MPC Introduction

• Overview
  • Basic Concept of MPC
  • History

• Optimization Formulation
  • Models
  • Analytical Solution to Unconstrained Problem

• Summary
  • Limitations & a Look Ahead

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Chemical and Biological Engineering
Motivation: Complex Processes
Important Issues in Petroleum Refining

- **Multivariable, Large Scale**
  - Challenge to tune individual SISO controllers

- **Operation at Constraints**
  - Anti-reset windup and other strategies for PID

- **Economic Payout for Advanced Control**
  - Economic return justifies capital and on-going maintenance costs

- **Model Predictive Control**
  - Evolved independently in the US and France refining industry
How is MPC used?

Unit 1 - PID Structure

- Plant-Wide Optimization
- Unit 1 Local Optimization
- High/Low Select Logic
- PID
- Lead/Lag
- SUM
- Unit 1 Distributed Control System (PID)
  - FC
  - PC
  - TC
  - LC

Unit 2 - MPC Structure

- Plant-Wide Optimization
- Unit 2 Local Optimization
- Model Predictive Control (MPC)
- Unit 2 Distributed Control System (PID)
  - FC
  - PC
  - TC
  - LC

From Tom Badgwell, 2003 Spring AIChE Meeting, New Orleans
Model Predictive Control (MPC)

Find current and future manipulated inputs that best meet a desired future output trajectory. Implement first “control move”.

- Type of model for predictions?
- Information needed at step k for predictions?
- Objective function and optimization technique?
- Correction for model error?
Model Predictive Control (MPC)

Find current and future manipulated inputs that best meet a desired future output trajectory. Implement first “control move”.

At next sample time:

Correct for model mismatch, then perform new optimization.

This is a major issue – “disturbances” vs. model uncertainty
MPC History

- **Intuitive**
  - Basically arose in two different “camps”

- **Dynamic Matrix Control (DMC)**
  - 1960’s and 1970’s – Shell Oil - US
    - Related to techniques developed in France (IDCOM)
    - Large-scale MIMO
    - Formulation for constraints important

- **Generalized Predictive Control (GPC)**
  - Evolved from adaptive control
  - Focus on SISO, awkward for MIMO

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**Model Predictive Heuristic Control: Applications to Industrial Processes**

J. Richalet,† A. Rault,† J. L. Testud† and J. Papon†

*Automatica, Vol. 14, pp. 413–428 (1976)*

**Clarke, D. W., C. Mohtadi and P. S. Tuffs (1987a).**


**Clarke, D. W., C. Mohtadi and P. S. Tuffs (1987b).**


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Objective Functions

Quadratic Objective Function, Prediction Horizon (P) = 3, Control Horizon (M) = 2

\[ J = \left( r_{k+1} - \hat{y}_{k+1} \right)^2 + \left( r_{k+2} - \hat{y}_{k+2} \right)^2 + \left( r_{k+3} - \hat{y}_{k+3} \right)^2 \]
\[ + w \Delta u_k^2 + w \Delta u_{k+1}^2 \]

Weight

3 steps into future

2 control moves

General Representation of a Quadratic Objective Function

\[ J = \sum_{i=1}^{P} \left( r_{k+i} - \hat{y}_{k+i} \right)^2 + w \sum_{i=0}^{M-1} \Delta u_{k+i}^2 \]

With linear models, results in analytical solution (w/o constraints)

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Alternative Objective Functions

Penalize $u$ rather than $\Delta u$

$$J = \sum_{i=1}^{P} (r_{k+i} - \hat{y}_{k+i})^2 + w \sum_{i=0}^{M-1} u_{k+i}^2$$

Will usually result in “offset”

Sum of absolute values (results in LP)

$$J = \sum_{i=1}^{P} |r_{k+i} - \hat{y}_{k+i}| + w \sum_{i=0}^{M-1} |\Delta u_{k+i}|$$

Existing LP methods are efficient, but solutions hop from one constraint to another

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Models

- State Space
- ARX (auto-regressive, exogenous input)
- Step Response
- Impulse (Pulse) Response
- Nonlinear, Fundamental (First-Principles)
- ANN (Artificial Neural Networks)
- Hammerstein (static NL with linear dynamics)
- Volterra
- Multiple Model
Discrete Linear Models used in MPC

\[ x_{k+1} = \Phi x_k + \Gamma u_k \]
\[ y_k = C x_k \]

State Space

Some texts/papers have different sign conventions

\[ y_k = -a_1 y_{k-1} - a_2 y_{k-2} - \cdots - a_n y_{k-n} + b_0 u_k + b_1 u_{k-1} + b_2 u_{k-2} + \cdots + b_m u_{k-m} \]

Input-Output (ARX)

usually \( b_0 = 0 \)

\[ y_k = \sum_{i=1}^{\infty} s_i \Delta u_{k-i} \]
\[ = s_1 \Delta u_{k-1} + \cdots + s_N \Delta u_{k-N} + s_{N+1} \Delta u_{k-N-1} + \cdots + s_{N+\infty} \Delta u_{k-\infty} \]

Step Response

\[ y_k = \sum_{i=1}^{\infty} h_i u_{k-i} \]
\[ = h_1 u_{k-1} + \cdots + h_N u_{k-N} + h_{N+1} u_{k-N-1} + \cdots + h_{N+\infty} u_{k-\infty} \]

Impulse Response

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Example Step Response Model

\[
S = \begin{bmatrix}
s_1 & s_2 & s_3 & s_4 & s_5 & \cdots & s_N \\
\end{bmatrix}^T
\]

Used in DMC
Example Impulse Response Model

Impulse and step response coefficients are related

\[ h_i = s_i - s_{i-1} \]

\[ s_i = \sum_{j=1}^{i} h_j \]
Step & Impulse Models from State Space Models

\[ x_{k+1} = \Phi x_k + \Gamma u_k \]
\[ y_k = C x_k \]

\[ H_i = C \Phi^{i-1} \Gamma \]

\[ S_k = \sum_{i=1}^{k} C \Phi^{i-1} \Gamma = \sum_{i=1}^{k} H_i \]
MPC based on State Space Models

\[ x_{k+1} = \Phi x_k + \Gamma u_k \]

\[ y_k = C x_k \]

with known current state, easy to propagate estimates

\[ x_{k+1} = \Phi x_k + \Gamma u_k \]

\[ y_{k+1} = C x_{k+1} = C\Phi x_k + C\Gamma u_k \]

and, using control changes

\[ u_k = u_{k-1} + \Delta u_k \]

\[ y_{k+1} = C\Phi x_k + C\Gamma u_{k-1} + C\Gamma \Delta u_k \]
Use \( ^{\wedge} \) notation for model states

\[
\hat{y}_{k+1|k} = C\Phi \hat{x}_{k|k} + C\Gamma u_{k-1} + C\Gamma \Delta u_k
\]

Now, propagate the prediction for \( P \) steps into the future
Output Predictions

\[
\begin{bmatrix}
\hat{y}_{k+1|k} \\
\hat{y}_{k+2|k} \\
\vdots \\
\hat{y}_{k+P|k}
\end{bmatrix}
= \begin{bmatrix}
C\Phi \\
C\Phi^2 \\
\vdots \\
C\Phi^P
\end{bmatrix}
\hat{x}_{k|k} + \begin{bmatrix}
C\Gamma \\
C\Phi \Gamma + C\Gamma \\
\vdots \\
\sum_{i=1}^{P} C\Phi^{i-1} \Gamma
\end{bmatrix} u_{k-1}
\]

"free" or "unforced response" (if no more control moves are made)

\[
f,
\begin{bmatrix}
0 & \cdots & 0 \\
C\Phi \Gamma + C\Gamma & C\Gamma & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^{P} C\Phi^{i-1} \Gamma & \sum_{i=1}^{P-1} C\Phi^{i-1} \Gamma & \cdots
\end{bmatrix}
\begin{bmatrix}
\Delta u_k \\
\Delta u_{k+1} \\
\vdots \\
\Delta u_{k+M-1}
\end{bmatrix}
\]

"forced" response

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Output Predictions

\[
\begin{bmatrix}
\hat{y}_{k+1|k} \\
\hat{y}_{k+2|k} \\
\vdots \\
\hat{y}_{k+P|k}
\end{bmatrix}
= 
\begin{bmatrix}
C\Phi \\
C\Phi^2 \\
\vdots \\
C\Phi^P
\end{bmatrix}
\hat{x}_{k|k} + 
\begin{bmatrix}
S_1 \\
S_2 \\
\vdots \\
S_P
\end{bmatrix}
\begin{bmatrix}
\Delta u_k \\
\Delta u_{k+1} \\
\vdots \\
\Delta u_{k+M-1}
\end{bmatrix}
\]

"free" or "unforced response" (if no more control moves are made)

\[
\begin{bmatrix}
S_1 & 0 & \cdots & 0 \\
S_2 & S_1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
S_P & S_{P-1} & \cdots & S_{P-M+1}
\end{bmatrix}
\]

"forced"response

\[\Delta u_f\]
Optimization Problem

\[
\min_{\Delta u_f} \quad J = \sum_{i=1}^{P} (r_{k+i|k} - \hat{y}_{k+i|k})^T W^y (r_{k+i|k} - \hat{y}_{k+i|k}) + \sum_{i=0}^{M-1} \Delta u_{k+i}^T W^u \Delta u_{k+i} \\
\quad \Delta u_f^T W^U \Delta u_f
\]

Where

\[
W^y = \begin{bmatrix}
W^y & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & W^y
\end{bmatrix}
\quad W^U = \begin{bmatrix}
W^u & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & W^u
\end{bmatrix}
\]

future setpoints

and

\[
\widehat{E} = r - \hat{Y} = r - f - S_f \Delta u_f
\]

“unforced” (free response) error

so

\[
\widehat{E} = E - S_f \Delta u_f
\]

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Optimization Problem

\[
\hat{E}^T W^Y \hat{E} = (E - S_f \Delta u_f)^T W^Y (E - S_f \Delta u_f)
\]

\[
= E^T W^Y E - 2\Delta u_f^T S_f^T W^Y E + \Delta u_f^T S_f^T W^Y S_f \Delta u_f
\]

so

\[
\min_{\Delta u_f} \quad J = \hat{E}^T W^Y \hat{E} + \Delta u_f^T W^U \Delta u_f
\]

can be written

\[
\min_{\Delta u_f} \quad J = \Delta u_f^T \left(S_f^T W^Y S_f + W^U\right) \Delta u_f - 2\Delta u_f^T S_f^T W^Y E
\]

and the unconstrained solution is found from

\[
\frac{\partial J}{\partial \Delta u_f} = 0
\]
Unconstrained Solution

Analytical Solution for Unconstrained System

\[ \Delta u_f = \left( S_f^T W^Y S_f + W^U \right)^{-1} S_f^T W^Y E \]

“unforced” error

In practice, do not actually invert a matrix. Solve as set of simultaneous equations (or use \ in MATLAB)

\[ \Delta u_f = \left( S_f^T W^Y S_f + W^U \right) \backslash S_f^T W^Y E \]
Vector of Control Moves

\[ \Delta u_f = \begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+M-1} \end{bmatrix} \]

current and future moves

Although a set of control moves is computed, only the first move \( \Delta u_k \) is implemented

The next output at \( k+1 \) is obtained, then a new optimization problem is solved
MPC Tuning Parameters

- Prediction Horizon, $P$
- Control Horizon, $M$
- Manipulated Input Weighting, $W^u$

Usually, $P >> M$ for robustness (less aggressive action). Sometimes $M = 1$, with $P$ varied for desired performance.

Sometimes larger input weights for robustness.
Pre-Summary

- Concise overview of MPC
- State space model, unconstrained solution
- Have not discussed
  - State estimation and “corrected outputs”
    - The additive disturbance assumption of DMC is covered in the slides that follow (this is identical to the plant-model mismatch term in IMC)
  - Disturbances
  - Constraints
  - Other model forms
Original DMC Approach to Plant-Model Mismatch

\[
\hat{x}_{k|k-1} = \Phi \hat{x}_{k-1|k-1} + \Gamma u_{k-1}
\]
\[
\hat{y}_{k|k-1} = C \hat{x}_{k|k-1}
\]

Prediction at step k, based on information at k-1

\[\hat{p}_{k|k} = y_k - \hat{y}_{k|k-1}\]
\[\hat{y}_{k|k} = \hat{y}_{k|k-1} + \hat{p}_{k|k}\]

Measured output

Model output predicted from k-1

“additive output” disturbance assumption (previously \(d_k\))

Forces the model “corrected output” equal to measured output

Notice that Model States are Not “Corrected”

\[\hat{x}_{k|k} = \hat{x}_{k|k-1}\]

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Model Prediction to $k+1$

\[
\begin{align*}
\hat{x}_{k+1|k} &= \Phi \hat{x}_{k|k} + \Gamma u_k \\
\hat{p}_{k+1|k} &= \hat{p}_{k|k} \\
\hat{y}_{k+1|k} &= C \hat{x}_{k+1|k} + \hat{p}_{k+1|k} = \\
&= C \Phi \hat{x}_{k|k} + CT u_k + \hat{p}_{k+1|k} \\
&= C \Phi \hat{x}_{k|k} + CT u_{k-1} + CT \Delta u_k + \hat{p}_{k+1|k}
\end{align*}
\]

Assumes future corrections equal to current correction
Continue Output Predictions

\[
\begin{bmatrix}
\hat{y}_{k+1|k} \\
\hat{y}_{k+2|k} \\
\vdots \\
\hat{y}_{k+P|k}
\end{bmatrix} = \begin{bmatrix}
C\Phi \\
C\Phi^2 \\
\vdots \\
C\Phi^P
\end{bmatrix} \hat{x}_{k|k} + \begin{bmatrix}
I \\
I \\
\vdots \\
I
\end{bmatrix} \hat{p}_{k|k} + \begin{bmatrix}
C\Gamma \\
C\Phi\Gamma + C\Gamma \\
\vdots \\
\sum_{i=1}^{P} C\Phi^{i-1}\Gamma
\end{bmatrix} u_{k-1}
\]

"free" or "unforced response" (if no more control moves are made)

\[
f = \begin{bmatrix}
C\Gamma & 0 & \cdots & 0 \\
C\Phi\Gamma + C\Gamma & C\Gamma & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^{P} C\Phi^{i-1}\Gamma & \sum_{i=1}^{P-1} C\Phi^{i-1}\Gamma & \cdots & \Delta u_{k+M-1}
\end{bmatrix}
\]

"forced" response

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Output Predictions

\[
\begin{bmatrix}
\hat{y}_{k+1|k} \\
\hat{y}_{k+2|k} \\
\vdots \\
\hat{y}_{k+P|k}
\end{bmatrix}
= \underbrace{egin{bmatrix}
C\Phi \\
C\Phi^2 \\
\vdots \\
C\Phi^P
\end{bmatrix}}_{\hat{Y}}
\begin{bmatrix}
x_{k|k} \\
I \\
I \\
S_1
\end{bmatrix} + \hat{P}_{k|k} + \begin{bmatrix}
S_1 \\
S_2 \\
\vdots \\
S_P
\end{bmatrix} u_{k-1}
\]

"free" or "unforced response" (if no more control moves are made)

\[
\begin{bmatrix}
S_1 & 0 & \cdots & 0 \\
S_2 & S_1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
S_P & S_{P-1} & \cdots & S_{P-M+1}
\end{bmatrix}
\begin{bmatrix}
\Delta u_k \\
\Delta u_{k+1} \\
\vdots \\
\Delta u_{k+M-1}
\end{bmatrix}
\]

"forced" response

\[\Delta u_f\]

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Example Inverse Response Process: Van de Vusse

\[
\frac{dCa}{dt} = -k_1 Ca - k_3 Ca^2 + (Cain - Ca)u
\]

\[
\frac{dCb}{dt} = k_1 Ca - k_2 Cb - Cbu \quad \text{where} \quad u = F/V
\]

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Example: Inverse Response Process

The graph shows the step response coefficients as a function of the discrete time index, \( i \).

- The first four coefficients are negative.
- The sum of the first eight coefficients is positive.

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Closed-Loop: Compare $P=10$, $M=1$ with $P=25$, $M=1$

Short prediction horizons & long control horizons lead to more aggressive action

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Results

Control Horizon: $M = 1$, Weighting: $W = 0$

$P = 8$

$P = 7$

P=7: Unstable

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Stability of Inverse Response Systems with DMC

- For a control horizon, $M = 1$, closed-loop MPC will be stable for a prediction horizon where the sum of the impulse response coefficients has the same sign as the process gain.

$$\sum_{i=1}^{P_{\text{min}}} S_i > 0$$


Paul R. Maurath,† Duncan A. Mellichamp, and Dale E. Seborg*

Department of Chemical and Nuclear Engineering, University of California, Santa Barbara, California 93106


For the example, $P_{\text{min}} = 8$, so $P = 7 = \text{unstable}$
Summary

- Concise overview of MPC
- State space model, unconstrained solution
- The additive disturbance assumption of DMC is used for plant-model mismatch

Upcoming topics
- Disturbances and state estimation (Kalman filtering framework)
- Constraints
- Other model forms