-E_1/RT}$ and $k_2 = k_{20}e^{-E_2/RT}$.

The concentrations of A and B are denoted by x_1 and x_2, respectively. The reactor model is

$$\frac{dx_1}{dt} = -k_{10}x_1e^{-E_1/RT}$$

$$\frac{dx_2}{dt} = k_{10}x_1e^{-E_1/RT} - k_{20}x_2e^{-E_2/RT}$$

Thus, the ultimate values of x_1 and x_2 depend on the reactor temperature as a function of time. For

$$k_{10} = 1.335 \times 10^{10} \text{ min}^{-1}, \quad k_{20} = 1.149 \times 10^{17} \text{ min}^{-1}$$
$$E_1 = 75{,}000 \text{ J/g mol}, \quad E_2 = 125{,}000 \text{ J/gmol}$$
$$R = 8.31 \text{ J/(gmol } K) \quad x_{10} = 0.7 \text{ mol/L}, \quad x_{20} = 0$$

Find the constant temperature that maximizes the amount of B, for $0 \le t \le 8$ min. Next allow the temperature to change as a cubic function of time

$$T(t) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Find the values of a_0, a_1, a_2, a_3 that maximize x_2 by integrating the model and using a suitable optimization method.

18.11 Suppose a batch reactor such as the one in Fig. 18.12 has a gas ingredient added to the liquid feed. As long as the reaction is proceeding normally, the gas is absorbed in the liquid (where it reacts), keeping the pressure low. However, if the reaction slows or the gas feed is greater than can be absorbed, the pressure will start to rise. The pressure rise can be compensated by an increase in liquid feed, but this may cause the cooling capacity to be exceeded. Describe a solution to this problem using overrides (see Chapter 15).

18.12 Fogler[2] describes a safety accident in which a batch reactor was used to produce nitroanaline from ammonia and o-nitro chlorobenzene. On the day of the accident, the feed composition was changed from the normal operating value. Using the material/energy balances and data provided by Fogler, show that the maximum cooling rate will not be sufficient to prevent a temperature runaway under conditions of the new feed composition. Use a simulator to solve the model equations.

18.13 Consider the batch reactor system simulated by Aziz et al.[3] The two reactions, $A + B \rightarrow C$ and $A + C \rightarrow D$, are carried out in a jacketed batch reactor, where C is the desired product and D is a waste product. The manipulated variable is the temperature of the coolant in the cooling jacket. There are two inequality constraints: input bounds on the coolant temperature and an upper limit on the maximum reactor temperature. Using the model parameters specified by Aziz et al., evaluate the following control strategies for a set-point change from 20 °C to 92 °C.

(a) PID controller

(b) Batch unit

(c) Batch unit with preload

(d) Dual-mode controller

[2] *Elements of Chemical Reaction Engineering*, 4th ed., Prentice Hall, Upper Saddle River, NJ, 2005, Chapter 9.

[3] N. Aziz, M. A. Hussain, and I. M. Mujtaba, Performance of Different Types of Controllers in Tracking Optimal Temperature Profiles in Batch Reactors, *Comput. Chem. Eng*, **24**, 1069 (2000).

Appendix A

Laplace Transforms

In Chapter 2 we developed a number of mathematical models that describe the dynamic operation of selected processes. Solving such models—that is, finding the output variables as functions of time for some change in the input variable(s)—requires either analytical or numerical integration of the differential equations. Sometimes considerable effort is involved in obtaining the solutions. One important class of models includes systems described by linear ordinary differential equations (ODEs). Such *linear systems* represent the starting point for many analysis techniques in process control.

In this Appendix, we introduce a mathematical tool, the *Laplace transform*, which can significantly reduce the effort required to solve and analyze linear differential equation models. A major benefit is that this transformation converts ordinary differential equations to algebraic equations, which can simplify the mathematical manipulations required to obtain a solution or perform an analysis.

First, we define the Laplace transform and show how it can be used to derive the Laplace transforms of simple functions. Then we show that linear ODEs can be solved using Laplace transforms, along with a technique called *partial fraction expansion*. Some important general properties of Laplace transforms are presented, and we illustrate the use of these techniques with a series of examples.

A.1 THE LAPLACE TRANSFORM OF REPRESENTATIVE FUNCTIONS

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt \qquad \text{(A-1)}$$

where $F(s)$ is the symbol for the Laplace transform, s is a complex independent variable, $f(t)$ is some function of time to be transformed, and $\mathcal{L}$ is an operator, defined by the integral. The function $f(t)$ must satisfy mild conditions that include being piecewise continuous for $0 < t < \infty$ (Churchill, 1971); this requirement almost always holds for functions that are useful in process modeling and control. When the integration is performed, the transform becomes a function of the Laplace transform variable s. The *inverse Laplace transform* ($\mathcal{L}^{-1}$) operates on the function $F(s)$ and converts it to $f(t)$. Notice that $F(s)$ contains no information about $f(t)$ for $t < 0$. Hence, $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is not defined for $t < 0$ (Schiff, 1999).

One of the important properties of the Laplace transform and the inverse Laplace transform is that they are linear operators; a linear operator satisfies the *superposition principle*:

$$\mathcal{F}(ax(t) + by(t)) = a\mathcal{F}(x(t)) + b\mathcal{F}(y(t)) \qquad \text{(A-2)}$$

where $\mathcal{F}$ denotes a particular operation to be performed, such as differentiation or integration with respect to time. If $\mathcal{F} \equiv \mathcal{L}$, then Eq. A-2 becomes

$$\mathcal{L}(ax(t) + by(t)) = aX(s) + bY(s) \qquad \text{(A-3)}$$

Therefore, the Laplace transform of a sum of functions $x(t)$ and $y(t)$ is the sum of the individual Laplace transforms $X(s)$ and $Y(s)$; in addition, multiplicative constants can be factored out of the operator, as shown in (A-3).

In this book we are more concerned with operational aspects of Laplace transforms—that is, using them to obtain solutions or the properties of solutions of *linear differential equations*. For more details on mathematical aspects of the Laplace transform, the texts by Churchill (1971) and Dyke (1999) are recommended.

Before we consider solution techniques, the application of Eq. A-1 should be discussed. The Laplace transform can be derived easily for most simple functions, as shown below.

Constant Function. For $f(t) = a$ (a constant),

$$\mathcal{L}(a) = \int_0^\infty a e^{-st}\,dt = -\frac{a}{s}e^{-st}\Big|_0^\infty \qquad \text{(A-4)}$$

$$= 0 - \left(-\frac{a}{s}\right) = \frac{a}{s}$$

Step Function. The unit step function, defined as

$$S(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \qquad \text{(A-5)}$$

is an important input that is used frequently in process dynamics and control. The Laplace transform of the unit step function is the same as that obtained for the constant above when $a = 1$:

$$\mathcal{L}[S(t)] = \frac{1}{s} \qquad \text{(A-6)}$$

If the step magnitude is a, the Laplace transform is a/s. The step function incorporates the idea of *initial time*, *zero time*, or *time zero* for the function, which refers to the time at which $S(t)$ changes from 0 to 1. To avoid any ambiguity concerning the value of the step function at $t = 0$ (it is discontinuous), we will consider $S(t = 0)$ to be the value of the function approached from the positive side, $t = 0^+$.

Derivatives. The transform of a first derivative of f is important because such derivatives appear in dynamic models:

$$\mathcal{L}(df/dt) = \int_0^\infty (df/dt)e^{-st}\,dt \qquad \text{(A-7)}$$

Integrating by parts,

$$\mathcal{L}(df/dt) = \int_0^\infty f(t)e^{-st}s\,dt + f(t)e^{-st}\Big|_0^\infty \qquad \text{(A-8)}$$

$$= s\mathcal{L}(f(t)) - f(0) = sF(s) - f(0) \qquad \text{(A-9)}$$

where $F(s)$ is the Laplace transform of $f(t)$. Generally, the point at which we start keeping time for a solution is arbitrary. Model solutions are most easily obtained assuming that time *starts* (i.e., $t = 0$) at the moment the process model is first perturbed. For example, if the process initially is assumed to be at steady state and an input undergoes a unit step change, *zero time* is taken to be the moment at which the input changes in magnitude. In many process modeling applications, functions are defined so that they are zero at initial time—that is, $f(0) = 0$. In these cases, (A-9) simplifies to $\mathcal{L}(df/dt) = sF(s)$.

The Laplace transform for higher-order derivatives can be found using Eq. A-9. To derive $\mathcal{L}[f''(t)]$, we define a new variable ($\phi = df/dt$) such that

$$\mathcal{L}\left(\frac{d^2f}{dt^2}\right) = \mathcal{L}\left(\frac{d\phi}{dt}\right) = s\phi(s) - \phi(0) \qquad \text{(A-10)}$$

$$\phi(s) = sF(s) - f(0) \qquad \text{(A-11)}$$

Substituting into Eq. A-10

$$\mathcal{L}\left(\frac{d^2f}{dt^2}\right) = s[sF(s) - f(0)] - \phi(0) \qquad \text{(A-12)}$$

$$= s^2F(s) - sf(0) - f'(0) \qquad \text{(A-13)}$$

where $f'(0)$ denotes the value of df/dt at $t = 0$. The Laplace transform for derivatives higher than second order can be found by the same procedure. An nth-order derivative, when transformed, yields a series of $(n + 1)$ terms:

$$\mathcal{L}\left(\frac{d^nf}{dt^n}\right) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots$$

$$- sf^{(n-2)}(0) - f^{(n-1)}(0) \qquad \text{(A-14)}$$

where $f^i(0)$ is the ith derivative evaluated at $t = 0$. If $n = 2$, Eq. A-13 is obtained.

Exponential Functions. The Laplace transform of an exponential function is important because exponential functions appear in the solution to all linear differential equations. For a negative exponential, e^{-bt}, with $b > 0$,

$$\mathcal{L}(e^{-bt}) = \int_0^\infty e^{-bt}e^{-st}dt = \int_0^\infty e^{-(b+s)t}dt \qquad \text{(A-15)}$$

$$= \frac{1}{b+s}\left[-e^{-(b+s)t}\right]\Big|_0^\infty = \frac{1}{s+b} \qquad \text{(A-16)}$$

The Laplace transform for $b < 0$ is unbounded if $s < b$; therefore, the real part of s must be restricted to be larger than $-b$ for the integral to be finite. This condition is satisfied for all problems we consider in this book.

Trigonometric Functions. In modeling processes and in studying control systems, there are many other important time functions, such as the trigonometric functions, cos ωt and sin ωt, where ω is the frequency in radians per unit time. The Laplace transform of cos ωt

Table A.1 Laplace Transforms for Various Time-Domain Functions[a]

$f(t)$	$F(s)$
1. $\delta(t)$ (unit impulse)	1
2. $S(t)$ (unit step)	$\frac{1}{s}$
3. t (ramp)	$\frac{1}{s^2}$
4. t^{n-1}	$\frac{(n-1)!}{s^n}$
5. e^{-bt}	$\frac{1}{s+b}$
6. $\frac{1}{\tau}e^{-t/\tau}$	$\frac{1}{\tau s+1}$
7. $\frac{t^{n-1}e^{-bt}}{(n-1)!}$ $(n > 0)$	$\frac{1}{(s+b)^n}$
8. $\frac{1}{\tau^n(n-1)!}t^{n-1}e^{-t/\tau}$	$\frac{1}{(\tau s+1)^n}$
9. $\frac{1}{b_1-b_2}(e^{-b_2t} - e^{-b_1t})$	$\frac{1}{(s+b_1)(s+b_2)}$
10. $\frac{1}{\tau_1-\tau_2}(e^{-t/\tau_1} - e^{-t/\tau_2})$	$\frac{1}{(\tau_1s+1)(\tau_2s+1)}$
11. $\frac{b_3-b_1}{b_2-b_1}e^{-b_1t} + \frac{b_3-b_2}{b_1-b_2}e^{-b_2t}$	$\frac{s+b_3}{(s+b_1)(s+b_2)}$
12. $\frac{1}{\tau_1}\frac{\tau_1-\tau_3}{\tau_1-\tau_2}e^{-t/\tau_1} + \frac{1}{\tau_2}\frac{\tau_2-\tau_3}{\tau_2-\tau_1}e^{-t/\tau_2}$	$\frac{\tau_3s+1}{(\tau_1s+1)(\tau_2s+1)}$
13. $1 - e^{-t/\tau}$	$\frac{1}{s(\tau s+1)}$
14. $\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
15. $\cos \omega t$	$\frac{s}{s^2+\omega^2}$
16. $\sin(\omega t + \phi)$	$\frac{\omega\cos\phi + s\sin\phi}{s^2+\omega^2}$
17. $e^{-bt}\sin \omega t$ } b, ω real	$\frac{\omega}{(s+b)^2+\omega^2}$
18. $e^{-bt}\cos \omega t$ } b, ω real	$\frac{s+b}{(s+b)^2+\omega^2}$
19. $\frac{1}{\tau\sqrt{1-\zeta^2}}e^{-\zeta t/\tau}\sin(\sqrt{1-\zeta^2}\,t/\tau)$ $(0 \leq \lvert\zeta\rvert < 1)$	$\frac{1}{\tau^2s^2+2\zeta\tau s+1}$
20. $1 + \frac{1}{\tau_2-\tau_1}(\tau_1e^{-t/\tau_1} - \tau_2e^{-t/\tau_2})$ $(\tau_1 \neq \tau_2)$	$\frac{1}{s(\tau_1s+1)(\tau_2s+1)}$

(continued)

Table A.1 (*Continued*)

$f(t)$	$F(s)$
21. $1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin[\sqrt{1-\zeta^2}\, t/\tau + \psi]$ $\psi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$, $(0 \le \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22. $1 - e^{-\zeta t/\tau}[\cos(\sqrt{1-\zeta^2}\, t/\tau) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\, t/\tau)]$ $(0 \le \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23. $1 + \frac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ $(\tau_1 \neq \tau_2)$	$\frac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
24. $\frac{df}{dt}$	$sF(s) - f(0)$
25. $\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots$ $- s f^{(n-2)}(0) - f^{(n-1)}(0)$
26. $f(t - t_0)S(t - t_0)$	$e^{-t_0 s} F(s)$

[a]Note that $f(t)$ and $F(s)$ are defined for $t \ge 0$ only.

or sin ωt can be calculated using integration by parts. An alternative method is to use the Euler identity[1]

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}, \quad j \triangleq \sqrt{-1} \qquad \text{(A-17)}$$

and to apply (A-1). Because the Laplace transform of a sum of two functions is the sum of the Laplace transforms,

$$\mathcal{L}(\cos \omega t) = \tfrac{1}{2}\mathcal{L}(e^{j\omega t}) + \tfrac{1}{2}\mathcal{L}(e^{-j\omega t}) \qquad \text{(A-18)}$$

Using Eq. A-16 gives

$$\mathcal{L}(\cos \omega t) = \frac{1}{2}\left(\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right)$$
$$= \frac{1}{2}\left(\frac{s + j\omega + s - j\omega}{s^2 + \omega^2}\right) = \frac{s}{s^2 + \omega^2} \qquad \text{(A-19)}$$

Note that the use of imaginary variables above was merely a device to avoid integration by parts; imaginary numbers do not appear in the final result. To find $\mathcal{L}(\sin \omega t)$, we can use a similar approach.

Table A.1 lists some important Laplace transform pairs that occur in the solution of linear differential equations. For a more extensive list of transforms, see Dyke (1999).

Note that in all the transform cases derived above, $F(s)$ is a ratio of polynomials in s, that is, a *rational form*. There are some important cases when nonpolynomial (nonrational) forms occur. One such case is discussed next.

The Rectangular Pulse Function. An illustration of the rectangular pulse is shown in Fig. A.1. The pulse has height h and width t_w. This type of signal might be used to depict the opening and closing of a valve regulating flow into a tank. The flow rate would be held at h for a duration of t_w units of time. The area under the curve in Fig. A.1 could be interpreted as the amount of material delivered to the tank ($= ht_w$). Mathematically, the function $f(t)$ is defined as

$$f(t) = \begin{cases} 0 & t < 0 \\ h & 0 \le t < t_w \\ 0 & t \ge t_w \end{cases} \qquad \text{(A-20)}$$

[1]The symbol j, rather than i, is traditionally used for $\sqrt{-1}$ in the control engineering literature.

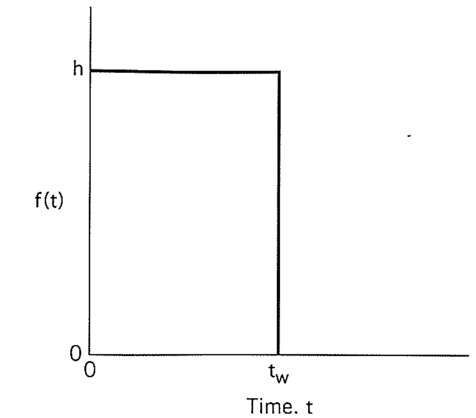

Figure A.1 The rectangular pulse function.

The Laplace transform of the rectangular pulse can be derived by evaluating the integral (A-1) between $t = 0$ and $t = t_w$ because $f(t)$ is zero everywhere else:

$$F(s) = \int_0^{\infty} f(t)e^{-st}\, dt = \int_0^{t_w} he^{-st}\, dt \qquad \text{(A-21)}$$

$$F(s) = -\frac{h}{s} e^{-st}\Big|_0^{t_w} = \frac{h}{s}(1 - e^{-t_w s}) \qquad \text{(A-22)}$$

Note that an exponential term in $F(s)$ results. For a *unit rectangular pulse*, $h = 1/t_w$ and the area under the pulse is unity.

Impulse Function. A limiting case of the unit rectangular pulse is the *impulse* or *Dirac delta function*, which has the symbol $\delta(t)$. This function is obtained when $t_w \to 0$ while keeping the area under the pulse equal to unity. A pulse of infinite height and infinitesimal width results. Mathematically, this can be accomplished by substituting $h = 1/t_w$ into (A-22); the Laplace transform of $\delta(t)$ is

$$\mathcal{L}(\delta(t)) = \lim_{t_w \to 0} \frac{1}{t_w s}(1 - e^{-t_w s}) \qquad \text{(A-23)}$$

Equation A-23 is an indeterminate form that can be evaluated by application of L'Hospital's rule (also spelled L'Hôpital), which involves taking derivatives of both numerator and denominator with respect to t_w:

$$\mathcal{L}(\delta(t)) = \lim_{t_w \to 0} \frac{se^{-t_w s}}{s} = 1 \qquad \text{(A-24)}$$

If the impulse magnitude (i.e., area $t_w h$) is a constant a rather than unity, then

$$\mathcal{L}(a\delta(t)) = a \qquad \text{(A-25)}$$

as given in Table A.1. The unit impulse function may also be interpreted as the time derivative of the unit step function $S(t)$. The response of a process to a unit impulse is called its *impulse response*, which is illustrated in Example A.7.

A physical example of an impulse function is the rapid injection of dye or tracer into a fluid stream, where $f(t)$ corresponds to the concentration or the flow rate of the tracer. This type of signal is sometimes used in process testing, for example, to obtain the residence time distribution of a piece of equipment, as illustrated in Section A.5.

A.2 SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORM TECHNIQUES

In the previous section we developed the techniques required to obtain the Laplace transform of each term in a linear ordinary differential equation. Table A.1 lists important functions of time, including derivatives, and their Laplace transform equivalents. Because the Laplace transform converts any function $f(t)$ to $F(s)$ and the inverse Laplace transform converts $F(s)$ back to $f(t)$, the table provides an organized way to carry out these transformations.

The procedure used to solve a differential equation is quite simple. First Laplace transform both sides of the differential equation, substituting values for the initial conditions in the derivative transforms. Rearrange the resulting algebraic equation, and solve for the transform of the dependent (output) variable. Finally, find the inverse of the transformed output variable. The solution method is illustrated by means of several examples.

EXAMPLE A.1

Solve the differential equation,

$$5\frac{dy}{dt} + 4y = 2 \quad y(0) = 1 \qquad \text{(A-26)}$$

using Laplace transforms.

SOLUTION

First, take the Laplace transform of both sides of Eq. A-26:

$$\mathcal{L}\left(5\frac{dy}{dt} + 4y\right) = \mathcal{L}(2) \qquad \text{(A-27)}$$

Using the principle of superposition, each term can be transformed individually:

$$\mathcal{L}\left(5\frac{dy}{dt}\right) + \mathcal{L}(4y) = \mathcal{L}(2) \qquad \text{(A-28)}$$

$$\mathcal{L}\left(5\frac{dy}{dt}\right) = 5\mathcal{L}\left(\frac{dy}{dt}\right) = 5(sY(s) - 1) = 5sY(s) - 5 \qquad \text{(A-29)}$$

$$\mathcal{L}(4y) = 4\mathcal{L}(y) = 4Y(s) \qquad \text{(A-30)}$$

$$\mathcal{L}(2) = \frac{2}{s} \qquad \text{(A-31)}$$

Substitute the individual terms:

$$5sY(s) - 5 + 4Y(s) = \frac{2}{s} \qquad \text{(A-32)}$$

Rearrange (A-32) and factor out $Y(s)$:

$$Y(s)(5s + 4) = 5 + \frac{2}{s} \qquad \text{(A-33)}$$

or

$$Y(s) = \frac{5s + 2}{s(5s + 4)} \qquad \text{(A-34)}$$

Take the inverse Laplace transform of both sides of Eq. A-34:

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{5s + 2}{s(5s + 4)}\right] \qquad \text{(A-35)}$$

The inverse Laplace transform of the right side of (A-35) can be found by using Table A.1. First, divide the numerator and denominator by 5 to put all factors in the $s + b$ form corresponding to the table entries:

$$y(t) = \mathcal{L}^{-1}\left(\frac{s + 0.4}{s(s + 0.8)}\right) \qquad \text{(A-36)}$$

Because entry 11 in the table, $(s + b_3)/[(s + b_1)(s + b_2)]$, matches (A-36) with $b_1 = 0.8$, $b_2 = 0$, and $b_3 = 0.4$, the solution can be written immediately:

$$y(t) = 0.5 + 0.5e^{-0.8t} \qquad \text{(A-37)}$$

Note that in solving (A-26), both the forcing function (the constant 2 on the right side) and the initial condition have been incorporated easily and directly. As for any differential equation solution, (A-37) should be checked to make sure it satisfies the initial condition and the original differential equation for $t \geq 0$.

Next we apply the Laplace transform solution to a higher-order differential equation.

EXAMPLE A.2

Solve the ordinary differential equation

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 1 \qquad \text{(A-38)}$$

with initial conditions $y(0) = y'(0) = y''(0) = 0$.

SOLUTION

Take Laplace transforms, term by term, using Table A.1:

$$\mathcal{L}\left(\frac{d^3y}{dt^3}\right) = s^3Y(s)$$

$$\mathcal{L}\left(6\frac{d^2y}{dt^2}\right) = 6s^2Y(s)$$

$$\mathcal{L}\left(11\frac{dy}{dt}\right) = 11sY(s)$$

$$\mathcal{L}(6y) = 6Y(s)$$

$$\mathcal{L}(1) = \frac{1}{s}$$

Rearranging and factoring $Y(s)$, we obtain

$$Y(s)(s^3 + 6s^2 + 11s + 6) = \frac{1}{s} \qquad \text{(A-39)}$$

$$Y(s) = \frac{1}{s(s^3 + 6s^2 + 11s + 6)} \qquad \text{(A-40)}$$

To invert (A-40) to find $y(t)$, we must find a similar expression in Table A.1. Unfortunately, no formula in the table has a fourth-order polynomial in the denominator. This example will be continued later, after we develop the techniques necessary to generalize the solution method in Section A.3.

In general, a transform expression may not exactly match any of the entries in Table A.1. This problem always arises for higher-order differential equations, because the order of the denominator polynomial (characteristic polynomial) of the transform is equal to the order of the original differential equation, and no table entries are higher than third order in the denominator. It is simply not practical to expand the number of entries in the table ad infinitum. Instead, we use a procedure based on elementary transform building blocks. This procedure, called *partial fraction expansion*, is presented in the next section.

A.3 PARTIAL FRACTION EXPANSION

The high-order denominator polynomial in a Laplace transform solution arises from the differential equation terms (its *characteristic polynomial*) plus terms contributed by the inputs. The factors of the characteristic polynomial correspond to the roots of the characteristic polynomial set equal to zero. The input factors may be quite simple. Once the factors are obtained, the Laplace transform is then expanded into *partial fractions*. As an example, consider

$$Y(s) = \frac{s + 5}{s^2 + 5s + 4} \qquad \text{(A-41)}$$

The denominator can be factored into a product of first-order terms, $(s + 1)(s + 4)$. This transform can be expanded into the sum of two partial fractions:

$$\frac{s + 5}{(s + 1)(s + 4)} = \frac{\alpha_1}{s + 1} + \frac{\alpha_2}{s + 4} \qquad \text{(A-42)}$$

where α_1 and α_2 are unspecified coefficients that must satisfy Eq. A-42. The expansion in (A-42) indicates that the original denominator polynomial has been factored into a product of first-order terms. In general, for every partial fraction expansion, there will be a unique set of α_i that satisfy the equation.

There are several methods for calculating the appropriate values of α_1 and α_2 in (A-42):

Method 1. Multiply both sides of (A-42) by $(s + 1)(s + 4)$:

$$s + 5 = \alpha_1(s + 4) + \alpha_2(s + 1) \qquad \text{(A-43)}$$

Equating coefficients of each power of s gives

$$s^1\text{:} \quad \alpha_1 + \alpha_2 = 1 \qquad \text{(A-44a)}$$

$$s^0\text{:} \quad 4\alpha_1 + \alpha_2 = 5 \qquad \text{(A-44b)}$$

Solving for α_1 and α_2 simultaneously yields $\alpha_1 = \frac{4}{3}$, $\alpha_2 = -\frac{1}{3}$.

Method 2. Because Eq. A-42 must be valid for all values of s, we can specify two values of s and solve for the two constants:

$$s = -5\text{:} \quad 0 = -\tfrac{1}{4}\alpha_1 - \alpha_2 \qquad \text{(A-45a)}$$

$$s = -3\text{:} \quad -\tfrac{2}{2} = -\tfrac{1}{2}\alpha_1 + \alpha_2 \qquad \text{(A-45b)}$$

Solving, $\alpha_1 = \frac{4}{3}$, $\alpha_2 = -\frac{1}{3}$.

Method 3. The fastest and most popular method is called the *Heaviside expansion*. In this method multiply both sides of the equation by one of the denominator terms $(s + b_i)$ and then set $s = -b_i$, which causes all terms except one to be multiplied by zero. Multiplying Eq. A-42 by $s + 1$ and then letting $s = -1$ gives

$$\alpha_1 = \left.\frac{s + 5}{s + 4}\right|_{s=-1} = \frac{4}{3}$$

Similarly, after multiplying by $(s + 4)$ and letting $s = -4$, the expansion gives

$$\alpha_2 = \left.\frac{s + 5}{s + 1}\right|_{s=-4} = -\frac{1}{3}$$

As seen above, the coefficients can be found by simple calculations.

For a more general transform, where the factors are real and distinct (no complex or repeated factors appear), the following expansion formula can be used:

$$Y(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_{i=1}^{n}(s + b_i)} = \sum_{i=1}^{n}\frac{\alpha_i}{s + b_i} \qquad \text{(A-46)}$$

where $D(s)$, an nth-order polynomial, is the denominator of the transform. $D(s)$ is the characteristic polynomial. The numerator $N(s)$ has a maximum order of $n - 1$. The ith coefficient can be calculated using the Heaviside expansion

$$\alpha_i = (s + b_i)\left.\frac{N(s)}{D(s)}\right|_{s=-b_i} \qquad \text{(A-47)}$$

Alternatively, an expansion for real, distinct factors may be written as

$$Y(s) = \frac{N'(s)}{D'(s)} = \frac{N'(s)}{\prod_{i=1}^{n}(\tau_i s + 1)} = \sum_{i=1}^{n}\frac{\alpha_i'}{\tau_i s + 1} \qquad \text{(A-48)}$$

Using Method 3, calculate the coefficients by

$$\alpha_i' = (\tau_i s + 1)\left.\frac{N'(s)}{D'(s)}\right|_{s=-\frac{1}{\tau_i}} \qquad \text{(A-49)}$$

Note that several entries in Table A.1 have the $\tau s + 1$ format.

We now can use the Heaviside expansion to complete the solution of Example A.2.

EXAMPLE A.2 *(Continued)*

First factor the denominator of Eq. A-40 into a product of first-order terms ($n = 4$ in Eq. A-46). Simple factors, as in this case, rarely occur in actual applications.

$$s(s^3 + 6s^2 + 11s + 6) = s(s + 1)(s + 2)(s + 3) \qquad \text{(A-50)}$$

This result determines the four terms that will appear in the partial fraction expansion—namely,

$$Y(s) = \frac{1}{s(s+1)(s+2)(s+3)}$$
$$= \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} + \frac{\alpha_4}{s+3} \quad \text{(A-51)}$$

The Heaviside expansion method gives $\alpha_1 = 1/6$, $\alpha_2 = -1/2$, $\alpha_3 = 1/2$, $\alpha_4 = -1/6$.

After the transform has been expanded into a sum of first-order terms, invert each term individually using Table A.1:

$$y(t) = \mathcal{L}^{-1}[Y(s)]$$
$$= \mathcal{L}^{-1}\left(\frac{1/6}{s} - \frac{1/2}{s+1} + \frac{1/2}{s+2} - \frac{1/6}{s+3}\right)$$
$$= \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$
$$+ \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s+3}\right)$$
$$= \frac{1}{6} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t} \quad \text{(A-52)}$$

Equation A-52 is thus the solution $y(t)$ to the differential equation (A-38). The α_i's are simply the coefficients of the solution. Equation A-52 also satisfies the three initial conditions of the differential equation. The reader should verify the result.

A.3.1 General Procedure for Solving Differential Equations

We now state a general procedure to solve ordinary differential equations using Laplace transforms. The procedure consists of four steps, as shown in Fig. A.2.

Note that solution for the differential equation involves use of Laplace transforms as an intermediate step. Step 3 can be bypassed if the transform found in Step 2 matches an entry in Table A.1. In order to factor $D(s)$ in Step 3, software such as MATLAB, Mathematica, or Mathcad can be utilized (Chapra and Canale, 2010).

In Step 3, other types of situations can occur. Both repeated factors and complex factors require modifications of the partial fraction expansion procedure.

Repeated Factors

If a term $s + b$ occurs r times in the denominator, r terms must be included in the expansion that incorporate the $s + b$ factor

$$Y(s) = \frac{\alpha_1}{s+b} + \frac{\alpha_2}{(s+b)^2} + \cdots + \frac{\alpha_r}{(s+b)^r} + \cdots \quad \text{(A-53)}$$

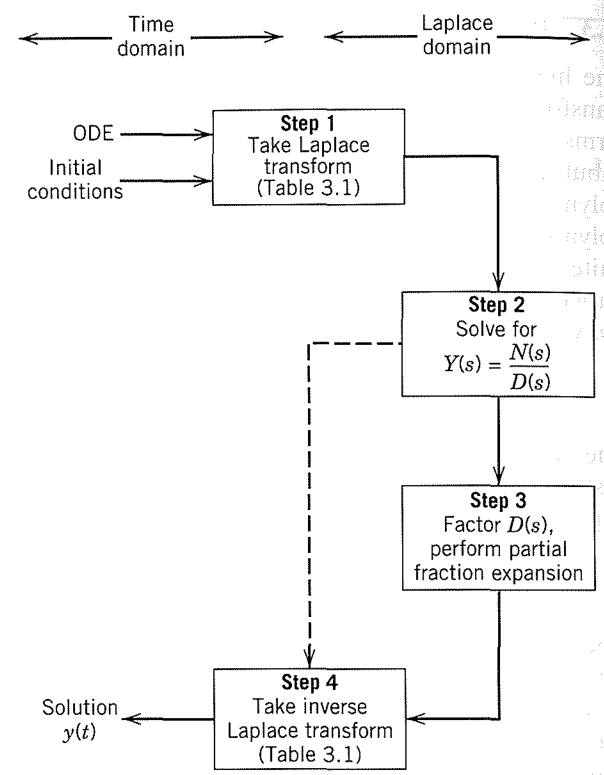

Figure A.2 The general procedure for solving an ordinary differential equation using Laplace transforms.

in addition to the other factors. Repeated factors arise infrequently in process models of real systems, mainly for a process that consists of a series of identical units or stages.

EXAMPLE A.3

For

$$Y(s) = \frac{s+1}{s(s^2+4s+4)} = \frac{\alpha_1}{s+2} + \frac{\alpha_2}{(s+2)^2} + \frac{\alpha_3}{s} \quad \text{(A-54)}$$

evaluate the unknown coefficients α_i.

SOLUTION

To find α_1 in (A-54), the Heaviside rule cannot be used for multiplication by $(s + 2)$, because $s = -2$ causes the second term on the right side to be unbounded, rather than 0 as desired. We therefore employ the Heaviside expansion method for the other two coefficients (α_2 and α_3) that can be evaluated normally and then solve for α_1 by arbitrarily selecting some other value of s. Multiplying (A-54) by $(s + 2)^2$ and letting $s = -2$ yields

$$\alpha_2 = \left.\frac{s+1}{s}\right|_{s=-2} = \frac{1}{2} \quad \text{(A-55)}$$

Multiplying (A-54) by s and letting $s = 0$ yields

$$\alpha_3 = \left.\frac{s+1}{s^2+4s+4}\right|_{s=0} = \frac{1}{4} \quad \text{(A-56)}$$

Substituting the value $s = -1$ in (A-54) gives

$$0 = \alpha_1 + \alpha_2 - \alpha_3 \quad \text{(A-57)}$$
$$\alpha_1 = -\frac{1}{4} \quad \text{(A-58)}$$

An alternative approach to find α_1 is to use differentiation of the transform. Equation A-54 is multiplied by $s(s + 2)^2$,

$$s + 1 = \alpha_1(s+2)s + \alpha_2 s + \alpha_3(s+2)^2 \quad \text{(A-59)}$$

Then (A-59) is differentiated twice with respect to s,

$$0 = 2\alpha_1 + 2\alpha_3; \quad \text{so that } \alpha_1 = -\alpha_3 = -\frac{1}{4} \quad \text{(A-60)}$$

Note that differentiation in this case is tantamount to equating powers of s, as demonstrated earlier.

The differentiation approach illustrated above can be used as the basis of a more general method to evaluate the coefficients of repeated factors. If the denominator polynomial $D(s)$ contains the repeated factor $(s + b)^r$, first form the quantity

$$Q(s) = \frac{N(s)}{D(s)}(s+b)^r = (s+b)^{r-1}\alpha_1 + (s+b)^{r-2}\alpha_2 + \ldots$$
$$+ \alpha_r + (s+b)^r[\text{other partial fractions}] \quad \text{(A-61)}$$

Setting $s = -b$ will generate α_r directly. Differentiating $Q(s)$ with respect to s and letting $s = -b$ generates α_{r-1}. Successive differentiation a total of $r - 1$ times will generate all α_i, $i = 1, 2, \ldots, r$ from which we obtain the general expression

$$\alpha_{r-i} = \left.\frac{1}{i}\frac{d^{(i)}Q(s)}{ds^{(i)}}\right|_{s=-b} \quad i = 0, \ldots, r-1 \quad \text{(A-62)}$$

For $i = 0$ in (A-62), 0! is defined to be 1 and the zeroth derivative of $Q(s)$ is defined to be simply $Q(s)$ itself.

Returning to the problem in Example A.3,

$$Q(s) = \frac{s+1}{s} \quad \text{(A-63)}$$

from which

$$i = 0{:}\ \alpha_2 = \left.\frac{s+1}{s}\right|_{s=-2} = \frac{1}{2} \quad \text{(A-64a)}$$

$$i = 1{:}\ \alpha_1 = \left.\frac{d\left(\frac{s+1}{s}\right)}{ds}\right|_{s=-2} = \left.\frac{-1}{s^2}\right|_{s=-2} = -\frac{1}{4} \quad \text{(A-64b)}$$

Complex Factors

An important case occurs when the factored characteristic polynomial yields terms of the form

$$\frac{c_1 s + c_0}{s^2 + d_1 s + d_0}$$

where

$$\frac{d_1^2}{4} < d_0$$

Here the denominator cannot be written as the product of two real factors, which can be determined by using the quadratic formula.

For example, consider the transform

$$Y(s) = \frac{s+2}{s^2+s+1} \quad \text{(A-65)}$$

To invert (A-65) to the time domain, we complete the square of the first two terms in the denominator:

$$Y(s) = \frac{s+2}{(s+0.5)^2 - 0.25 + 1} = \frac{(s+0.5)+1.5}{(s+0.5)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \quad \text{(A-66)}$$

Dividing the numerator of $Y(s)$ into two terms,

$$Y(s) = \frac{s+0.5}{(s+0.5)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1.5}{(s+0.5)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \quad \text{(A-67)}$$

To determine $y(t)$, we invert each term separately. Note that in Table A.1, $\frac{s+b}{(s+b)^2+\omega^2}$ transforms to $e^{-bt}\cos\omega t$, while $\frac{\omega}{(s+b)^2+\omega^2}$ transforms to $e^{-bt}\sin\omega t$.

Therefore the corresponding time-domain solution is

$$y(t) = e^{-0.5t}\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\,e^{-0.5t}\sin\frac{\sqrt{3}}{2}t \quad \text{(A-68)}$$

If the denominator is factored into a pair of complex terms (complex conjugates) in the partial fraction equation, we can alternatively express the transform as

$$Y(s) = \frac{\alpha_1 + j\beta_1}{s + b + j\omega} + \frac{\alpha_2 + j\beta_2}{s + b - j\omega} \quad \text{(A-69)}$$

Appearance of these complex factors implies oscillatory behavior in the time domain. Terms of the form $e^{-bt}\sin\omega t$ and $e^{-bt}\cos\omega t$ arise after combining the inverse transforms $e^{-(b+j\omega)t}$ and $e^{-(b-j\omega)t}$. Dealing with complex factors is more tedious than analyzing real factors.

A partial fraction form that avoids complex algebra is

$$Y(s) = \frac{a_1(s + b) + a_2}{(s + b)^2 + \omega^2} + \cdots \quad \text{(A-70)}$$

Using Table A.1, the corresponding expression for $y(t)$ is

$$y(t) = a_1 e^{-bt}\cos\omega t + \frac{a_2}{\omega}e^{-bt}\sin\omega t + \cdots \quad \text{(A-71)}$$

However, the coefficients a_1 and a_2 must be found by solving simultaneous equations, rather than by the Heaviside expansion, as shown as follows in Example A.4.

EXAMPLE A.4

Find the inverse Laplace transform of

$$Y(s) = \frac{s + 1}{s^2(s^2 + 4s + 5)} \quad \text{(A-72)}$$

SOLUTION

The roots of the denominator term $(s^2 + 4s + 5)$ are imaginary $(s + 2 + j, s + 2 - j)$, so we know the solution will involve oscillatory terms (sin, cos). The partial fraction form for (A-72) that avoids using complex factors or roots is

$$Y(s) = \frac{s + 1}{s^2(s^2 + 4s + 5)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \frac{\alpha_5 s + \alpha_6}{s^2 + 4s + 5} \quad \text{(A-73)}$$

Multiply both sides of Eq. A-73 by $s^2(s^2 + 4s + 5)$ and collect terms:

$$s + 1 = (\alpha_1 + \alpha_5)s^3 + (4\alpha_1 + \alpha_2 + \alpha_6)s^2 + (5\alpha_1 + 4\alpha_2)s + 5\alpha_2 \quad \text{(A-74)}$$

Equate coefficients of like powers of s:

$$s^3: \alpha_1 + \alpha_5 = 0 \quad \text{(A-75a)}$$

$$s^2: 4\alpha_1 + \alpha_2 + \alpha_6 = 0 \quad \text{(A-75b)}$$

$$s^1: 5\alpha_1 + 4\alpha_2 = 1 \quad \text{(A-75c)}$$

$$s^0: 5\alpha_2 = 1 \quad \text{(A-75d)}$$

Solving simultaneously gives $\alpha_1 = 0.04$, $\alpha_2 = 0.2$, $\alpha_5 = 0.04$, $\alpha_6 = -0.36$. The inverse Laplace transform of $Y(s)$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{0.04}{s}\right) + \mathcal{L}^{-1}\left(\frac{0.2}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{-0.04s - 0.36}{s^2 + 4s + 5}\right) \quad \text{(A-76)}$$

Before using Table A.1, the denominator term $(s^2 + 4s + 5)$ must be converted to the standard form by completing the square to $(s + 2)^2 + 1^2$; the numerator is $-0.04(s + 9)$. In order to match the expressions in Table A.1, the argument of the last term in (A-76) must be written as

$$\frac{-0.04s - 0.36}{(s + 2)^2 + 1} = \frac{-0.04(s + 2)}{(s + 2)^2 + 1} + \frac{-0.28}{(s + 2)^2 + 1} \quad \text{(A-77)}$$

This procedure yields the following time-domain expression:

$$y(t) = 0.04 + 0.2t - 0.04e^{-2t}\cos t - 0.28e^{-2t}\sin t$$

It is clear from this example that the Laplace transform solution for complex or repeated roots can be quite cumbersome for transforms of ODEs higher than second order. In this case, using numerical simulation techniques may be more efficient to obtain a solution, as discussed in Chapters 4 and 5.

A.4 OTHER LAPLACE TRANSFORM PROPERTIES

In this section, we consider several Laplace transform properties that are useful in process dynamics and control.

A.4.1 Final Value Theorem

The asymptotic value of $y(t)$ for large values of time $y(\infty)$ can be found from (A-78), providing that $\lim_{s\to 0}[sY(s)]$ exists for all $Re(s) \geq 0$:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}[sY(s)] \quad \text{(A-78)}$$

Equation A-78 can be proved using the relation for the Laplace transform of a derivative (Eq. A-9):

$$\int_0^\infty \frac{dy}{dt}e^{-st}\,dt = sY(s) - y(0) \quad \text{(A-79)}$$

Taking the limit as $s \to 0$ and assuming that dy/dt is continuous and that $sY(s)$ has a limit for all $Re(s) \geq 0$,

$$\int_0^\infty \frac{dy}{dt}\,dt = \lim_{s\to 0}[sY(s)] - y(0) \quad \text{(A-80)}$$

Integrating the left side and simplifying yields

$$\lim_{t\to\infty} y(t) = \lim_{s\to\infty}[sY(s)] \quad \text{(A-81)}$$

If $y(t)$ is unbounded for $t \to \infty$, Eq. A-81 gives erroneous results. For example, if $Y(s) = 1/(s - 5)$, Eq. A-81 predicts $y(\infty) = 0$. Note that Eq. A-79, which is the basis of (A-79), requires that $\lim y(t \to \infty)$ exists. In this case, $y(t) = e^{5t}$, which is unbounded for $t \to \infty$. However, Eq. A-79 does not apply here, because $sY(s) = s/(s - 5)$ does not have a limit for some real value of $s \geq 0$, in particular, for $s = 5$.

A.4.2 Initial Value Theorem

Analogous to the final value theorem, the initial value theorem can be stated as

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty}[sY(s)] \quad \text{(A-82)}$$

The proof of this theorem is similar to the development in (A-78) through (A-81). It also requires that $y(t)$ is continuous. The proof is left to the reader as an exercise.

EXAMPLE A.5

Apply the initial and final value theorems to the transform derived in Example A.1:

$$Y(s) = \frac{5s + 2}{s(5s + 4)}$$

SOLUTION

Initial Value

$$y(0) = \lim_{s\to\infty}[sY(s)] = \lim_{s\to\infty}\frac{5s + 2}{5s + 4} = 1 \quad \text{(A-83a)}$$

Final Value

$$y(\infty) = \lim_{s\to 0}[sY(s)] = \lim_{s\to 0}\frac{5s + 2}{5s + 4} = 0.5 \quad \text{(A-83b)}$$

The initial value of 1 corresponds to the initial condition given in Eq. A-26. The final value of 0.5 agrees with the time-domain solution in Eq. A-37. Both theorems are useful for checking mathematical errors that may occur in obtaining Laplace transform solutions.

EXAMPLE A.6

A process is described by a third-order ODE:

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 4u \quad \text{(A-84)}$$

with all initial conditions on y, dy/dt, and d^2y/dt^2 equal to zero. Show that for a step change in u of 2 units, the steady-state result in the time domain is the same as applying the final value theorem.

SOLUTION

If $u = 2$ the steady-state result for y can be found by setting all derivatives to zero and substituting for u. Therefore

$$6y = 8 \quad \text{or} \quad y = 4/3 \quad \text{(A-85)}$$

The transform of (A-84) is

$$(s^3 + 6s^2 + 11s + 6)Y(s) = 8/s \quad \text{(A-86)}$$

$$Y(s) = \frac{8}{s^4 + 6s^3 + 11s^2 + 6s} \quad \text{(A-87)}$$

One of the benefits of the final value theorem is that we do not have to solve for the analytical solution of $y(t)$. Instead, simply apply Eq. A-81 to the transform $Y(s)$ as follows:

$$\lim_{s\to 0} sY(s) = \lim_{s\to 0}\frac{8}{s^3 + 6s^2 + 11s + 6} = \frac{8}{6} = \frac{4}{3} \quad \text{(A-88)}$$

This is the same answer as obtained in Eq. A-85. The time-domain solution obtained from a partial fraction expansion is

$$y = 4/3 - 2e^{-t} + 2e^{-2t} - 2/3e^{-3t} \quad \text{(A-89)}$$

As $t \to \infty$, only the first term remains, which is the same result as in Eq. A-90 (using the final value theorem).

A.4.3 Transform of an Integral

Occasionally, it is necessary to find the Laplace transform of a function that is integrated with respect to time. By applying the definition (Eq. A-1) and integrating by parts,

$$\mathcal{L}\left\{\int_0^t f(t^*)\,dt^*\right\} = \int_0^\infty\left\{\int_0^t f(t^*)\,dt^*\right\}e^{-st}\,dt \quad \text{(A-90)}$$

$$= -\frac{1}{s}\left[e^{-st}\int_0^t f(t^*)\,dt^*\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}f(t)\,dt \quad \text{(A-91)}$$

The first term in (A-93) yields 0 when evaluated at both the upper and lower limits, as long as $f(t^*)$ possesses a transform (is bounded). The integral in the second term is simply the definition of the Laplace transform of $f(t)$. Hence,

$$\mathcal{L}\left\{\int_0^t f(t^*)\,dt^*\right\} = \frac{1}{s}F(s) \qquad \text{(A-92)}$$

Note that Laplace transformation of an integral function of time leads to division of the transformed function by s. We have already seen in (A-9) that transformation of time derivatives leads to an inverse relation—that is, multiplication of the transform by s.

A.4.4 Time Delay (Translation in Time)

Functions that exhibit time delay play an important role in process modeling and control. Time delays commonly occur as a result of the transport time required for a fluid to flow through piping. Consider the stirred-tank heating system example presented in Chapter 2. Suppose one thermocouple is located at the outflow point of the stirred tank, and a second thermocouple is immersed in the fluid a short distance (L= 10 m) downstream. The heating system is off initially, and at time zero it is turned on. If there is no fluid mixing in the pipe (the fluid is in plug flow) and if no heat losses occur from the pipe, the shapes of the two temperature responses should be identical. However, the second sensor response will be translated in time; that is, it will exhibit a *time delay*. If the fluid velocity is 1 m/s, the time delay ($t_0 = L/v$) is 10 s. If we denote $f(t)$ as the transient temperature response at the first sensor and $f_d(t)$ as the temperature response at the second sensor, Fig. A.3 shows how they are related. The function $f_d = 0$ for $t < t_0$. Therefore, f_d and f are related by

$$f_d(t) = f(t - t_0)S(t - t_0) \qquad \text{(A-93)}$$

Note that f_d is the function $f(t)$ delayed by t_0 time units. The unit step function $S(t - t_0)$ is included to denote explicitly that $f_d(t) = 0$ for all values of $t < t_0$. If $\mathcal{L}(f(t)) = F(s)$, then

$$\begin{aligned}\mathcal{L}(f_d(t)) &= \mathcal{L}(f(t - t_0)S(t - t_0))\\ &= \int_0^\infty f(t - t_0)S(t - t_0)e^{-st}\,dt\\ &= \int_0^{t_0} f(t - t_0)(0)e^{-st}\,dt + \int_{t_0}^\infty f(t - t_0)e^{-st}\,dt\\ &= \int_{t_0}^\infty f(t - t_0)e^{-s(t-t_0)}e^{-st_0}\,d(t - t_0) \qquad \text{(A-94)}\end{aligned}$$

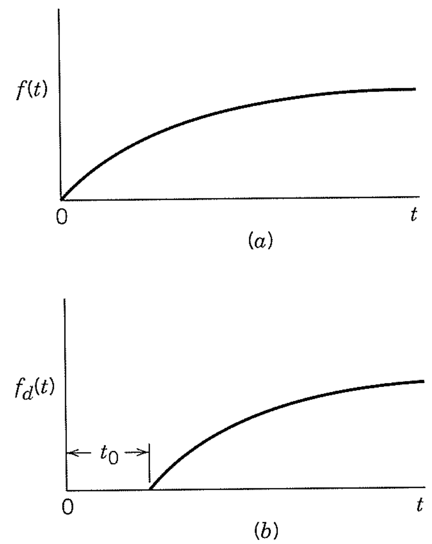

Figure A.3 A time function with and without time delay. (*a*) Original function (no delay); (*b*) function with delay t_0.

Because $(t - t_0)$ is now the artificial variable of integration, it can be replaced by t^*.

$$\mathcal{L}(f(t)) = e^{-st_0}\int_0^\infty f(t^*)e^{-st^*}\,dt^* \qquad \text{(A-95)}$$

yielding the *Real Translation Theorem:*

$$F_d(s) = \mathcal{L}(f(t - t_0)S(t - t_0)) = e^{-st_0}F(s) \qquad \text{(A-96)}$$

In inverting a transform that contains an e^{-st_0} element (time-delay term), the following procedure will easily yield results and also avoid the pitfalls of dealing with translated (shifted) time arguments. Starting with the Laplace transform

$$Y(s) = e^{-st_0}F(s) \qquad \text{(A-97)}$$

1. Invert $F(s)$ in the usual manner; that is, perform partial fraction expansion, and so forth, to find $f(t)$.
2. Find $y(t) = f(t - t_0)S(t - t_0)$ by replacing the argument t, wherever it appears in $f(t)$, by $(t - t_0)$; then multiply the entire function by the shifted unit step function, $S(t - t_0)$.

EXAMPLE A.6

Find the inverse transform of

$$Y(s) = \frac{1 + e^{-2s}}{(4s + 1)(3s + 1)} \qquad \text{(A-98)}$$

SOLUTION

Equation A-100 can be split into two terms:

$$\begin{aligned}Y(s) &= Y_1(s) + Y_2(s) &\text{(A-99)}\\ &= \frac{1}{(4s + 1)(3s + 1)} + \frac{e^{-2s}}{(4s + 1)(3s + 1)} &\text{(A-100)}\end{aligned}$$

The inverse transform of $Y_1(s)$ can be obtained directly from Table A.1:

$$y_1(t) = e^{-t/4} - e^{-t/3} \qquad \text{(A-101)}$$

Because $Y_2(s) = e^{-2s}Y_1(s)$, its inverse transform can be written immediately by replacing t by $(t - 2)$ in (A-101), and then multiplying by the shifted step function:

$$y_2(t) = [e^{-(t-2)/4} - e^{-(t-2)/3}]S(t - 2) \qquad \text{(A-102)}$$

Thus, the complete inverse transform is

$$y(t) = e^{-t/4} - e^{-t/3} + [e^{-(t-2)/4} - e^{-(t-2)/3}]S(t - 2) \qquad \text{(A-103)}$$

Equation A-103 can be numerically evaluated without difficulty for particular values of t, noting that the term in brackets is multiplied by 0 (the value of the unit step function) for $t < 2$, and by 1 for $t \geq 2$. An equivalent and simpler method is to evaluate the contributions from the bracketed time functions only when the time arguments are nonnegative. An alternative way of writing Eq. A-105 is as two equations, each one applicable over a particular interval of time:

$$0 \leq t < 2: \quad y(t) = e^{-t/4} - e^{-t/3} \qquad \text{(A-104)}$$

and

$$\begin{aligned}t \geq 2: \quad y(t) &= e^{-t/4} - e^{-t/3} + [e^{-(t-2)/4} - e^{-(t-2)/3}]\\ &= e^{-t/4}(1 + e^{2/4}) - e^{-t/3}(1 + e^{2/3})\\ &= 2.6487e^{-t/4} - 2.9477e^{-t/3} \qquad \text{(A-105)}\end{aligned}$$

Note that (A-104) and (A-105) give equivalent results for $t = 2$, because in this case, $y(t)$ is continuous at $t = 2$.

A.5 A TRANSIENT RESPONSE EXAMPLE

In Chapter 3 we will develop a standardized approach for using Laplace transforms to calculate transient responses. That approach will unify the way process models are manipulated after transforming them, and it will further simplify the way initial conditions and inputs (forcing functions) are handled. However, we already have the tools to analyze an example of a transient response situation in some detail. Example A.7 illustrates many features of Laplace transform methods in investigating the dynamic characteristics of a physical process.

EXAMPLE A.7

The Ideal Gas Company has two fixed-volume, stirred-tank reactors connected in series as shown in Fig. A.4. The three IGC engineers who are responsible for reactor operations—Kim Ng, Casey Gain, and Tim Delay—are concerned about the adequacy of mixing in the two tanks and want to run a tracer test on the system to determine whether dead zones and/or channeling exist in the reactors.

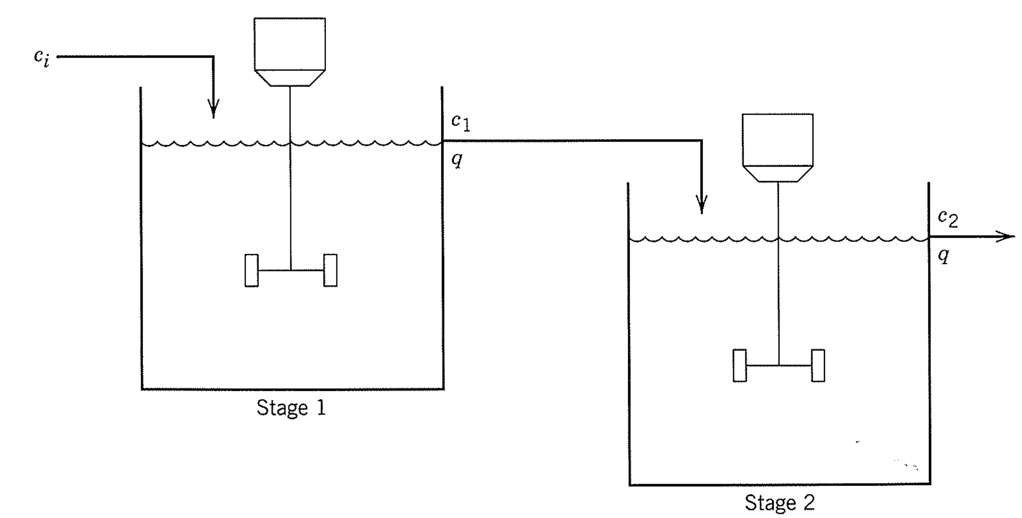

Figure A.4 Two-stage stirred-tank reactor system.

Table A.2 Two-Stage Stirred-Tank Reactor Process and Operating Data

Volume of Stage 1	$= 4\ \text{m}^3$
Volume of Stage 2	$= 3\ \text{m}^3$
Total flow rate q	$= 2\ \text{m}^3/\text{min}$
Nominal feed reactant concentration (c_i)	$= 1\ \text{kg mol/m}^3$

Their idea is to operate the reactors at a temperature low enough that reaction will not occur, and to apply a rectangular pulse in the reactant concentration to the first stage for test purposes. In this way, available instrumentation on the second-stage outflow line can be used without modification to measure reactant (tracer) concentration.

Before performing the test, the engineers would like to have a good idea of the results that should be expected if perfect mixing actually is accomplished in the reactors. A rectangular pulse input for the change in reactant concentration will be used with the restriction that the resulting output concentration changes must be large enough to be measured precisely.

The process data and operating conditions required to model the reactor tracer test are given in Table A.2. Figure A.5 shows the proposed pulse change of 0.25 min duration that can be made while maintaining the total reactor input flow rate constant. As part of the theoretical solution, Kim, Casey, and Tim would like to know how closely the rectangular pulse response can be approximated by the system response to an impulse of equivalent magnitude. Based on all of these considerations, they need to obtain the following information:

(a) The magnitude of an impulse input equivalent to the rectangular pulse of Fig. A.5.
(b) The impulse and pulse responses of the reactant concentration leaving the first stage.
(c) The impulse and pulse responses of the reactant concentration leaving the second stage.

SOLUTION

The reactor model for a single-stage CSTR was given in Eq. 2-66 as

$$V\frac{dc}{dt} = q(c_i - c) - Vkc$$

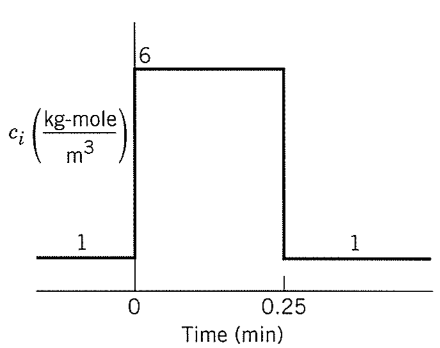

Figure A.5 Proposed input pulse in reactant concentration.

where c is the reactant concentration of component A. Because the reaction term can be neglected in this example ($k = 0$), the stages are merely continuous-flow mixers. Two material balance equations are required to model the two stages:

$$4\frac{dc_1}{dt} + 2c_1 = 2c_i \tag{A-106}$$

$$3\frac{dc_2}{dt} + 2c_2 = 2c_1 \tag{A-107}$$

If the system initially is at steady state, all concentrations are equal to the feed concentration:

$$c_2(0) = c_1(0) = c_i(0) = 1\ \text{kg mol/m}^3 \tag{A-108}$$

(a) The pulse input is described by

$$c_i^P = \begin{cases} 1 & t < 0 \\ 6 & 0 \le t < 0.25\ \text{min} \\ 1 & t \ge 0.25\ \text{min} \end{cases} \tag{A-109}$$

A convenient way to interpret (A-109) is as a constant input of 1 added to a rectangular pulse of height = 5 kg mol/m³:

$$c_i^P = 6 \quad \text{for} \quad 0 \le t < 0.25\ \text{min} \tag{A-110}$$

The magnitude of an impulse input that is equivalent to the time-varying portion of (A-110) is simply the integral of the rectangular pulse:

$$M = 5\frac{\text{kg mol}}{\text{m}^3} \times 0.25\ \text{min} = 1.25\frac{\text{kg mol}\cdot\text{min}}{\text{m}^3}$$

Therefore, the equivalent impulse input is

$$c_i^\delta(t) = 1 + 1.25\delta(t) \tag{A-111}$$

Although the units of M have little physical meaning, the product

$$qM = 2\frac{\text{m}^3}{\text{min}} \times 1.25\frac{\text{kg mol}\cdot\text{min}}{\text{m}^3} = 2.5\ \text{kg mol}$$

can be interpreted as the amount of additional reactant fed into the reactor as either the rectangular pulse or the impulse.

(b) The impulse response of Stage 1 is obtained by Laplace transforming (A-106), using $c_1(0) = 1$:

$$4sC_1(s) - 4(1) + 2C_1(s) = 2C_i(s) \tag{A-112}$$

By rearranging (A-112), we obtain $C_1(s)$:

$$C_1(s) = \frac{4}{4s+2} + \frac{2}{4s+2}C_i(s) \tag{A-113}$$

The transform of the impulse input in feed concentration in (A-111) is

$$C_i^\delta(s) = \frac{1}{s} + 1.25 \tag{A-114}$$

Substituting (A-114) into (A-113), we have

$$C_1^\delta(s) = \frac{2}{s(4s+2)} + \frac{6.5}{4s+2} \tag{A-115}$$

Equation A-115 does not correspond exactly to any entries in Table A.1. However, putting the denominator in $\tau s + 1$ form yields

$$C_1^\delta(s) = \frac{1}{s(2s+1)} + \frac{3.25}{2s+1} \tag{A-116}$$

which can be directly inverted using the table, yielding

$$c_1^\delta(t) = 1 - e^{-t/2} + 1.625e^{-t/2} = 1 + 0.625e^{-t/2} \tag{A-117}$$

The rectangular pulse response is obtained in the same way. The transform of the input pulse (A-109) is given by (A-22), so that

$$C_i^P(s) = \frac{1}{s} + \frac{5(1 - e^{-0.25s})}{s} \tag{A-118}$$

Substituting (A-118) into (A-113) and solving for $C_1^P(s)$ yields

$$C_1^P(s) = \frac{4}{4s+2} + \frac{12}{s(4s+2)} - \frac{10e^{-0.25s}}{s(4s+2)} \tag{A-119}$$

Again, we have to put (A-121) into a form suitable for inversion:

$$C_1^P(s) = \frac{2}{2s+1} + \frac{6}{s(2s+1)} - \frac{5e^{-0.25s}}{s(2s+1)} \tag{A-120}$$

Before inverting (A-120), note that the term containing $e^{-0.25s}$ will involve a translation in time. Utilizing the procedure discussed above, we obtain the following inverse transform:

$$c_1^P(t) = e^{-t/2} + 6(1 - e^{-t/2}) - 5[1 - e^{-(t-0.25)/2}]S(t - 0.25) \tag{A-121}$$

Note that there are two solutions; for $t < 0.25$ min (or t_w) the rightmost term, including the time delay, is zero in the time solution. Thus, for

$$t < 0.25\ \text{min:} \quad c_1^P(t) = e^{-t/2} + 6(1 - e^{-t/2}) = 6 - 5e^{-t/2} \tag{A-122}$$

$$\begin{aligned} t \ge 0.25\ \text{min:} \quad c_1^P(t) &= e^{-t/2} + 6(1 - e^{-t/2}) \\ &\quad - 5(1 - e^{-(t-0.25)/2}) \\ &= 1 - 5e^{-t/2} + 5e^{-t/2}e^{+0.25/2} \\ &= 1 + 0.6657e^{-t/2} \end{aligned} \tag{A-123}$$

Plots of (A-117), (A-122), and (A-123) are shown in Fig. A.6. Note that the rectangular pulse response approximates the impulse response fairly well for $t > 0.25$ min. Obviously, the approximation cannot be very good before $t = 0.25$ min, because the full effect of the rectangular pulse is not felt until that time, while the full effect of the hypothetical impulse begins immediately at $t = 0$.

(c) For the impulse response of Stage 2, Laplace transform (A-107), using $c_2(0) = 1$:

$$3sC_2(s) - 3(1) + 2C_2(s) = 2C_1(s) \tag{A-124}$$

Rearrange to obtain $C_2(s)$:

$$C_2(s) = \frac{3}{3s+2} + \frac{2}{3s+2}C_1(s) \tag{A-125}$$

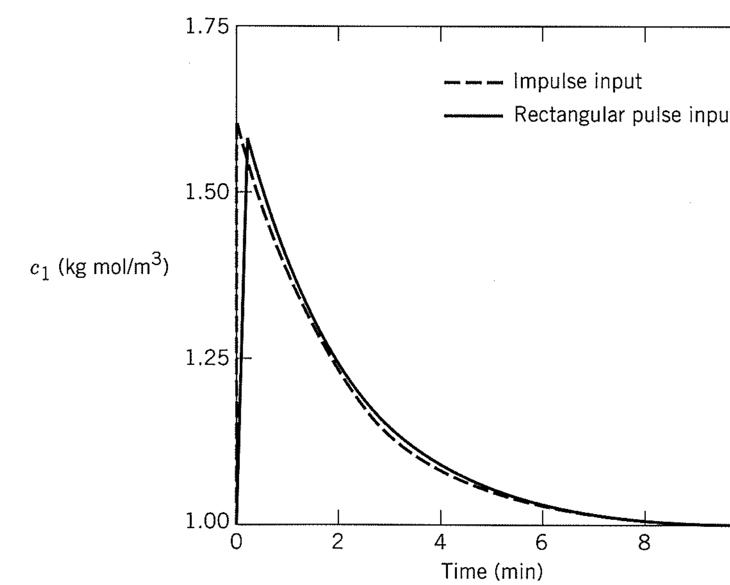

Figure A.6 Reactor Stage 1 response.

For the input to (A-127), substitute the Laplace transform of the output from Stage 1—namely, (A-116):

$$C_2^S(s) = \frac{3}{3s+2} + \frac{2}{3s+2}\left[\frac{1}{s(2s+1)} + \frac{3.25}{2s+1}\right] \tag{A-126}$$

which can be rearranged to

$$C_2^S(s) = \frac{1.5}{1.5s+1} + \frac{1}{s(1.5s+1)(2s+1)} + \frac{3.25}{(1.5s+1)(2s+1)} \tag{A-127}$$

Because each term in (A-127) appears as an entry in Table A.1, partial fraction expansion is not required:

$$c_2^S(t) = e^{-t/1.5} + \left[1 + \frac{1}{0.5}(1.5e^{-t/1.5} - 2e^{-t/2})\right] + \frac{3.25}{0.5}[e^{-t/2} - e^{-t/1.5}]$$
$$= 1 - 2.5e^{-t/1.5} + 2.5e^{-t/2} \tag{A-128}$$

For the rectangular pulse response of Stage 2, substitute the Laplace transform of the appropriate stage output, Eq. A-120, into Eq. A-125 to obtain

$$C_2^P(s) = \frac{1.5}{1.5s+1} + \frac{2}{(1.5s+1)(2s+1)} + \frac{6}{s(1.5s+1)(2s+1)} - \frac{5e^{-0.25s}}{s(1.5s+1)(2s+1)} \tag{A-129}$$

Again, the rightmost term in (A-129) must be excluded from the inverted result or included, depending on whether or not $t < 0.25$ min. The calculation of the inverse transform of (A-129) gives

$$t < 0.25: \quad c_2^P(t) = 6 + 15e^{-t/1.5} - 20e^{-t/2} \tag{A-130}$$
$$t \geq 0.25: \quad c_2^P(t) = 1 - 2.7204e^{-t/1.5} + 2.663e^{-t/2} \tag{A-131}$$

Plots of Eqs. A-128, A-130, and A-131 are shown in Fig. A.7. The rectangular pulse response is virtually indistinguishable from the impulse response. Hence, Kim, Casey, and Tim can use the simpler impulse response solution to compare with real data obtained when the reactor is forced by a rectangular pulse. The maximum expected value of $c_2(t)$ is approximately 1.25 kg mol/m^3. This value should be compared with the nominal concentration before and after the test ($\bar{c}_2 = 1.0$ kg mol/m^3) to determine if the instrumentation is precise enough to record the change in concentration. If the change is too small, then the pulse amplitude, pulse width, or both must be increased.

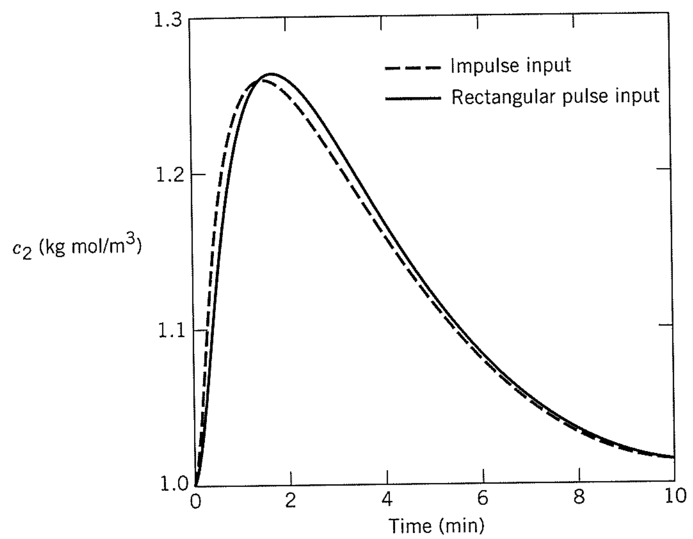

Figure A.7 Reactor Stage 2 response.

Because this system is linear, multiplying the pulse magnitude (h) by a factor of four would yield a maximum concentration of reactant in the second stage of about 2.0 (the difference between initial and maximum concentration will be four times as large). On the other hand, the solutions obtained above strictly apply only for $t_w = 0.25$ min. Hence, the effect of a fourfold increase in t_w can be predicted only by re-solving the model response for $t_w = 1$ min. Qualitatively, we know that the maximum value of c_2 will increase as t_w increases. Because the impulse response model is a reasonably good approximation with $t_w = 0.25$ min, we expect that *small changes* in the pulse width will yield an approximately proportional effect on the maximum concentration change. This argument is based on a proportional increase in the approximately equivalent impulse input. A quantitative verification using numerical simulation is left as an exercise.

SUMMARY

In this chapter we have considered the application of Laplace transform techniques to solve linear differential equations. Although this material may be a review for some readers, an attempt has been made to concentrate on the important properties of the Laplace transform and its inverse, and to point out the techniques that make manipulation of transforms easier and less prone to error.

The use of Laplace transforms can be extended to obtain solutions for models consisting of simultaneous differential equations. However, before addressing such extensions, we introduce the concept of input-output models described by transfer functions. The conversion of differential equation models into transfer function models, covered in the next chapter, represents an important simplification in the methodology, one that can be exploited extensively in process modeling and control system design.

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EXERCISES

A.1 The differential equation (dynamic) model for a chemical process is as follows:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = 2u(t)$$

where $u(t)$ is the single input function of time. $y(0)$ and dy/dt (0) are both zero.

What are the functions of the time (e.g., $e^{-t/\tau}$) in the solution to the ODE for output $y(t)$ for each of the following cases?

(a) $u(t) = be^{-2t}$

(b) $u(t) = ct$

b and c are constants.

Note: You do not have to find $y(t)$ in these cases. Just determine the functions of time that will appear in $y(t)$.

A.2 Solve the ODE

$$\frac{d^4y}{dt^4} + 16\frac{d^3y}{dt^3} + 86\frac{d^2y}{dt^2} + 176\frac{dy}{dt} + 105y = 1$$

using partial fraction expansion. Note you need to calculate the roots of a fourth-order polynomial in s. All initial conditions on y and its derivatives are zero.

A.3 Figure EA.3 shows a pulse function.

(a) From details in the drawing, calculate the pulse width, t_w.

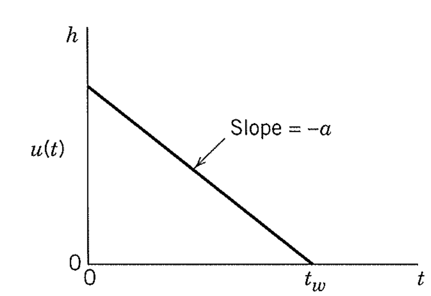

Figure EA.3 Triangular pulse function.

(b) Construct this function as the sum of simpler time elements, some perhaps translated in time, whose transforms can be found directly from Table A.1.

(c) Find $U(s)$.

(d) What is the area under the pulse?

A.4 Derive Laplace transforms of the input signals shown in Figs. EA.4*a* and EA.4*b* by summing component functions found in Table A.1.

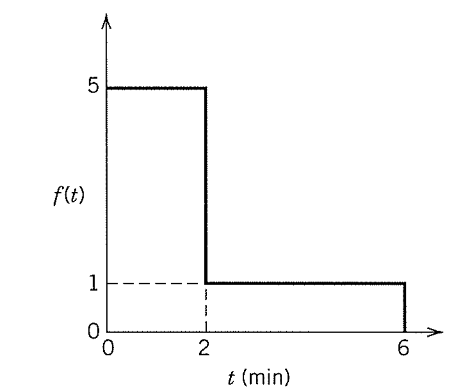

Figure EA.4*a*

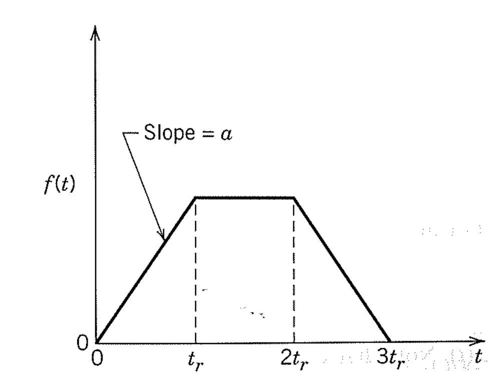

Figure EA.4*b*

A.5 The start-up procedure for a batch reactor includes a heating step where the reactor temperature is gradually heated to the nominal operating temperature of 75°C. The desired temperature profile $T(t)$ is shown in Fig. EA.5. What is $T(s)$?

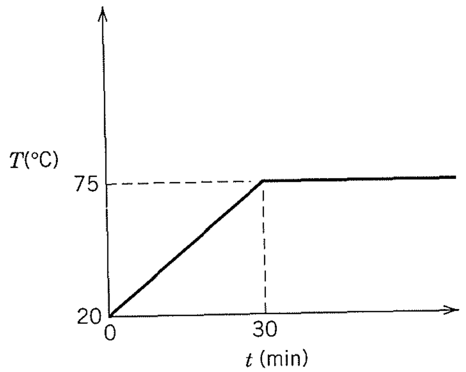

Figure EA.5

A.6 Using partial fraction expansion where required, find $x(t)$ for

(a) $X(s) = \dfrac{s(s+1)}{(s+2)(s+3)(s+4)}$

(b) $X(s) = \dfrac{s+1}{(s+2)(s+3)(s^2+4)}$

(c) $X(s) = \dfrac{s+4}{(s+1)^2}$

(d) $X(s) = \dfrac{1}{s^2+s+1}$

(e) $X(s) = \dfrac{s+1}{s(s+2)(s+3)}e^{-0.5s}$

A.7 Expand each of the following s-domain functions into partial fractions:

(a) $Y(s) = \dfrac{6(s+1)}{s^2(s+1)}$

(b) $Y(s) = \dfrac{12(s+2)}{s(s^2+9)}$

(c) $Y(s) = \dfrac{(s+2)(s+3)}{(s+4)(s+5)(s+6)}$

(d) $Y(s) = \dfrac{1}{[(s+1)^2+1]^2(s+2)}$

A.8 (a) For the integro-differential equation

$$\ddot{x} + 3\dot{x} + 2x = 2\int_0^t e^{-\tau}\,d\tau$$

find $x(t)$. Note that $\dot{x} = dx/dt$, etc.

(b) What is the value of $x(t)$ as $t \to \infty$?

A.9 For each of the following functions $X(s)$, what can you say about $x(t)$ $(0 \le t \le \infty)$ without solving for $x(t)$? In other words, what are $x(0)$ and $x(\infty)$? Is $x(t)$ converging, or diverging? Is $x(t)$ smooth, or oscillatory?

(a) $X(s) = \dfrac{6(s+2)}{(s^2+9s+20)(s+4)}$

(b) $X(s) = \dfrac{10s^2-3}{(s^2-6s+10)(s+2)}$

(c) $X(s) = \dfrac{16s+5}{s^2+9}$

A.10 For each of the following cases, determine what functions of time, e.g., $\sin 3t$, e^{-8t}, will appear in $y(t)$. (Note that you do not have to find $y(t)$!) Which $y(t)$ are oscillatory? Which exhibit a constant value of $y(t)$ for large values of t?

(i) $Y(s) = \dfrac{2}{s(s^2+4s)}$

(ii) $Y(s) = \dfrac{2}{s(s^2+4s+3)}$

(iii) $Y(s) = \dfrac{2}{s(s^2+4s+4)}$

(iv) $Y(s) = \dfrac{2}{s(s^2+4s+8)}$

(v) $Y(s) = \dfrac{2(s+1)}{s(s^2+4)}$

A.11 Which solutions of the following equations will exhibit convergent behavior? Which are oscillatory?

(a) $\dfrac{d^3x}{dt^3} + 2\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 3$

(b) $\dfrac{d^2x}{dt^2} - x = 2e^t$

(c) $\dfrac{d^3x}{dt^3} + x = \sin t$

(d) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} = 4$

Note: All of the above differential equations have one common factor in their characteristic equations.

A.12 The differential equation model for a particular chemical process has been found by testing to be as follows:

$$\tau_1\tau_2\frac{d^2y}{dt^2} + (\tau_1+\tau_2)\frac{dy}{dt} + y = Ku(t)$$

where τ_1 and τ_2 are constant parameters and $u(t)$ is the input function of time.

What are the functions of time (e.g., e^{-t}) in the solution for each output $y(t)$ for the following cases? (Optional: find the solutions for $y(t)$.)

(a) $u(t) = aS(t)$ unit step function

(b) $u(t) = be^{-t/\tau}$ $\tau \ne \tau_1 \ne \tau_2$

(c) $u(t) = ce^{-t/\tau}$ $\tau = \tau_1 \ne \tau_2$

(d) $u(t) = d\sin\omega t$ $\tau_1 \ne \tau_2$

A.13 Find the complete time-domain solutions for the following differential equations using Laplace transforms:

(a) $\dfrac{d^3x}{dt^3} + 4x = e^t$ with $x(0) = 0$, $\dfrac{dx(0)}{dt} = 0$, $\dfrac{d^2x(0)}{dt^2} = 0$

(b) $\dfrac{dx}{dt} - 12x = \sin 3t$ $x(0) = 0$

(c) $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 25x = e^{-t}$ $x(0) = 0$, $\dfrac{dx(0)}{dt} = 0$

(d) A process is described by two differential equations:

$$\frac{dy_1}{dt} + y_2 = x_1$$

$$\frac{dy_2}{dt} - 2y_1 + 3y_2 = 2x_2$$

If $x_1 = e^{-t}$ and $x_2 = 0$, what can you say about the form of the solution for y_1? For y_2?

A.14 The dynamic model between an output variable y and an input variable u can be expressed by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = 4\frac{du(t-2)}{dt} - u(t-2)$$

(a) Will this system exhibit an oscillatory response after an arbitrary change in u?

(b) What is the steady-state gain?

(c) For a step change in u of magnitude 1.5, what is $y(t)$?

A.15 Find the solution of

$$\frac{dx}{dt} + 4x = f(t)$$

$$\text{where } f(t) = \begin{cases} 0 & t < 0 \\ h & 0 \le t < 1/h \\ 0 & t \ge 1/h \end{cases}$$

$$x(0) = 0$$

Plot the solution for values of $h = 1, 10, 100$, and the limiting solution ($h \to \infty$) from $t = 0$ to $t = 2$. Put all plots on the same graph.

A.16 (a) The differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = \cos t$$

has initial conditions $y(0) = 1$, $y'(0) = 2$. Find $Y(s)$ and, without finding $y(t)$, determine what functions of time will appear in the solution.

(b) If $Y(s) = \dfrac{s+1}{s(s^2+4s+8)}$, find $y(t)$.

A.17 A stirred-tank blending system initially is full of water and is being fed pure water at a constant flow rate, q. At a particular time, an operator shuts off the pure water flow and adds caustic solution at the same volumetric flow rate q but with concentration $\bar{c}_i$. If the liquid volume V is constant, the dynamic model for this process is

$$V\frac{dc}{dt} + qc = q\bar{c}_i$$

with $c(0) = 0$.

What is the concentration response of the reactor effluent stream, $c(t)$? Sketch it as a function of time.

Data: $V = 2\ \text{m}^3$; $q = 0.4\ \text{m}^3/\text{min}$; $\bar{c}_i = 50\ \text{kg/m}^3$

A.18 For the dynamic system

$$2\frac{dy}{dt} = -y + 5u$$

y and u are deviation variables—y in degrees, u in flow rate units.

(a) u is changed from 0.0 to 2.0 at $t = 0$. Sketch the response and show the value of y_{ss}. How long does it take for y to reach within 0.1 degree of the final steady state?

(b) If u is changed from 0.0 to 4.0 at $t = 0$, how long does it take to cross the same steady state that was determined in part (a)? What is the new steady state?

(c) Suppose that after step (a) that the new temperature is maintained at 10 degrees for a long time. Then, at $t = t_1$, u is returned to zero. What is the new steady-state value of y? Use Laplace transformation to show how to obtain the analytical solution to the above ODE for this case. (Hint: select a new time, $t = 0$, where $y(0) = 10$.)

A.19 Will the solution to the ODE that follows reach a steady state? Will it oscillate?

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = 4$$

Show appropriate calculations using partial fraction expansion and Laplace transforms.

A.20 Three stirred-tanks in series are used in a reactor train (see Fig. EA.20). The flow rate into the system of some inert species is maintained constant while tracer test are conducted. Assuming that mixing in each tank is perfect and volumes are constant:

(a) Derive model expressions for the concentration of tracer leaving each tank, c_i is the concentration of tracer entering the first tank.

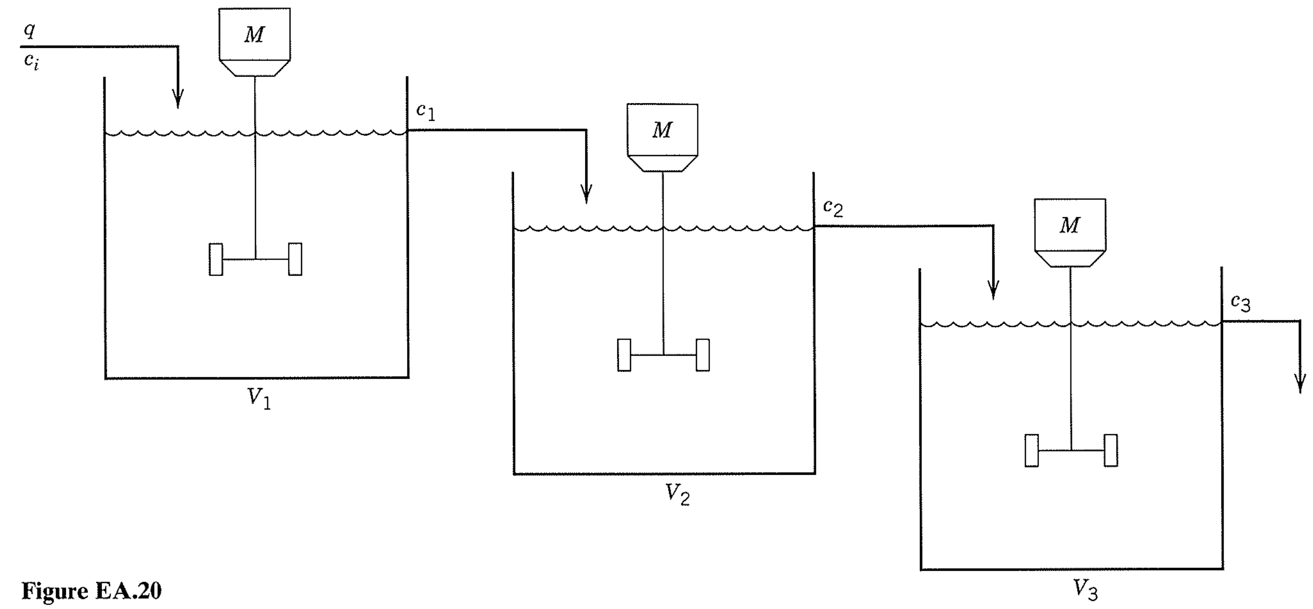

Figure EA.20

(b) If c_i has been constant and equal to zero for a long time and an operator suddenly injects a large amount of tracer material in the inlet to tank 1, what will be the form of $c_3(t)$ (i.e., what kind of time functions will be involved) if

1. $V_1 = V_2 = V_3$
2. $V_1 \neq V_2 \neq V_3$.

(c) If the amount of tracer injected is unknown, is it possible to back-calculate the amount from experimental data? How?

A.21 A stirred-tank reactor is operated with a feed mixture containing reactant A at a mass concentration C_{Ai}. The feed flow rate is w_i, as shown in Fig. EA.21. Under certain conditions the system operates according to the model

$$\frac{d(\rho V)}{dt} = w_i - w$$

$$\frac{d(\rho V c_A)}{dt} = w_i c_{Ai} - w c_A - \rho V k c_A$$

(a) For cases where the feed flow rate and feed concentration may vary and the volume is not fixed, simplify the model to one or more equations that do not contain product derivatives. The density may be assumed to be constant. Is the model in a satisfactory form for Laplace transform operations? Why or why not?

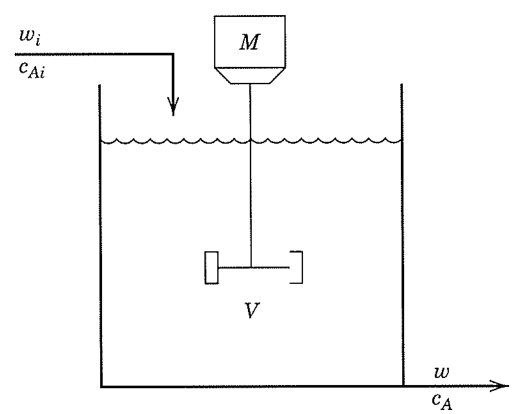

Figure EA.21

(b) For the case where the feed flow rate has been steady at $\overline{w}_i$ for some time, determine how c_A changes with time if a step change in c_{Ai} is made from c_{A1} to c_{A2}. List all assumptions necessary to solve the problem using Laplace transform techniques.

Appendix B

Digital Process Control Systems: Hardware and Software

Process control implemented by computers has undergone extensive changes in both concepts and equipment during the past 50 years. The feasibility of digital computer control in the chemical process industries was first investigated in the mid-1950s. During that period, studies were performed to identify chemical processes that were suitable for process monitoring and control by computers. These efforts culminated in several successful applications, the first ones being a Texaco refinery and a Monsanto chemical plant (both on the Gulf Coast) using mainframe computers. The first commercial systems were slow in execution and massive in size compared with the computers available today. They also had very limited capacity. For example, a typical first-generation process control computer had 32K RAM and disk storage of 1MB.

The functionalities of these early control systems were limited by capabilities of the existing computers rather than the process characteristics. These limitations, coupled with inadequate operator training and an unfriendly user interface, led to designs that were difficult to operate, maintain, and expand. In addition, many systems had customized specifications, making them extremely expensive. Although valuable experience was gained in systems design and implementation, the lack of financial success hindered the infusion of digital system applications into the process industries until about 1970, when inexpensive microprocessors became available commercially (Lipták, 2005).

During the past 40 years, developments in microelectronics and software technologies have led to the widespread application of computer control systems. Digital control systems have largely replaced traditional analog instrument panels, allowing computers to control process equipment while monitoring process conditions. Technological advancements, such as VLSI (very large-scale integrated) circuitry, object-oriented programming techniques, and distributed configurations have improved system reliability and maintainability while reducing manufacturing and implementation cost. This cost reduction has allowed small-scale applications in new areas, for example, microprocessors in single-loop controllers and smart instruments (Herb, 1999). Programmable logic controllers have also gained a strong foothold in the process industries.

Increased demand for digital control systems created a new industry, consisting of systems engineering and service organizations. Manufacturing companies moved toward enterprise-wide computer networks by interfacing process control computers with business computer networks. These networks permit all computers to use the same databases in planning and scheduling (see Chapter 19), and they also allow access to operator station information from locations outside the plant.

In the following sections, we provide an overview of the hardware and software used for process control. The distributed control system configuration is described first, followed by data acquisition for different signal types. Digital hardware is then considered, and concluding with a description of control system software organization and architectures.