# Manifolds

**Defn.:** A (topological) <u>manifold</u> of dim. $n$ ($n$-manifold) is a second countable Hausdorff space $M$ s.t. $\forall\, x \in M$ $\exists$ an open nbhd. $U$, $x \in U$ s.t. $U$ is homeomorphic to $\mathbb{R}^n$.

<u>Second countable</u> for a top. sp. $X$ means $\exists$ countable set of opens $U_1, U_2, \ldots$ s.t. every open subset of $X$ is a union of $U_i$'s.

E.g. in $\mathbb{R}^n$ can take open balls w/ rational radius & w/ centre having rational coords.

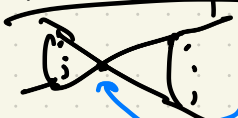
(Not really relevant, except allows us to use induction instead of transfinite induction at one point.)

Compact mfd.s in this sense are often called closed mfd.s (to contrast w/ compact mfd.s w/ boundary, but we won't talk about those).

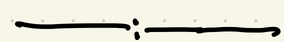
**Examples:**

- $S^n$ spheres
- torus
- Klein bottle
- orientable surfaces of genus $g$
- $\mathbb{RP}^n$, $\mathbb{CP}^n$ $(n<\infty)$
- $M$ $m$-mfd., $N$ $n$-mfd. then $M\times N$ is an $(n+m)$-mfd.

**Non-examples:**



no open nbhd. of cone pt. is homeomorphic to $\mathbb{R}^2$



$\mathbb{R} \cup_{\mathbb{R}\setminus\{0\}} \mathbb{R}$

not Hausdorff, but satisfies other conds for a 1-mfd.

**Goal:** Poincaré duality – for $M$ an $R$-orientable compact $n$-mfd., there is a fundamental class $[M] \in H_n(M;R)$ s.t. cap product w/ $[M]$ gives isomor.s

$$-\cap [M] : H^k(M;R) \xrightarrow{\sim} H_{n-k}(M;R)$$

**Lemma:** $M$ an $n$-mfd., $x \in M$, then $H_*(M, M \setminus \{x\};R) \cong \begin{cases} \cancel{\mathbb{Z}}\ R, & * = n \\ 0, & * \neq n \end{cases}$

**Note:** This implies if $M$ is an $n$-mfd. then it's not an $m$-mfd. for $n \neq m$. (Generalizes $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$.)

**Proof:** $\exists$ open nbhd. $U$ of $x$ s.t. $U \cong \mathbb{R}^n$ via homeomor. $\varphi : U \xrightarrow{\sim} \mathbb{R}^n$.

Then $H_*(M, M\setminus\{x\}) \underset{\text{by excision of } M\setminus U}{\cong} H_*(U, U\setminus\{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{\varphi x\})$

So remains to compute $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$

$(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is ht. py eq.t to $(D^n, D^n \setminus \{0\})$

$(D^n \hookrightarrow \mathbb{R}^n$ is a deformation retract$)$

$(D^n, S^{n-1}) \overset{i}{\hookrightarrow} (D^n, D^n \setminus \{0\})$ gives $H_*$-iso

- $S^{n-1} \hookrightarrow D^n \setminus \{0\}$ is a deformation retract

$\Rightarrow$ iso. for $i$ by LES for pairs & 5-lemma

$$\Rightarrow H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong H_*(D^n, S^{n-1}) \cong \tilde{H}_*(S^n) = \begin{cases} \mathbb{Z}, & * = n \\ 0, & * \neq n. \end{cases} \quad \square$$

$K \subset M$

<u>Notation:</u> $H_*(M|K;R) := H_*(M, M\setminus K; R)$

"homology in a small nbhd. of K"

$K \subset L \subset M \rightsquigarrow (M, M\setminus L) \hookrightarrow (M, M\setminus K)$

$\rightsquigarrow \rho^L_K : H_*(M|L;R) \to H_*(M|K;R)$

- if $K = \{x\}$ write $H_*(M|x;R)$, $\rho^L_x$

<u>Propn.</u>: $M$ an $n$-mfd., $K \subset M$ cpt.

(i) $H_*(M|K;R) = 0$, $* > n$

(ii) $\alpha \in H_n(M|K;R)$ is $0$ iff $\rho_x \alpha = 0$ in $H_n(M|x;R)$ $\forall x \in K$

(Or: $H_n(M|K;R) \to \prod_{x \in K} H_n(M|x;R)$ is injective.)

Cor.: If $M$ is a cpt. mfd. then $H_*(M;R) = 0$ for $* > n$

Proof: Can take $K = M$ in Prop. & $H_*(M|M;R) = H_*(M, \emptyset; R) = H_*(M;R)$.

$\square$

For the proof we need a variant of the Mayer-Vietoris sequence:

Defn.: A pair of subspaces $A, B \subset X$ is reasonable if in $A \cup B$ $\exists$ opens $U, V$ w/ $A \subset U$, $B \subset V$, $U \cup V = A \cup B$ s.t. $A \hookrightarrow U$, $B \hookrightarrow V$, $A \cap B \hookrightarrow U \cap V$ give $H_*$-isomorphs

(E.g. $A, B$ any open subsets of $X$)

Write $S_\cdot(X, A+B) := S_\cdot X / (S_\cdot A + S_\cdot B)$

Propn.: For $A, B \subset X$ reasonable, $S_\cdot(X, A+B)$ is q.isomorphic as $S_\cdot(X, A \cup B)$.

Proof:

Have diagram of SESs

$$\begin{tikzcd} S_\cdot A + S_\cdot B \arrow[r] \arrow[d] & S_\cdot X \arrow[r] \arrow[d, equal] & S_\cdot(X, A+B) \arrow[d] \\ S_\cdot(A \cup B) \arrow[r] & S_\cdot X \arrow[r] & S_\cdot(X, A \cup B) \end{tikzcd}$$

Using 5-lemma, it's enough to show $S_\cdot A + S_\cdot B \to S_\cdot(A \cup B)$ is isomor. on $H_*$.

Factors as $S_\cdot A + S_\cdot B \to \underbrace{S_\cdot U + S_\cdot V}_{\text{subcx. of "small chains" for cover of } A \cup B \text{ by } U, V} \to S_\cdot(A \cup B)$

can show using SESs & 5-lemma again that this gives isomor. on $H_*$

Locality Thm. says this is a q.isomor.

∎

**Lemma:** $A, B \subset X$ reasonable

$$\begin{tikzcd}
(X, A\cap B) \arrow[r, "i"] \arrow[d, "i'"'] & (X, A) \arrow[d, "j"] \\
(X, B) \arrow[r, "j'"'] & (X, A\cup B)
\end{tikzcd}$$

There's a LES

$$\cdots H_n(X, A\cap B) \xrightarrow[(i_*, i'_*)]{} H_n(X,A) \oplus H_n(X,B) \xrightarrow[j_* - j'_*]{} H_n(X, A\cup B) \to H_{n-1}(X, A\cap B) \cdots$$

**Proof:** Given $N, N' \subset M$ subgroups of $M \in \mathrm{Ab}$, we have a comm. square

$$\begin{tikzcd}
M/N\cap N' \arrow[r, "p"] \arrow[d, "p'"'] & M/N \arrow[d, "q"] \\
M/N' \arrow[r, "q'"'] & M/(N+N')
\end{tikzcd}$$

& SES

$$0 \to M/N\cap N' \xrightarrow{(p,p')} M/N \oplus M/N' \xrightarrow{q-q'} M/(N+N') \to 0$$

Apply to singular chains, get SES of ch. cx-s

$$0 \to S_\bullet(X, A\cap B) \to S_\bullet(X,A) \oplus S_\bullet(X,B) \to S_\bullet(X, A+B) \to 0$$

this gives LES in result since $H_k S_\bullet(X, A+B) \cong H_*(X, A\cup B)$. $\square$

Special case: $M$ an $n$-mfd., $K, L \subset M$ cpt. ($\Rightarrow$ closed)
apply LES to opens $M\setminus K$, $M \setminus L$ to get

$$\cdots H_k(M|K\cup L) \xrightarrow[(\rho^{K\cup L}_K,\ \rho^{K\cup L}_L)]{} H_k(M|K) \oplus H_k(M|L) \xrightarrow[\rho^K_{K\cap L} - \rho^L_{K\cap L}]{} H_k(M|K\cap L) \longrightarrow H_{k-1}(M|K\cup L) \cdots$$

Propn.: $M$ an $n$-mfd., $K \subset M$ cpt.

(i) $H_*(M|K;R) = 0$, $* > n$

(ii) $\alpha \in H_n(M|K;R)$ is $0$ iff $\rho_x \alpha = 0$ in $H_n(M|x;R)$ $\forall\, x \in K$

(Or: $H_n(M|K;R) \to \prod_{x \in K} H_n(M|x;R)$ is injective.)

Proof:

First prove the case when $M = \mathbb{R}^n$.

Step 1: $K \subset \mathbb{R}^n$ compact & convex.

For $x \in K$ it's enough to show $(\mathbb{R}^n, \mathbb{R}^n \setminus K) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ gives $H_*$-iso. (This is stronger than (i) & (ii) but not true more generally)

Using LES for pairs & 5-lemma, enough to show

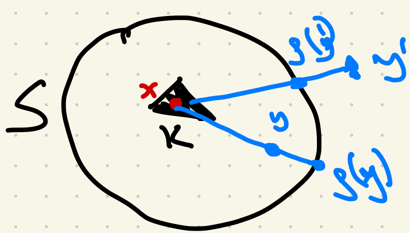
$$\mathbb{R}^n \setminus K \hookrightarrow \mathbb{R}^n \setminus \{x\}$$

gives $H_*$-isomor.s.

Choose a sphere $S \subset \mathbb{R}^n$ centred at $x$ w/ $K$ inside the open ball bounded by $S$

Then $S \hookrightarrow \mathbb{R}^n \setminus K$ is a deformation retract

Define $\mathbb{R}^n \setminus K \xrightarrow{s} S$ by $y$ to point where line from $x$ to $y$ intersects $S$. Moving from $y$ to $s(y)$ along this line




gives a ht.py
(can't hit $K$ because then the line between $x$ & the point where it does would lie in $K$)

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ \mathbb{R}^n\setminus K \arrow[rr] & & \mathbb{R}^n\setminus\{x\} \end{tikzcd}$$

give $H_*$-iso.s $\Rightarrow$ so does the 3rd, by the 2-out-of-3 property for isomorphisms.

Step 2: $K \subset \mathbb{R}^n$ is a finite union $K_1 \cup \cdots \cup K_r$ w/ $K_i$ compact & convex

Induct on $r$ ($r=1$ is step 1). Set $K' = K_1 \cup \cdots \cup K_{r-1}$

Then $K' \cap K_r$ = union of cpt. convex subsets $K_1 \cap K_r, \ldots, K_{r-1} \cap K_r$

Have LES (w/ $K = K' \cup K_r$)

$$\cdots\ H_{k+1}(\mathbb{R}^n | K' \cap K_r) \to H_k(\mathbb{R}^n | K) \to H_k(\mathbb{R}^n | K') \oplus H_k(\mathbb{R}^n | K_r) \to \cdots$$

If $k > n$ then both groups next to $H_k(\mathbb{R}^n | K)$ are $0$

$\Rightarrow H_k(\mathbb{R}^n | K) = 0$.

For $k=n$. $H_{n+1}(\mathbb{R}^n|K'\cap K_v) = 0$ so

$$H_n(\mathbb{R}^n|K) \to H_n(\mathbb{R}^n|K') \oplus H_n(\mathbb{R}^n|K_v)$$

is injective

So $\alpha \in H_n(\mathbb{R}^n|K)$ is $0$ iff $\rho^K_{K'}\alpha = 0$ & $\rho^K_{K_v}\alpha = 0$

$\iff \rho^K_x\alpha = \rho^{K'}_x \rho^K_{K'}\alpha$ is $0$ $\forall x \in K'$

& $\rho^K_x\alpha = \rho^{K_v}_x \rho^K_{K_v}\alpha$ is $0$ $\forall x \in K_v$

i.e. $\rho^K_x\alpha = 0 \quad \forall x \in K$.

<u>Step 3:</u> $K \subset \mathbb{R}^n$ any compact set

Given $\alpha \in H_i(\mathbb{R}^n|K)$ can lift to a chain $\gamma \in S_i\mathbb{R}^n$ s.t. image in $S_i(\mathbb{R}^n, \mathbb{R}^n\setminus K)$ is a cycle representing $\alpha$.

This means $\partial\gamma$ is in $S_{i-1}(\mathbb{R}^n\setminus K)$

$\Rightarrow$ $\partial\gamma$ is linear comb. of finitely many simplices in $\mathbb{R}^n\setminus K$

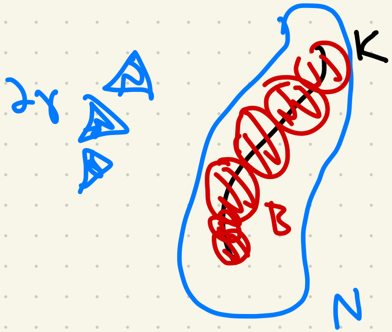
- can choose $L$ compact, disjoint from $K$ s.t. the images of these simplices lie in $L$


choose a $\underbrace{\text{cpt. nbhd.}}_{\text{cpt. \& contains an open nbhd. of every pt. on } K}$ $N$ of $K$ s.t. $N\cap L=\emptyset$

Then $\partial\gamma$ is in $S_{i-1}(\mathbb{R}^n\setminus N)$

$\Rightarrow$ image of $\gamma$ in $S_i(\mathbb{R}^n, \mathbb{R}^n\setminus N)$ is a cycle representing $\alpha' \in H_i(\mathbb{R}^n|N)$ s.t. $\alpha = \rho^N_K\alpha'$

Can cover $K$ by finitely many closed balls $B_1, \dots, B_r$ s.t. $B_i\subset N$ & $B_i\cap K\neq\emptyset$.




$B := B_1 \cup \cdots \cup B_r$

$K \subset B \subset N$ and Step 2 applies to $B$

If $i > n$ then $\rho^N_B \alpha' = 0$ by Step 2 $\Rightarrow$ $\alpha = \rho^B_K \rho^N_B \alpha' = 0$

$\Rightarrow H_i(\mathbb{R}^n \mid K) = 0$ for $i > n$.

If $i = n$ & $\rho_x \alpha = 0 \ \forall \ x \in K$, want to show $\alpha = 0$.

We know $\rho^{B_i}_b$ is iso. $\forall \ b \in B_i$ by Step 1.

$B_i \cap K \neq \emptyset$ so $\exists \ x \in B_i \cap K$ & $\rho_x \rho^N_{B_i} \alpha' = \rho_x \alpha = 0$

$\Rightarrow \rho_{B_i} \alpha' = 0$. Then $\rho^B_b \rho^N_B \alpha' = 0 \ \forall \ b \in B$.

So $\rho^N_B \alpha' = 0$ by Step 2. $\Rightarrow \alpha = \rho^B_K \rho^N_B \alpha' = 0$.

Now consider general $M$.

<u>Step 4:</u> $K \subset M$ compact s.t. $\exists$ $U$ open, $U \cong \mathbb{R}^n$ w/ $K \subset U$.

Then $H_*(M|K) \cong H_*(U|K)$ by excision

& now can apply Step 3.

<u>Step 5:</u> $K \subset M$ arbitrary. Can write $K = K_1 \cup \cdots \cup K_r$

where each $K_i$ is as in Step 4.

Apply LES as in Step 2 to induct on $r$. $\square$

<u>Propn.</u> : $M$ an $n$-mfd., $K \subset M$ cpt.

(i) $H_*(M|K;R) = 0$, $* > n$

(ii) $\alpha \in H_n(M|K;R)$ is $0$ iff $\rho_x \alpha = 0$ in $H_n(M|x;R)$ $\forall\, x \in K$

(Or: $H_n(M|K;R) \xrightarrow{(\rho_x)_{x\in K}} \prod_{x\in K} H_n(M|x;R)$ is injective.)

<u>Defn.</u>: $M$ $n$-mfd., $T \subset M$, $\Gamma(M|T;R)$ := subgroup of $\prod_{x\in T} H_n(M|x;R)$ consisting of $(\alpha_x)_{x\in T}$ s.t. $\forall\, x \in T$ $\exists$ compact nbhd. $\overset{\subset T}{N}$ of $x$, $\alpha_N \in H_n(M|N;R)$ s.t. $\forall\, y \in N$ we have $\alpha_y = \rho_y \alpha_N$.

<u>Propn.:</u> $M$ $n$-mfd., $K \subset M$ cpt. Then

$$H_n(M|K;R) \xrightarrow{\sim} \Gamma(M|K;R)$$

<u>Proof:</u> The map $(\rho_x)_{x\in K} : H_n(M|K;R) \to \prod_{x\in X} H_n(M|x;R)$

is injective and factors through $\Gamma(M|K;R)$ since $K$ gives the required cpt. nbhd. for every point for a class in the image.

Remains to show that $(\alpha_x)_{x\in K} \in \Gamma(M|K;R)$ is in image.

For $x \in K$ $\exists$ cpt. nbhd. $N^{\subset K}$ of $x$ & $\alpha_N \in H_n(M|N;R)$ s.t.

$\alpha_y = \rho_y \alpha_N$ $\forall\, y \in N$. Since $K$ is compact, can cover it by finitely many such nbhd.s $K_1, \dots, K_r$ s.t. $\alpha_{K_i}$ exists

Set $K_i' = K_1 \cup \dots \cup K_i$

We prove by induction that $\exists\, \alpha_{K_i'} \in H_n(M | K_i' ; R)$

s.t. $\alpha_y = \rho_y \alpha_{K_i'}$ for $y \in K_i'$.

Have LES (for $K_{i-1}', K_i$)

$$\cdots \underset{\substack{\| \\ 0}}{H_{n+1}(M | K_{i-1}' \cap K_i)} \to H_n(M | K_i') \xrightarrow{\left(\rho^{K_i'}_{K_{i-1}'},\ \rho^{K_i'}_{K_i}\right)} H_n(M | K_{i-1}') \oplus H_n(M | K_i) \xrightarrow{\rho^{K_{i-1}'}_{K_{i-1}' \cap K_i} - \rho^{K_i}_{K_{i-1}' \cap K_i}} H_n(M | K_{i-1}' \cap K_i)$$

$$\ni \left(\alpha_{K_{i-1}'}, \alpha_{K_i}\right) \longmapsto 0$$

because get $0$ when restrict to any point $x$ in $K_{i-1}' \cap K_i$:

$$\rho_x\left(\rho^{K_{i-1}'}_{K_{i-1}' \cap K_i} \alpha_{K_{i-1}'} - \rho^{K_i}_{K_{i-1}' \cap K_i} \alpha_{K_i}\right) = \alpha_x - \alpha_x = 0$$

By exactness $\exists\ \alpha_{K_i'} \longmapsto (\alpha_{K_{i-1}'}, \alpha_{K_i})$

Then $\rho_x \alpha_{K_i'} = \alpha_x$ for all $x \in K_i' = K_{i-1}' \cup K_i$. $\square$

<u>Cor.</u>: If $M$ is a cpt. $n$-mfd. then

- $H_i(M;R) = 0$ if $i > n$
- $H_n(M;R) \cong \Gamma(M|M;R)$.

<u>Defn.</u>: $M$ an $n$-mfd. (not necessarily compact), $R$ a comm. ring

An <u>$R$-orientation</u> of $M$ is $(\alpha_x)_{x \in M} \in \Gamma(M|M;R)$

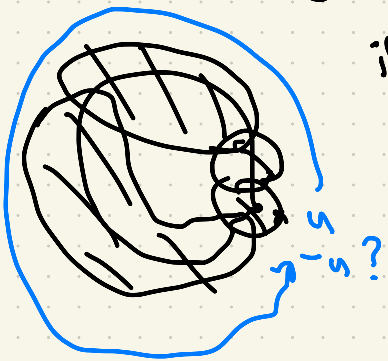
s.t. $\alpha_x$ is a generator of $H_n(M|x;R) \cong R$ $\forall x \in M$

$M$ is <u>$R$-orientable</u> if $\exists$ an $R$-orientation,
<u>$R$-oriented</u> if we've chosen one.
For $R = \mathbb{Z}$, we just say that $M$ is orientable / oriented.

<u>Remark:</u> for $\mathbb{Z}$, this agrees w/ more geometric defn.s of orientations.

Given a generator $u \in H_n(M|x)$ can always extend uniquely to a nbhd. of $x$: take $U \subset M$ open nbhd. of $x$, $U \cong \mathbb{R}^n$, & $B \subset U$ image of closed ball around image of $x \in \mathbb{R}^n$, then $\rho^B_x : H_n(M|B) \to H_n(M|x)$ is an isomor. (by excising $M \setminus U$) & can assign $\rho^B_y (\rho^B_x)^{-1} u \in H_n(M|y)$ for $y \in B$.

__But__ we may not be able to do this consistently on all of $M$:




if we make a sequence of such extensions going around a closed loop at $x$, might get back $-\mu$ instead of $\mu$.

So not every mfd. is ($\mathbb{Z}$-) orientable.

But w/ $R = \mathbb{Z}/2$ the non-zero elt. $\mu$ is __unique__ — no choices involved $\Rightarrow$ all mfd.s are (uniquely) $\mathbb{Z}/2$-orientable.

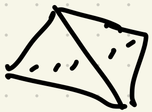
__Defn.:__ $M$ an $R$-oriented compact $n$-mfd. The __fundamental class__ $[M] \in H_n(M;R)$ is the h'l.ogy class corresponding to the orientation under the isomor. from above.

Think of $[M]$ as a homology class representing "all of M"

- if we can represent M as a $\Delta$-set of dim. $n$ then $[M]$ is represented as a cycle by a sum of all $n$-simplices

Every $(n-1)$-simplex is a face of exactly two $n$-simplices & have to choose signs in the sum so that these cancel out - w/ $\mathbb{Z}$-coeff.s such a choice of signs exists iff M is orientable.

W/ $\mathbb{Z}/2$-coeff.s these pairs always cancel so sum of $n$-simplices is automatically a cycle.

 $\pm \sum (-1)^i \partial_i \Delta^3$ is a cycle that represents $[S^2]$

$\partial \Delta^3 \simeq S^2$

We saw this represents a generator of $H_2 S^2$.

**Propn.:** $M$ compact, connected $n$-mfd. Then

$$s_x : H_n(M;R) \to H_n(M|x;R) \cong R$$

is injective, & an isomor. if $M$ orientable.

**Proof:** $H_n(M;R) \cong \Gamma(M|M;R)$ so enough to show

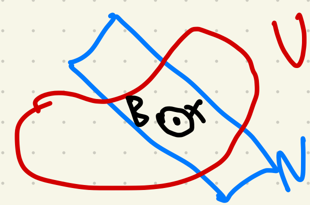
$\Gamma(M|M;R) \longrightarrow H_n(M|x;R)$ is injective.

Given $(\alpha_x)_{x \in M} \in \Gamma(M|M;R)$ let $V \subset M$ be the set of $x \in M$ s.t. $\alpha_x = 0$

We will show $V$ is both open & closed - since $M$ connected this means $V = M$ or $\emptyset$ - implies $\alpha_y = 0 \; \forall y$ iff $\alpha_x = 0$ for a single $x$.

For $x \in M$ $\exists$ $N$ cpt. nbhd. of $x$ and $\alpha_N \in H_n(M|N;R)$ s.t. $\alpha_y = \rho^N_y \alpha_N \; \forall y \in N$.

Can choose $U$ open nbhd. of $x$, $U \cong \mathbb{R}^n$ & $B \subset U \cap N$ corr. to closed ball around image of $x$ in $\mathbb{R}^n$




$\rho^B_y : H_n(M|B;R) \xrightarrow{\sim} H_n(M|y;R) \; \forall y \in B$

$\Rightarrow$ if $x \in V$ then $\alpha_y = 0 \quad \forall y \in B$

- so some open nbhd. of $x$ lies in $V$

& if $x \notin V$ then $\alpha_y \neq 0 \quad \forall y \in B$

- so some open nbhd. of $x$ lies in $M \setminus V$

$\Rightarrow$ $V$ & $M \setminus V$ are open $\Rightarrow$ $V$ is open & closed.

Implies injectivity.

If $M$ is orientable then orientation maps to a generator

so $\rho_x$ is surjective. $\square$

Upshot: $M$ a compact, connected, $R$-orientable $n$-mfd., then

$$H_n(M;R) \cong R,$$

Can show if $M$ is not $R$-orientable, then the image of $\rho_x$ is 2-torsion subgroup of $H_n(M;R)$

for $R = \mathbb{Z}$, this means

$$H_n(M) = \begin{cases} \mathbb{Z}, & M \text{ orientable} \\ 0, & M \text{ not orientable} \end{cases}$$

E.g. $\mathbb{RP}^n$ is orientable iff $n$ odd

# Cap products

**Defn.** : $C_\bullet \in Ch$, $M \in Ab$

$\exists$ a natural evaluation pairing $ev : Hom(C,M)_\bullet \otimes C_\bullet \longrightarrow \underbrace{M[0]}$

$M$ in deg. 0, $0$ in deg. $\neq 0$

given in deg. 0 by

$$\varphi \in Hom(C,M)_{-n} = Hom(C_n, M), \quad x \in C_n \qquad ev(\varphi \otimes x) = (-1)^{\lambda(n)} \varphi(x)$$

$$\lambda(n) = \begin{cases} 0, & n \equiv 0,3 \pmod 4 \\ 1, & n \equiv 1,2 \pmod 4 \end{cases}$$

Needed to get a chain map: $ev(d(\varphi \otimes x)) = ev(d\varphi \otimes x + (-1)^n \varphi \otimes dx)$

$\varphi \in Hom(C_n, M)$, $x \in C_{n+1}$ $\quad = (-1)^{\lambda(n+1)} \varphi(dx) + (-1)^{\lambda(n)+n} \varphi(dx) = 0$

As a special case, for $X \in \mathrm{Top}$ have

$$ev : S^\bullet(X;M) \otimes S_\bullet(X) \longrightarrow M[0]$$

$$\overset{\shortparallel}{\mathrm{Hom}(S_\bullet X, M)}$$

For $R$ a ring, have also

$$ev_R : S^\bullet(X;R) \otimes \underbrace{S_\bullet(X;R)}_{S_\bullet X \otimes R[0]} \longrightarrow R[0] \otimes R[0] \longrightarrow R[0]$$

mult. in $R$

In homology, this gives the <u>Kronecker pairing</u>

$$K_R : H^*(X;R) \otimes H_*(X;R) \longrightarrow R[0]$$

or: $H^n(X;R) \otimes H_n(X;R) \longrightarrow R$

<u>Defn.</u>: Diagonal + Eilenberg–Zilber map give natural chain map

$$\begin{array}{ccccc} S_\cdot(X;R) & \xrightarrow{\Delta_*} & S_\cdot(X\times X;R) & \longrightarrow & S_\cdot X \otimes S_\cdot(X;R) \\ & & \wr\!\wr & & \wr\!\wr \\ & & S_\cdot(X\times X)\otimes R & \longrightarrow & (S_\cdot X\otimes S_\cdot X)\otimes R \end{array}$$

Combine w/ ev:

$$S^\cdot(X;R)\otimes S_\cdot(X;R) \longrightarrow S^\cdot(X;R)\otimes S_\cdot X\otimes S_\cdot(X;R) \xrightarrow{ev\otimes id} R[0]\otimes S_\cdot(X;R) \xrightarrow[\text{mult. in } R]{} S_\cdot(X;R)$$

This is the (chain-level) cap product

In homology, get

$$H^*(X;R)\otimes H_*(X;R) \xrightarrow{-\cap-} H_*(X;R)$$ – independent of choice of E–Z map

For $\varphi \in H^n(X;R)$, $\alpha \in H_m(X;R)$ get $\varphi \cap \alpha \in H_{m-n}(X;R)$.

<u>Key properties:</u> $\varphi \in H^n(X;R)$, $\psi \in H^m(X;R)$, $\alpha \in H_k(X;R)$

- $\kappa_R(\underbrace{\varphi}, \underbrace{\psi \cap \alpha}) = \kappa_R(\underbrace{\varphi \cup \psi}, \underbrace{\alpha})$

  if $n = k-m$ $\qquad\qquad$ $n+m = k$

- $\underbrace{(\varphi \cup \psi) \cap \alpha}_{k-(n+m)} = \underbrace{\varphi \cap (\underbrace{\psi \cap \alpha}_{k-m})}_{(k-m)-n}$

- $1 \cap \alpha = \alpha$, $\quad 1 \in H^0(X;R)$.

- for $f: X \to Y$ cts., $\eta \in H^n(Y;R)$,

  $$\eta \cap f_*\alpha = f_*(f^*\eta \cap \alpha)$$

Proofs: Either draw a bunch of big diagrams

or use Alexander-Whitney map to get explicit formulas (Exercise).

<u>Thm.</u> (Compact Poincaré duality): $M$ an $R$-oriented compact $n$-mfd., then capping w/ fund. class $[M] \in H_n(M;R)$ gives isomor.s

$$- \cap [M] : H^k(M;R) \xrightarrow{\sim} H_{n-k}(M;R).$$

If $k$ is a field, then UCT implies $H^i(M;R) \cong \mathrm{Hom}_k(H_i(M;k), k)$

$$= H_i(M;k)^\vee \text{ (dual v. sp.)}$$

If we know $H_i(M;k)$ are finite-dimensional then we have (non-canonical) isomor.s $H_i(M;k) \cong H_{n-i}(M;k)$.

This is true if $M$ is smooth (& in general if $n \neq 4$) since then $M$ can be described as a finite cell cx.

<u>Example:</u> $S^1 \vee S^3$ is not ht.py-equivalent to any cpt. mfd.:

$$H_*(S^1 \vee S^3; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & * = 0, 1, 3 \\ 0, & \text{otherwise} \end{cases}$$

- so can only be ht.py-eq.t to a 3-mfd.

(since for any cpt. n-mfd $M$ we know $H_n(M; \mathbb{Z}/2) \cong \mathbb{Z}/2$ & $H_i(M; \mathbb{Z}/2) = 0$ for $i > n$)

But then Poincaré duality implies $H_1 \cong H_{3-1} = H_2$

but $H_1 = \mathbb{Z}/2$, $H_2 = 0$.

## Application to cup products

Can use connection between $\cap$ & $\cup$ to extract information about $\cup$ from PD.

<u>Defn.</u>: For a homomor. of R-modules $\varphi : M \otimes_R N \to P$ ($M, N, P$ R-modules)

can define two adjoint homomorphisms

$$M \to \mathrm{Hom}_R(N, P)$$
$$m \longmapsto \varphi(m \otimes -)$$

$$N \to \mathrm{Hom}_R(M, P)$$
$$n \mapsto \varphi(- \otimes n)$$

$\varphi$ is a <u>perfect pairing</u> if these are both isomor.s

Propn.: $M$ a compact $R$-oriented $n$-manifold. The pairing

$$H^{n-k}(M;R) \otimes_R H^k(M;R) \xrightarrow{\cup} H^n(X;R) \xrightarrow{\kappa_R(-,[M])} R$$

is perfect if either $R$ is a field or $R=\mathbb{Z}$ and $H_*(M)$ are free ab. groups.

Note: We have $\kappa_R(\varphi \cup \psi, [M]) = \kappa_R(\varphi, \psi \cap [M])$

So the pairing is equivalently given as

$$H^{n-k}(M;R) \otimes_R H^k(M;R) \xrightarrow{\mathrm{id} \otimes (- \cap [M])} H^{n-k}(M;R) \otimes_R H_{n-k}(M;R) \xrightarrow{\kappa_R} R$$

The adjoint $H^{n-k}(M;R) \to \mathrm{Hom}_R(H^k(M;R), R)$ can be described as

$$H^{n-k}(M;R) \xrightarrow[\text{- adjoint of } \kappa_R]{\kappa'_R} \mathrm{Hom}_R(H_{n-k}(M;R),R) \xrightarrow[\text{isomor. by Poincaré duality}]{(-\frown[M])^*} \mathrm{Hom}_R(H^k(M;R),R)$$

$\kappa'_R$ is <u>almost</u> the map in UCT for $H^*$ - only difference is sign $(-1)^{\gamma(n)}$

So by UCT isomor. if $R$ is a field or $R=\mathbb{Z}$, $H_*M$ free.

Same argument works for the other adjoint. $\square$

Poincaré duality and UCT give:

<u>Propn.</u>: $M$ a cpt. $R$-oriented $n$-mfd., $R$ = field or $R = \mathbb{Z}$ and $H_* M$ is free, then

$$H^{n-k}(M;R) \otimes_R H^k(M;R) \xrightarrow{\cup} H^n(M;R) \xrightarrow{\kappa_R(-,[M])} R$$

is a perfect pairing, i.e.

$$\begin{aligned} H^k(M;R) &\xrightarrow{\sim} \mathrm{Hom}_R(H^{n-k}(M;R), R) \\ \varphi &\longmapsto \kappa_R(\varphi \cup -, [M]) = \kappa_R(-, \varphi \cap [M]) \end{aligned}$$

**Cor.:** $M$ cpt. connected $R$-oriented $n$-mfd.

(i) $R=\mathbb{Z}$, $H_* M$ free. If $\alpha \in H^k(M)$ (free) generates a summand $\mathbb{Z}\alpha$ then $\exists\ \beta \in H^{n-k}(M)$ s.t. $\alpha \cup \beta$ generates $H^n(M) \cong \mathbb{Z}$.

(ii) $R$ a field. For any $\alpha \in H^k(M;R)$, $\alpha \neq 0$, then $\exists\ \beta \in H^{n-k}(M;R)$ s.t. $\alpha \cup \beta \neq 0$ in $H^n(M;R) \cong R$.

**Proof:** In (i), $H^k(M) \cong \mathbb{Z}S$, $\alpha \in S$

Projection to summand $\mathbb{Z}\alpha$ (or $\mathbb{Z}S \to \mathbb{Z}$, $\alpha \mapsto 1$, $s \mapsto 0,\ s \neq \alpha$)

is a homomor. $H^k(M) \xrightarrow{\pi} \mathbb{Z}$ s.t. $\pi(\alpha) = 1$.

$\exists!\ \beta \in H^{n-k}(M)$ s.t. $\pi(\xi) = \kappa(\xi \cup \beta, [M])$

In particular, $\kappa(\alpha \cup \beta, [M]) = 1$

$$\kappa(\alpha\cup\beta,[M]) = \kappa(1, (\alpha \cup \beta) \cap [M])$$

Since $M$ is connected, $\kappa(1, -) : H_0 M \to \mathbb{Z}$ is an isomor., so $(\alpha \cup \beta) \cap [M]$ generates $H_0 M$

$\Rightarrow \alpha \cup \beta$ generates $H^n M$ since $- \cap [M] : H^n M \to H_0 M$ is an isomor.

(ii) is the same. $\square$

<u>Example:</u> $S^2 \vee S^4$ is not htpy-eq.t to a compact mfd.:

$$H_*(S^2 \vee S^4; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & * = 0, 2, 4 \\ 0, & \text{otherwise} \end{cases}$$

So can only be htpy-eq.t to a 4-manifold

But then if $x \in H_2$ is the non-zero element,

we must have $x^2 = x \cup x$ is non-zero in $H^4$,

We saw $x^2 = 0$ so this contradicts the previous Cor.

Cor.: There are isomors of graded rings

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1}) \quad \text{w/ } \deg x = 1$$

$$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[y]/(y^{n+1}) \quad \text{w/ } \deg y = 2.$$

Proof: $\mathbb{RP}^n$ is a compact, connected $n$-mfd. ($\mathbb{Z}/2$ - oriented)

The inclusion $\mathbb{RP}^i \hookrightarrow \mathbb{RP}^n$ as the $i$-skeleton gives

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) \longrightarrow H^*(\mathbb{RP}^i; \mathbb{Z}/2) \quad \text{ring homomor.,}$$

isomor. in degrees $\leq i$ (by cellular cohlgy)

$\Rightarrow$ cup products on $\mathbb{RP}^n$ in deg. $< n$ are determined by those in $\mathbb{RP}^{n-1}$ - can proceed by induction.

Enough to show

$$H^i(\mathbb{RP}^n;\mathbb{Z}/2) \otimes H^{n-i}(\mathbb{RP}^n;\mathbb{Z}/2) \xrightarrow{\cup} H^n(\mathbb{RP}^n;\mathbb{Z}/2)$$

is an isomor. $(i<n)$

All 3 groups are $\mathbb{Z}/2$ so this means

for $x_i \neq 0$ in $H^i$, we have $x_i \cup x_{n-i} \neq 0$

$x_i = x_1^i$

— this follows from cup product pairing being perfect

$$x_n = x_i \cup x_{n-i} = x_1^i \cup x_1^{n-i} = x_1^n$$

I.e. the unique non-zero elt. in $H^i(\mathbb{RP}^n;\mathbb{Z}/2)$ w/ $i \leq n$ is $x_1^i$.

$\mathbb{CP}^n$ is a compact connected $\underbrace{\text{oriented}}$ $2n$-mfd.

because $H^{2n}(\mathbb{CP}^n) \cong \mathbb{Z}$

Again can use cell str. to induct on $n$.

$y_2$ generator of $H^2(\mathbb{CP}^n)$, then we may assume

$y_2^i$ is a generator of $H^{2i}(\mathbb{CP}^n)$, for $i < n$.

Need to show the same holds for $i = n$.

Since $y_2^i$ is a generator, $\exists\ y_{n-i} \in H^{2(n-i)}$ s.t.

$y_2^i \cup y_{n-i}$ is a generator of $H^{2n}(\mathbb{CP}^n) \cong \mathbb{Z}$.

We know $y_{n-i} = m\, y_1^{n-i}$ for some $m \in \mathbb{Z}$, since $y_1^{n-i}$ is a generator.

But then $y_1^i \cup m y_1^{n-i} = m y_1^n$ is a generator

This can only happen if $m = \pm 1$ and $y_1^n$ is a generator. $\square$

To prove Poincaré duality for a compact mfd. $M$ we want to work locally on $M$ - all we know about $M$ is that locally it looks like $\mathbb{R}^n$.

But open subsets of $M$ won't be compact.

Clearly PD as we stated it before fails for non-compact mfd.s: for $\mathbb{R}^n$ we have

$$H_*(\mathbb{R}^n) \cong H^*(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * \neq 0 \end{cases}$$

To fix this we need a new variant of cohlgy

# Cohomology with Compact Support

**Defn.:** $\varphi \in S^k(X;R) \cong R^{\mathrm{Sing}_k(X)}$ has compact support if $\exists\ K \subset X$ compact s.t. $\varphi(\sigma) = 0$ if $\sigma(\Delta^n) \subset X \setminus K$

— or $\varphi$ vanishes on $\mathrm{Sing}_k(X \setminus K)$

i.e. $\varphi$ is in $S^k(X, X\setminus K; R)$

The cochains w/ cpt. support form a subgroup $S^k_c(X;R) \subset S^k(X;R)$

This is $\displaystyle\bigcup_{\substack{K \subset X \\ \text{compact}}} S^k(X, X\setminus K; R)$.

If $\varphi$ has cpt. support, so does $\delta\varphi \Rightarrow S^\bullet_c(X;R)$ is a subcomplex of $S^\bullet(X;R)$.

Cohomology of X w/ compact support is

$$H_c^*(X;R) = H_*(S_c^\bullet(X;R)).$$

<u>Note</u>: If X is compact, then every cochain has compact support $\Rightarrow$ $H_c^*(X;R) = H^*(X;R)$.

Want to reformulate this using colimits:

<u>Defn.</u>: $F: \mathcal{J} \to \mathcal{C}$ a functor. The <u>colimit</u> of F, if it exists, is an ob. $\operatorname{colim}_\mathcal{J} F \in \mathcal{C}$ w/ maps $u_i : F(i) \to \operatorname{colim}_\mathcal{J} F$ s.t. for $f: i \to j$ in $\mathcal{J}$ the triangle

$$\begin{tikzcd} F(i) \arrow[rr, "F(f)"] \arrow[dr, "u_i"'] & & F(j) \arrow[dl, "u_j"] \\ & \operatorname{colim}_\mathcal{J} F & \end{tikzcd}$$

commutes,

w/ the universal property that given $x$, $\varphi_i : F(i) \to x$, s.t. $\varphi_i = \varphi_j F(f)$ $\forall f : i \to j$ in $J$, $\exists!$ map $\varphi : \mathrm{colim}_J F \to x$ s.t. $\varphi \psi_i = \varphi_i$.

Example: $M$ a mfd., $\mathrm{Cpt}(M)$ = set of compact subsets of $M$, partially ordered by $\subseteq$, viewed as a cat. (ob.s = elt.s of $\mathrm{Cpt}(M)$ & unique mor. $K \to L$ iff $K \subseteq L$).

$$S_c^\bullet(M;R) = \bigcup_{K \in \mathrm{Cpt}(M)} S^\bullet(M, M \setminus K; R) = \text{colim of } \begin{aligned} \mathrm{Cpt}(M) &\to \mathrm{Ch} \\ K &\longmapsto S^\bullet(M, M\setminus K; R) \\ &\qquad\quad \| \\ &\qquad S^\bullet(M|K; R) \end{aligned}$$

w/ $K \hookrightarrow L \longmapsto$ chain map from $(M, M\setminus L) \to (M, M \setminus K)$.

<u>Propn.:</u> $H^*_c(M;R) \cong \operatorname{colim}_{K \in \mathrm{Cpt}(M)} H^*(M|K;R)$.

<u>Fact:</u> A poset $I$ is <u>filtered</u> if $\forall\, i,j \in I\ \exists\, k$ s.t. $i \le k$, $j \le k$

$H_*: \mathrm{Ch} \to \mathrm{gr\,Ab}$ preserves filtered colimits.

Here $\mathrm{Cpt}(M)$ is filtered since $K, L \in \mathrm{Cpt}(M) \Rightarrow K \cup L$ also compact.

<u>Fact:</u> $I$ poset, $J \subset I$ is <u>cofinal</u> if $\forall\, i \in I\ \exists\, j \in J$ s.t. $i \le j$

For $F: I \to \mathcal{C}$ this implies $\operatorname{colim}_J F|_J \xrightarrow{\sim} \operatorname{colim}_I F$.

Example ($H^*_c(\mathbb{R}^n)$):

Any cpt. subset is contained in some closed ball centred at 0

$\Rightarrow$ these give a cofinal subset of $\mathrm{Cpt}(\mathbb{R}^n)$.

For such $B$, $B' \subset B$

$$\begin{tikzcd}
H^*(\mathbb{R}^n|0) \arrow[r, "\sim"] \arrow[dr, "\wr"'] & H^*(\mathbb{R}^n|B) \arrow[d, "\wr"] \\
& H^*(\mathbb{R}^n|B')
\end{tikzcd}$$

If we have a filtered poset & a diagram where every map is an isom. then colim is also isomorphic to the objects in the diagram.

$$\Rightarrow \quad H^*_c(\mathbb{R}^n) \cong H^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \tilde{H}^*(S^n) \cong \begin{cases} \mathbb{Z}, & * = n \\ 0, & * \neq n \end{cases}$$

$\Rightarrow H^k_c(\mathbb{R}^n) \cong H_{n-k}(\mathbb{R}^n)$ as required for Poincaré duality.

<u>Defn.:</u> $M$ an $R$-oriented $n$-mfd., $K \subset M$ cpt.

We proved that the orientation gives $\mu_K \in H_n(M|K;R)$

s.t. $\rho_x \mu_K$ is the orientation at $x$ $\forall\, x \in K$

$\Rightarrow$ if $K \subset L$ $\rho^L_K \mu_L = \mu_K$

$\exists$ relative cap products $H^i(M|K;R) \otimes H_j(M|K;R) \to H_{j-i}(M;R)$

We get $- \cap \mu_K : H^i(M|K;R) \longrightarrow H_{n-i}(M;R)$

$K \subset L$

$$\begin{tikzcd}
H^i(M|K;R) \arrow[r] \arrow[dr, "-\cap \mu_K"'] & H^i(M|L;R) \arrow[r, dashed] \arrow[d, "-\cap \mu_L"] & H^i_c(M;K) \arrow[dl, "\exists!\ D_M"] \\
& H_{n-i}(M;R) &
\end{tikzcd}$$

by univ. prop. of colim

(If $M$ is compact, then $D_M = - \frown [M]$ as $[M] = \mu_M$)

<u>Thm.</u>: $M$ an $R$-oriented $n$-mfd., then

$$D_M : H^k_c(M;R) \longrightarrow H_{n-k}(M;R)$$

is an isomor.

<u>Lemma</u>: Thm holds for $M = \mathbb{R}^n$.

(computation above $+ \varepsilon$)

- $M$ an $n$-mfd., $U, V$ open s.t. $M = U \cup V$

  Then $\exists$ "Mayer-Vietoris for $H^*_c$", i.e. a LES

$$\cdots H^i_c(U \cap V;R) \longrightarrow H^i_c(U;R) \oplus H^i_c(V;R) \longrightarrow H^i_c(M;R) \to H^{i+1}_c(U \cap V;R) \cdots$$

Note maps go the "wrong" way: $U \subset M$ gives $H^*_c(U) \to H^*_c(M)$

$K \subset U$ cpt.

$$H^*(U, U \setminus K) \xleftarrow{\sim} H^*(M, M \setminus K) \quad \text{by excision}$$

so

$$H^*_c(U) \cong \operatorname*{colim}_{K \in \mathrm{Cpt}(U)} H^*(M, M \setminus K) \longrightarrow \operatorname*{colim}_{K \in \mathrm{Cpt}(M)} H^*(M, M \setminus K) = H^*_c(M)$$

$$\mathrm{Cpt}(U) \hookrightarrow \mathrm{Cpt}(M)$$

- There is a commutative diagram relating this to M-V for $H_*$:

$$\begin{tikzcd}
\cdots H^i_c(U \cap V; R) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^i_c(U;R) \oplus H^i_c(V;R) \arrow[r] \arrow[d, "(D_U, D_V)"] & H^i_c(M;R) \cdots \arrow[d, "D_M"] \\
\cdots H_{n-i}(U \cap V; R) \arrow[r] & H_{n-i}(U;R) \oplus H_{n-i}(V;R) \arrow[r] & H_{n-i}(M;R) \cdots
\end{tikzcd}$$

(1) So 5-lemma gives: if $D_U, D_V, D_{U\cap V}$ are isomors, so is $D_M$

- If $M$ is a union of $U_1 \subset U_2 \subset \cdots \subset M$ then

$$\begin{tikzcd}
\operatorname{colim}_i H^*_c(U_i;R) \arrow[r, "\sim"] \arrow[d, "\operatorname{colim}_i D_{U_i}"] & H^*_c(M;R) \arrow[d, "D_M"] \\
\operatorname{colim}_i H_{n-*}(U_i;R) \arrow[r, "\sim"] & H_{n-*}(M;R)
\end{tikzcd}$$

(2) so $D_M$ is an isomor. if each $D_{U_i}$ is.

Now we can prove PD starting from the case of $\mathbb{R}^n$:

Step 1: $U = U_1 \cup \cdots \cup U_r \subseteq \mathbb{R}^n$ open w/ $U_i \subset \mathbb{R}^n$ open & convex. Induct on $r$.

$r=1$: $U_1 \cong \mathbb{R}^n$

$U' = U_1 \cup \cdots \cup U_{r-1}$, $\quad U' \cap U_r =$ union $U_i \cap U_r$ $\quad i \leq r-1$
convex open

Have $D_{U'}$, $D_{U' \cap U_r}$, $D_{U_r}$ isomor.s

$\Rightarrow D_U$ isomor. by (1).

Step 2: $U \subset \mathbb{R}^n$ any open

$U = \bigcup_{i=1}^{\infty} U_i$, $U_i$ open & convex. Set $V_j = U_1 \cup \cdots \cup U_j$

then $D_{V_j}$ is isomor. by Step 1 $\Rightarrow D_U$ isomor. by (2)

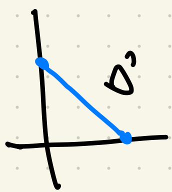
**Step 3:** $M = U_1 \cup \cdots \cup U_r$ w/ $U_i$ homeomorphic to open in $\mathbb{R}^n$

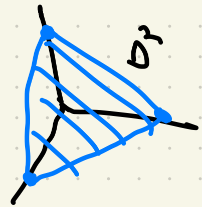
Induct on $r$ as in Step 1.

**Step 4:** $M$ arbitrary — since $M$ is 2nd countable
$\exists$ open cover $U_1, U_2, \ldots$ w/ $U_i$ is open in $\mathbb{R}^n$.
Use (2) & Step 3 as in Step 2. $\square$

# Review

$\Delta^n \subseteq \mathbb{R}^{n+1}$ set of $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ s.t. $x_i \geq 0$, $\sum x_i = 1$





$d^i : \Delta^{n-1} \hookrightarrow \Delta^n$ incl. of subset of $(x_0, \dots, x_n)$ where $x_i = 0$

or $(y_0, \dots, y_{n-1}) \longmapsto (y_0, \dots, y_{i-1}, 0, y_i, \dots, y_{n-1})$

$0 \leq j < i \leq n+1$, then $d^i d^j = d^j d^{i-1}$

Singular n-simplices $\mathrm{Sing}_n X = \{\Delta^n \to X \text{ cts.}\}$
$X$ top. sp.

$$\partial_i : \mathrm{Sing}_n X \longrightarrow \mathrm{Sing}_{n-1} X$$

$$\sigma : \Delta^n \to X \longmapsto \sigma \circ d^i : \Delta^{n-1} \hookrightarrow \Delta^n \longrightarrow X$$

$$\partial^j \partial^i = \partial^{i-1} \partial^j$$

Singular n-chains $S_n X = \mathbb{Z}\,\mathrm{Sing}_n X$

$$\partial : S_n X \longrightarrow S_{n-1} X$$

$\partial \sigma = \sum_{i=0}^{n} (-1)^i \partial_i \sigma$ for $\sigma \in \mathrm{Sing}_n X$, and extend linearly

$\partial^2 = 0$ — i.e. $(S_\bullet X, \partial)$ is a chain cx.

$(C_\bullet, d)$ chain cx. (i.e. $C_n \in Ab$, $n \in \mathbb{Z}$, $d : C_n \to C_{n-1}$, $d^2 = 0$)

$Z_n C \subset C_n$ <u>n-cycles</u> $= \ker d : C_n \to C_{n-1}$

$B_n C \subset C_n$ <u>n-boundaries</u> $= \operatorname{im} d : C_{n+1} \to C_n$

$d^2 = 0 \Rightarrow \quad B_n C \subset Z_n C$

$H_n C = Z_n C / B_n C$ <u>homology</u>

$X \in Top$,

$H_n X := H_n S_\bullet X$

$f : X \longrightarrow Y$ cts. $\rightsquigarrow$ $\mathrm{Sing}_n X \xrightarrow{f_*} \mathrm{Sing}_n Y$

$\sigma : \Delta^n \to X \longmapsto f \circ \sigma : \Delta^n \to X \to Y$

$\partial_i (f_* \sigma) = f_* (\partial_i \sigma)$

$\rightsquigarrow$ $f_* : S_n X \to S_n Y$ extending linearly

$\partial f_* = f_* \partial$ i.e. $f_*$ is a chain map

$\rightsquigarrow$ $f_* : H_n X \to H_n Y$, gives $H_* : \mathrm{Top} \xrightarrow{S_\cdot} \mathrm{Ch} \xrightarrow{H_*} \mathrm{grAb}$ functor

<u>Example:</u> $H_*(*)$

$\mathrm{Sing}_n(*) = \{ \Delta^n \xrightarrow{c_n} * \}$

$\partial_i c_n = c_{n-1} \quad \forall i$

$S_n(*) = \mathbb{Z} c_n \xrightarrow{\partial} \mathbb{Z} c_{n-1}$

$$\partial c_n = \sum_{i=0}^{n} (-1)^i \partial_i c_n = \left( \sum_{i=0}^{n} (-1)^i \right) c_{n-1} = \begin{cases} 0, & n \text{ odd} \\ 1 \cdot c_{n-1}, & n \text{ even} \end{cases}$$

$S_*(*)$:

$$\cdots \mathbb{Z} \xrightarrow{0} \overset{2}{\mathbb{Z}} \xrightarrow{1} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}} \to 0 \cdots$$

$$H_*(*) = \begin{cases} \mathbb{Z}/0 = \mathbb{Z}, & * = 0 \\ \mathbb{Z}/\mathbb{Z} = 0, & * > 0 \text{ odd} \\ 0/0 = 0, & * > 0 \text{ even} \end{cases}$$

$$= \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * \neq 0 \end{cases}$$

- $\sigma : \Delta^1 \to X$, $\quad rw : \Delta^1 \to \Delta^1$

reverse orientation

$$\sigma \circ rw = -\sigma + \text{bd. } ry$$

- $H_0 X \cong \mathbb{Z}\,\underbrace{\pi_0 X}_{\text{set of path-components}}$

  $= X \big/ x \sim x'$ if $\exists$ path between $x, x'$

$\Big( S_0 X = \mathbb{Z}X$

$S_1 X = \mathbb{Z}\{\text{paths in } X\}$

$p : \underset{\Delta^1}{\overset{[0,1]}{I}} \to X, \quad \partial p = p(1) - p(0) \Big)$

- $H_1 X \cong (\pi_1 X)^{ab} = \pi_1 X \big/ \underset{ghg^{-1}h^{-1}}{\text{commutators}}$

$\Big[\ l : S^1 \to X \quad \longmapsto \quad l_*[S^1] \in H_1 X$

ht.py-invariance: $l \simeq l' \Rightarrow l_* = l'_* \Rightarrow l_*[S^1] = l'_*[S^1]$

$\bigcirc \xrightarrow{p} 8 \xrightarrow{(l,l')} X$ — we saw $((l,l')\circ p)_* = l_* + l'_*$ so homomor. $\Big]$

- $H_*\left(\coprod_{i\in I} X_i\right) \cong \bigoplus_i H_* X_i$

Relative homology: $A \subset X$ subspace, $S_n A \subset S_n X$ subgroup

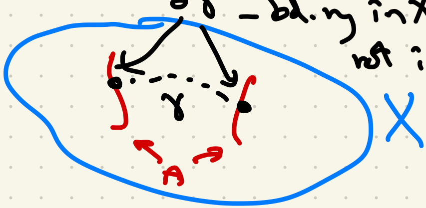
$$S_n(X,A) = S_n X / S_n A$$

$\partial$ on $S_\bullet X$ induces boundary map on $S_\bullet(X,A)$

$$H_*(X,A) = H_* S_\bullet(X,A) \text{ - relative homology}$$

Relative cycle in $S_n(X,A)$ - represented by $\gamma \in S_n X$ s.t. $\partial\gamma$ is in $S_{n-1}A$

$\partial\gamma$ - bdry in $X$, but not in $A$




- here $\partial\gamma$ is a cycle in $S_\bullet A$ but not necessarily a boundary in $S_\bullet A$

This gives $\quad H_n(X,A) \longrightarrow H_{n-1}A$

homology class
rep. by relative
cycle that is $\quad \longmapsto \quad [\partial\gamma]$
rep. by chain
$\gamma$ w/ $\partial\gamma \in S_{n-1}A$

This gives LES of a pair $(X,A)$:

$$\cdots H_nA \longrightarrow H_nX \longrightarrow H_n(X,A) \xrightarrow{\partial} H_{n-1}A \ \cdots$$

- exactness means at each group im = ker

Eilenberg – Steenrod axioms: A homology theory consists of

functors $h_n : \mathrm{Pair} \to \mathrm{Ab}$, natural maps $\partial : h_n(X,A) \to h_{n-1}A := h_{n-1}(A,\emptyset)$

- LES $\cdots h_n A \to h_n X \to h_n(X,A) \xrightarrow{\partial} h_{n-1} A \cdots$

- Htpy-invariance: if $f,g : (X,A) \to (Y,B)$ $(f,g : X \to Y$ cts., $fA \subset B)$ are homotopic $(\exists\, h : X \times I \to Y,\ h(-,0) = f,\ h(-,1) = g,\ h(a,t) \in B\ \forall a \in A,\ t \in I)$ then $f_* = g_* : h_n(X,A) \to h_n(Y,B)$

- Excision: if $U \subset A \subset X$, $\overline{U} \subset A^\circ$ then
$$h_n(X \setminus U, A \setminus U) \xrightarrow{\sim} h_n(X,A) \text{ isomor.}$$

- Additivity: $h_n\left(\coprod_{i \in I} X_i\right) \cong \bigoplus_{i \in I} h_n X_i$

- Dimension axiom: $h_n(*) = 0, \quad * \neq 0$

Excision & ht. py-inv. ce $\rightsquigarrow$ if $A \subset X$ & $\exists$ $A \subset B \subset X$ s.t. $\bar{A} \subset B^\circ$ & $A \overset{i}{\hookrightarrow} B$ a deformation retract ($\exists$ $\rho : B \to A$, $\rho i = id_A$, $\exists$ htpy between $i\rho$ & id taking $A \times I$ to $A$) = "good pair"

Then $\quad H_n(X, A) \cong \tilde{H}_*(X/A) = H_*(X/A, *)$

single point that is image of $A$ in quotient

<u>Example:</u> Compute $\tilde{H}_* S^1$ as $H_*(I, \partial I)$:

$$\cdots \; \underset{\substack{\| \\ 0}}{H_1 I} \to H_1(I,\partial I) \longrightarrow \underset{\substack{\| \\ \mathbb{Z}^2}}{H_0 \partial I} \xrightarrow{\varphi} \underset{\substack{\| \\ \mathbb{Z}}}{H_0 I} \to \underset{\substack{\| \\ 0}}{H_0(I,\partial I)} \to 0$$

$$\mathbb{Z}^2 \xrightarrow{\varphi} \mathbb{Z}$$

$$(1,0) \longmapsto 1$$

$$(0,1) \longmapsto 1$$

because both path-comp.s of $\partial I$ go to the single one of $I$

$$\varphi(i,j) = i+j \quad \text{– surjective – coker} = 0$$

kernel: $(i,j)$ s.t. $i+j=0$

$$\cong \mathbb{Z}\{(1,-1)\}$$

$\Rightarrow \; H_1(I,\partial I) \cong \mathbb{Z}, \quad H_0(I,\partial I) = 0.$

Mayer–Vietoris sequence:

$A, B \subset X, \quad A^\circ \cup B^\circ = X$

$$\begin{tikzcd}
A \cap B \arrow[r, hook, "j"] \arrow[d, hook, "j'"'] & A \arrow[d, hook, "i"] \\
B \arrow[r, hook, "i'"'] & X
\end{tikzcd}$$

$$\cdots H_n(A \cap B) \xrightarrow{(j_*, j'_*)} H_n A \oplus H_n B \xrightarrow{i_* - i'_*} H_n X \xrightarrow{\Delta} H_{n-1}(A \cap B)$$

$\Delta$ is $H_n X \longrightarrow H_n(X, B) \xrightarrow{\sim} H_n(A, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B)$

excision

Cellular homology:

A <u>cell complex</u> (or <u>CW-complex</u>) is a top. sp. $X$ w/ filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset \bigcup_i X_i = X$$

w/ colimit topology: $U \subset X$ open iff $U \cap X_i$ open in $X_i \ \forall i$

w/ pushouts

$$\begin{tikzcd} \coprod_{\alpha \in \Gamma_n} S^{n-1} \arrow[r, hook] \arrow[d, "(f_\alpha)"] & \coprod_{\alpha \in \Gamma_n} D^n \arrow[d] \\ X_{n-1} \arrow[r] & X_n \end{tikzcd}$$

$$\Rightarrow X_n / X_{n-1} \cong \bigvee_{\alpha \in \Gamma_n} S^n$$

<u>Example:</u> $\mathbb{RP}^n$ is $S^n/(x \sim -x)$ or $D^n/(x \sim -x \text{ for } x \in \partial D^n)$

— gives pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d] & D^n \arrow[d] \\ \mathbb{RP}^{n-1} \arrow[r, hook] & \mathbb{RP}^n \end{tikzcd}$$

This determines a cell str. on $\mathbb{RP}^n$ w/ one cell in each dim. $\leq n$.

Cellular chains: $C_n^{cell} X = H_n(X_n, X_{n-1}) = \tilde{H}_n(X_n / X_{n-1}) \cong \tilde{H}_n(\bigvee_{\Gamma_n} S^n)$

$$\cong \mathbb{Z}\Gamma_n$$

w/ differential $H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \longrightarrow H_{n-1}(X_{n-1}, X_{n-2})$

This is a chain cx. & $H_* C^{cell}_* X \cong H_* X$

$\partial^{cell} : \mathbb{Z}\Gamma_n \to \mathbb{Z}\Gamma_{n-1}$

$$\alpha \longmapsto \sum_{\beta \in \Gamma_{n-1}} \deg(q_\beta f_\alpha)\, \beta$$

send all spheres to $*$
except that indexed by $\beta$

where
$$\begin{tikzcd} S^{n-1} \arrow[r, "f_\alpha"] & X_{n-1} \arrow[r] \arrow[rr, bend right, "q_\beta"'] & X_{n-1}/X_{n-2} \cong \bigvee_{\Gamma_{n-1}} S^{n-1} \arrow[r] & S^{n-1} \end{tikzcd}$$

Example: $H_* \mathbb{RP}^n$ is computed by

$$0 \to \underset{n}{\mathbb{Z}c_n} \to \mathbb{Z}c_{n-1} \to \cdots \underset{1}{\mathbb{Z}c_1} \to \underset{0}{\mathbb{Z}c_0} \to 0$$

$$\partial^{cell} c_n = \deg\left( S^{n-1} \to \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \cong S^{n-1} \right) \cdot c_{n-1}$$

$$= \begin{cases} 0, & n \text{ odd} \\ 2c_{n-1}, & n \text{ even} \end{cases}$$

$$0 \to \mathbb{Z}_n \xrightarrow{\substack{0,\ n \text{ odd} \\ 2,\ n \text{ even}}} \cdots\ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}_0 \to 0$$

$$\rightsquigarrow \quad H_* \mathbb{RP}^n \cong \begin{cases} \mathbb{Z}, & * = 0 \\ \mathbb{Z}, & k = n, \ n \text{ odd} \\ \mathbb{Z}/2, & 0 < * < n, \ * \text{ odd} \end{cases}$$

($\Rightarrow \mathbb{RP}^n$ orientable for $n$ odd, not for $n$ even)

$S_\bullet X \otimes M \qquad M \in \mathrm{Ab}$

$\shortparallel$

$S_\bullet(X;M)$

$$H_*(X;M) := H_* S_\bullet(X;M)$$

Have canonical map $H_n(X) \otimes M \to H_n(X;M)$ but not isomor. in general

$\mathrm{Tor}(A,B) = \ker i \otimes B$ – well-defined

$0 \to F_1 \xrightarrow{i} F_0 \to A \to 0$

free

free resolution of A

$\mathrm{Tor}(\mathbb{Z},B) = 0$, $\mathrm{Tor}(\mathbb{Q},B) = 0$, $\mathrm{Tor}(\mathbb{Z}/n, B) =$ n-torsion subgroup in B

from resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$

Universal coefficient theorem: there are natural SESs

$$0 \to H_n X \otimes M \longrightarrow H_n(X; M) \longrightarrow \mathrm{Tor}(H_{n-1} X, M) \to 0$$

E.g. $H_*(X; \mathbb{Q}) \cong (H_* X) \otimes \mathbb{Q}$

<u>Example:</u> $H_*(\mathbb{RP}^2; \mathbb{Z}/2)$:

$$H_* \mathbb{RP}^2 = \begin{cases} 0, & * > 1 \\ \mathbb{Z}/2, & * = 1 \\ \mathbb{Z}, & * = 0 \end{cases}$$

$$H_0(\mathbb{RP}^2; \mathbb{Z}/2) \cong \mathbb{Z} \otimes \mathbb{Z}/2 = \mathbb{Z}/2$$

$$0 \to \underset{\substack{\parallel \\ \mathbb{Z}/2}}{\mathbb{Z}/2 \otimes \mathbb{Z}/2} \longrightarrow H_1(\mathbb{RP}^2; \mathbb{Z}/2) \longrightarrow \underset{\substack{\parallel \\ 0}}{\mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2)} \to 0$$

$H_2$:

$$0 \to 0 \to H_2(\mathbb{RP}^2; \mathbb{Z}/2) \to \mathrm{Tor}(\underbrace{H_1(\mathbb{RP}^2)}_{\cong\, \mathbb{Z}/2}, \mathbb{Z}/2) \to 0$$

$$\mathrm{Tor}(H_1(\mathbb{RP}^2), \mathbb{Z}/2) \cong \mathbb{Z}/2$$

$$\Rightarrow \quad H_*(\mathbb{RP}^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & 0 \le * \le 2 \\ 0, & * > 2 \end{cases}$$

## Cohomology:

$$S^\bullet(X;M) = \mathrm{Hom}(S_\bullet X, M)$$

$$S^n(X;M) = \mathrm{Hom}(S_n X, M) \quad \text{(lives in deg. } -n \text{ in } S^\bullet(X;M) \text{ as a chain cx.)}$$

singular $n$-cochains

$$= \mathrm{Hom}(\mathbb{Z}\,\mathrm{Sing}_n X, M)$$

$$= M^{\mathrm{Sing}_n X} \quad \text{(functions } \mathrm{Sing}_n X \to M\text{)}$$

Coboundary map $\delta : S^n(X;M) \to S^{n+1}(X;M)$

$$\varphi : S_n X \to M \longmapsto \varphi \circ d$$

- $H^0 X \cong \mathbb{Z}^{\pi_0 X}$

- Dual Eilenberg-Steenrod axioms, Mayer-Vietoris, ...

Universal coeff. thm.: $\exists$ natural SESs

$$0 \to \mathrm{Ext}(H_{n-1}X, M) \to H^n(X; M) \to \mathrm{Hom}(H_n X, M) \to 0$$

where $\mathrm{Ext}(A,B)$ is coker $\mathrm{Hom}(i,B) : \mathrm{Hom}(F_0, B) \to \mathrm{Hom}(F_1, B)$.

$$0 \to F_1 \xrightarrow{i} F_0 \to A \to 0$$

- $\mathrm{Ext}(A, \mathbb{Q}) = 0 \implies H^n(X; \mathbb{Q}) \cong \mathrm{Hom}(H_n X, \mathbb{Q})$
- $\mathrm{Ext}(\mathbb{Z}, B) = 0$
- $\mathrm{Ext}(\mathbb{Z}/n, B) = \mathrm{coker}\,(B \xrightarrow{n\cdot} B) = B/nB$

**Example:** $H^* \mathbb{RP}^2$ :

$$H^0 \mathbb{RP}^2 = \mathrm{Hom}(H_0 \mathbb{RP}^2, \mathbb{Z}) \cong \mathbb{Z}$$

$$0 \to \underset{\substack{\parallel \\ 0}}{\mathrm{Ext}(\mathbb{Z}, \mathbb{Z})} \to \underset{\substack{\parallel \\ 0}}{H^1 \mathbb{RP}^2} \to \underset{\substack{\parallel \\ 0}}{\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z})} \to 0$$

$$0 \to \underset{\substack{\parallel \\ \mathbb{Z}/2}}{\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z})} \to \underset{\substack{\parallel \\ \mathbb{Z}/2}}{H^2 \mathbb{RP}^2} \to \underset{\substack{\parallel \\ 0}}{\mathrm{Hom}(0, \mathbb{Z})} \to 0$$

Eilenberg–Zilber thm.: $S_\bullet(X \times Y)$ naturally chain ht.py eq't to $S_\bullet X \otimes S_\bullet Y$

Künneth thm.: for $X, Y$ have natural SESs

$$0 \to \underbrace{(H_* X \otimes H_* Y)_n}_{= \bigoplus_{p+q=n} H_p X \otimes H_q Y} \longrightarrow H_n(X \times Y) \longrightarrow \underbrace{\mathrm{Tor}(H_* X, H_* Y)_{n-1}}_{\bigoplus_{i+j=n-1} \mathrm{Tor}(H_i X, H_j Y)} \to 0$$