COMMENTS AND CORRECTIONS

RUNE HAUGSENG

Abstract. This is a (no doubt very incomplete) list of mistakes and errors in my papers that have not (yet?) been corrected in the arXiv versions.

Contents

1. Enriched ∞-categories via non-symmetric ∞-operads [GH15]

The proof that the tensor product of enriched ∞-categories preserves colimits in each variable in **Proposition 5.7.16** and **Corollary 4.3.16** does not make sense; see [Hau23] for a more details and a (hopefully) correct proof of this statement.

The proof in **Proposition A.5.11** that we get an adjunction on ∞-categories of algebras from a symmetric monoidal functor that is a left adjoint is unnecessarily complicated (and uses an assumption of presentability that is not needed); this in fact follows directly from the functoriality of ∞-categories of algebras with respect to natural transformations.

2. Iterated spans and classical topological field theories [Hau18]

In **Remark 10.7**, the construction does not work for a general n-fold category object as claimed, only for n-fold Segal objects. To extract an n-fold monoid in the more general case, one should instead take a right Kan extension from the full subcategory of $\Delta^{n,\mathrm{op}}$ containing all objects $([i_1],\ldots,[i_n])$ such that at least one component i_j is 0. (Thanks to Manuel Krannich for pointing this out.)

3. Homotopy-coherent algebra via Segal conditions [CH21]

Example 14.23 is not correct in general. For one thing, an ∞-operad $\mathcal{O}$ is not necessarily slim — the necessary objects of $\mathcal{O}$ can be described as those lists $(x_1,\ldots,x_n)$ of objects of $\mathcal{O}_{\langle 1\rangle}$ such that there exists an active morphism $(x_1,\ldots,x_n)\to y$ for $y\in\mathcal{O}_{\langle 1\rangle}$. The canonical saturated pattern $\overline{\mathcal{O}}$ is only the entire Lawvere theory if $\mathcal{O}$ is in fact slim, i.e. if every object admits such an active morphism. A simple counterexample is the ∞-operad with only unary operations obtained from an ∞-category $\mathcal{C}$ (explicitly, this is $\mathcal{C}^{\amalg}\times_{\mathbb{F}_*}\mathbb{F}_*^{\mathrm{int}}$ by [Lur17, Theorem 2.4.3.18]); the corresponding ∞-category of Segal objects in $\mathcal{S}$ is just $\mathsf{Fun}(\mathcal{C},\mathcal{S})$, and the canonical pattern is clearly $\mathcal{C}$ (with all objects elementary and only equivalences as inert morphisms), while the corresponding Lawvere theory is obtained by freely adding products to $\mathcal{C}$.

4. On distributivity in higher algebra I: The universal property of bispans [EH23]

The proof that the $(\infty,2)$-category of bispans in $\mathcal{S}$ is equivalent to that of polynomial functors in **Corollary 3.3.5** has a gap: to see that the functor is an equivalence one should really check that the cocartesian fibrations for mapping ∞-categories in $\mathrm{PolyFun}(\mathcal{S})$ agree with free fibrations as in **Corollary 2.3.18**. (Thanks to Maxime Ramzi for pointing this out.)

Date: June 1, 2024.

5. The AKSZ construction in derived algebraic geometry as an extended TFT [CHS21]

The discussion of quasicoherent sheaves in the appendix completely ignores an important size issue: if we define the ∞-category of quasicoherent sheaves for a derived stack $X\colon \mathrm{CAlg}^{c}_{\mathbb{K}} \to \widehat{\mathcal{S}}$ as a limit

$$\mathrm{QCoh}(X) := \lim_{\mathrm{Spec}\,A\to X} \mathsf{Mod}_A$$

over maps from affines to X, where Mod_A is the ordinary (large) ∞-category of A-modules, then we get the pullback functors $f^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ for any map of derived stacks $f\colon X \to Y$, but in general this does not have a right adjoint f_*: For example, if X is a *large* colimit of affines, then the global sections functor $\Gamma\colon \mathrm{QCoh}(X) \to \mathsf{Mod}_{\mathbb{K}}$ would involve taking a limit that is too large to exist in $\mathsf{Mod}_{\mathbb{K}}$. (We became aware of this issue from the discussion in Safronov's paper [Saf23, §2.2].)

For this reason, when discussing quasicoherent sheaves we should restrict to derived stacks that are *small*, meaning that they can be described as the colimit of a *small* diagram of affines. This includes all geometric stacks (by definition) as well as Betti stacks on small ∞-groupoids, so this does not affect any of the results of the paper. (When discussing (pre)orientations we must necessarily assume all stacks involved are small, since the definition refers to a pushforward functor.) Such "small" functors have previously been studied in the category theory literature; see for instance [DL07], where it is proved that small stacks are closed under small limits. In the ∞-categorical context, Hesselholt and Pstrągowski [HP24] develop the formal properties of small sheaves for their "Dirac geometry", and all of their results carry over to the setting of étale sheaves on derived rings needed for our paper.

References

[CHS21] Damien Calaque, Rune Haugseng, and Claudia Scheimbauer, *The AKSZ construction in derived algebraic geometry as an extended topological quantum field theory* (2021), available at arXiv:2108.02473.

[CH21] Hongyi Chu and Rune Haugseng, *Homotopy-coherent algebra via Segal conditions*, Advances in Mathematics **385** (2021), 107733, available at arXiv:1907.03977.

[DL07] Brian J. Day and Stephen Lack, *Limits of small functors*, J. Pure Appl. Algebra **210** (2007), no. 3, 651–663.

[EH23] Elden Elmanto and Rune Haugseng, *On distributivity in higher algebra I: the universal property of bispans*, Compos. Math. **159** (2023), no. 11, 2326–2415.

[GH15] David Gepner and Rune Haugseng, *Enriched ∞-categories via non-symmetric ∞-operads*, Adv. Math. **279** (2015), 575–716, available at arXiv:1312.3178.

[Hau18] Rune Haugseng, *Iterated spans and classical topological field theories*, Math. Z. **289** (2018), no. 3, 1427–1488, available at arXiv:1409.0837.

[Hau23] ———, *On the tensor product of enriched ∞-categories* (2023), available at arXiv:2311.13362.

[HP24] Lars Hesselholt and Piotr Pstrągowsky, *Dirac geometry II: coherent cohomology*, Forum Math. Sigma **12** (2024), paper no. e27, available at arXiv:2303.13444.

[Lur17] Jacob Lurie, *Higher Algebra*, 2017. Available at `http://math.ias.edu/~lurie/`.

[Saf23] Pavel Safronov, *Shifted geometric quantization*, J. Geom. Phys. **194** (2023), paper no. 104992, available at arXiv:2011.05730.

Norwegian University of Science and Technology (NTNU), Trondheim, Norway
URL: `http://folk.ntnu.no/runegha`