Tensor products & chain complexes

$C_\cdot, D_\cdot \in Ch$

$$(C_\cdot \otimes D_\cdot)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

$c \in C_p,\ d \in D_q \qquad \partial(c \otimes d) = \partial_C c \otimes d + (-1)^p c \otimes \partial_D d$

$C_\cdot$ levelwise free, $D_\cdot \in Ch$

$$0 \to (H_* C \otimes H_* D)_n \longrightarrow H_n(C \otimes D) \to \mathrm{Tor}(H_* C, H_* D)_{n-1} \to 0$$

$$(H_* C \otimes H_* D)_n = \bigoplus_{p+q=n} H_p C \otimes H_q D \qquad \mathrm{Tor}(H_* C, H_* D)_{n-1} = \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p C, H_q D)$$

$C_*, D_* \in \mathrm{grAb}$

$C_* \otimes D_* \longrightarrow E_* \iff$ bilinear maps $C_p \times D_q \xrightarrow{f_{pq}} E_{p+q}$

$C_\bullet, D_\bullet \in \mathrm{Ch}$

chain map $C_\bullet \otimes D_\bullet \xrightarrow{f} E_\bullet \iff \partial f_{pq}(c,d)$

$$= f_{p-1,q}(\partial c, d) + (-1)^p f_{p,q-1}(c, \partial d)$$

$[f_{p,q}(c,d) = f(c \otimes d)]$

The Eilenberg-Zilber Theorem

We will show $S_\bullet(X \times Y)$ is naturally chain htpc. to $S_\bullet X \otimes S_\bullet Y$

Will use acyclic models - $\exists$ also explicit chain maps

In deg. 0 we have a natural isomor:

$$(S_\bullet X \otimes S_\bullet Y)_0 = S_0 X \otimes S_0 Y = \mathbb{Z}X \otimes \mathbb{Z}Y \cong \mathbb{Z}(X \times Y) = S_0(X \times Y)$$

Propn.

(i) $\exists$ natural chain map $\varphi : S_\bullet(X \times Y) \to S_\bullet X \otimes S_\bullet Y$ given in deg. 0 by this isomorphism

(ii) Any two such are naturally chain homotopic.

Proof:

(i) Want to inductively define $\varphi_n^{X,Y} : S_n(X \times Y) \to (S_\cdot X \otimes S_\cdot Y)_n$

s.t. $\partial \varphi_n^{X,Y} = \varphi_{n-1}^{X,Y} \partial$, starting w/ φ_0 = isomor. above

For $(\sigma, \tau) : \Delta^n \to X \times Y$

$$\begin{tikzcd} \Delta^n \arrow[r] \arrow[d, "\delta_n"'] & X \times Y \\ \Delta^n \times \Delta^n \arrow[ur, "\sigma \times \tau"'] & \end{tikzcd}$$

$(\sigma \times \tau)_* : S_\cdot(\Delta^n \times \Delta^n) \to S_\cdot(X \times Y)$

so $(\sigma, \tau) = (\sigma \times \tau)_* \delta_n$ in $S_n(X \times Y)$

for $\delta_n \in S_n(\Delta^n \times \Delta^n)$

So naturality forces $\varphi_n^{X,Y}(\sigma, \tau) = \varphi_n^{X,Y}((\sigma \times \tau)_* \delta_n)$

$= (\sigma_* \otimes \tau_*) \varphi_n^{\Delta^n, \Delta^n}(\delta_n)$

So we start by defining $\varphi_n(\delta_n)$. This should satisfy

$\partial \varphi_n(\delta_n) = \varphi_{n-1}(\partial \delta_n)$

$\underline{n=1}$: Want $\varphi_1(\delta_1)$ s.t. $\partial \varphi_1 \delta_1 = \varphi_0(\partial \delta_1) = \varphi_0\big(([1],[1]) - ([0],[0])\big)$

$[0],[1] : \Delta^0 \to \Delta^1$ two faces of Δ^1 $\qquad = [1]\otimes[1] - [0]\otimes[0]$

$$\begin{tikzcd} 00 \arrow[r] \arrow[d] \arrow[dr, "\delta_1"] & 01 \arrow[d] \\ 10 \arrow[r] & 11 \end{tikzcd}$$

$$\varphi_1 \delta_1 = [0]\otimes \iota_1 + \iota_1 \otimes [1]$$

$$\partial \varphi_1 \delta_1 = [0]\otimes([1]-[0]) + ([1]-[0])\otimes[1]$$
$$= -[0]\otimes[0] + [1]\otimes[1]$$

$\underline{n>1}$: Want $\varphi_n \delta_n$ s.t. $\partial \varphi_n \delta_n = \varphi_{n-1}(\partial \delta_n)$

$$H_*(S.\Delta^n \otimes S.\Delta^n) = \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * \neq 0 \end{cases}$$

In particular $H_{n-1}(S.\Delta^n \otimes S.\Delta^n) = 0 \Rightarrow$ cycles are boundaries

$\varphi_{n-1}(\partial \delta_n)$ is a cycle: $\partial \varphi_{n-1}(\partial \delta_n) = \varphi_{n-2}(\partial^2 \delta_n) = 0$

so $\exists$ some chain whose boundary is $\varphi_{n-1}(\partial \delta_n)$ – we choose one & define this to be $\varphi_n \delta_n$.

Now we set $\varphi_n^{X,Y}(\sigma,\tau) = (\sigma_* \otimes \tau_*)\varphi_n\delta_n$ & extend linearly to chains.

This is natural: for $f: X \to X'$, $g: Y \to Y'$ we want

$$(f_* \otimes g_*)\varphi_n^{X,Y} = \varphi_n^{X',Y'} \circ (f \times g)_*$$

Enough to check on $(\sigma,\tau): \Delta^n \to X \times Y$ since both sides are linear

Then $(f_* \otimes g_*)\varphi_n^{X,Y}(\sigma,\tau) = (f_* \otimes g_*)(\sigma_* \otimes \tau_*)\varphi_n\delta_n$

$$= (f_*\sigma_* \otimes g_*\tau_*)\varphi_n\delta_n$$

$$= ((f\sigma)_* \otimes (g\tau)_*)\varphi_n\delta_n$$

$$= \varphi_n^{X',Y'}(f\sigma, g\tau)$$

$$= \varphi_n^{X',Y'}(f \times g)_*(\sigma,\tau)$$

In Ab:

$A \xrightarrow{\alpha} A' \xrightarrow{\beta} A''$

$B \xrightarrow{\gamma} B' \xrightarrow{\delta} B''$

Want: $\beta\alpha \otimes \delta\gamma = (\beta \otimes \delta)(\alpha \otimes \gamma)$

$A \otimes B \to A'' \otimes B''$

Here

$(\beta\alpha \otimes \delta\gamma)(a \otimes b)$

$= \beta\alpha a \otimes \delta\gamma b$

$(\beta \otimes \delta)(\alpha \otimes \gamma)(a \otimes b)$

$= (\beta \otimes \delta)(\alpha a \otimes \gamma b) = \beta\alpha a \otimes \delta\gamma b$

$\varphi_n^{X,Y}$ is a chain map:

$$\partial \varphi_n^{X,Y}(\sigma,\tau) = \partial(\sigma_* \otimes \tau_*)\varphi_n \delta_n$$
$$= (\sigma_* \otimes \tau_*)\partial \varphi_n \delta_n$$
$$= (\sigma_* \otimes \tau_*)\varphi_{n-1}(\partial \delta_n)$$
$$= \varphi_{n-1}^{X,Y}(\sigma \times \tau)_*(\partial \delta_n)$$
$$= \varphi_{n-1}^{X,Y}\partial(\sigma \times \tau)_* \delta_n$$
$$= \varphi_{n-1}^{X,Y}\partial(\sigma,\tau)$$

(ii) Given φ, φ' as in (i) (both given by standard isom. in deg. 0)
we want a natural chain ht.py, i.e. want

$$s_n^{X,Y} : S_n(X \times Y) \longrightarrow (S_\cdot X \otimes S_\cdot Y)_{n+1} \quad \text{s.t.}$$
$$\partial s_n + s_{n-1}\partial = \varphi' - \varphi$$

For $n=0$ we know $\varphi_0 = \varphi_0'$ so take $s_0 = 0$

Given s_k for $k<n$ start by defining $s_n(\delta_n)$

- this should satisfy

$$\partial s_n(\delta_n) = \varphi_n'\delta_n - \varphi_n\delta_n - s_{n-1}(\partial\delta_n)$$

$$H_n(S_\bullet\Delta^n \otimes S_\bullet\Delta^n) = 0 \text{ as } n>0$$

so RHS is a bdry iff it's a cycle:

$$\partial(\text{RHS}) = \partial(\varphi_n'\delta_n - \varphi_n\delta_n) + s_{n-2}(\partial^2\delta_n) - (\varphi_{n-1}'\partial\delta_n - \varphi_{n-1}\partial\delta_n)$$

$$= 0$$

So can choose $s_n\delta_n$ w/ this boundary & proceed as in (i).

$\square$

Propn.

(i) $\exists$ natural chain map $\mu : S_\bullet X \otimes S_\bullet Y \to S_\bullet(X \times Y)$ given in deg. 0 by this isomorphism

(ii) Any two such are naturally chain homotopic.

Remark: Earlier we defined exterior mult. maps

$$S_p X \times S_q Y \to S_{p+q}(X \times Y)$$

- these combine to a chain map $S_\bullet X \otimes S_\bullet Y \to S_\bullet(X \times Y)$ that satisfy (i).

Propn.:

(i) Any two natural chain maps $S_\bullet X \otimes S_\bullet Y \to S_\bullet X \otimes S_\bullet Y$ given by id in deg. 0 are naturally chain htpc.

(ii) Ditto for $S_\bullet(X \times Y) \to S_\bullet(X \times Y)$

Thm. (Eilenberg-Zilber): $\exists$ natural chain htpy eq. $S_\cdot(X\times Y) \simeq S_\cdot X \otimes S_\cdot Y$.

Proof: We know $\exists$ chain maps $S_\cdot(X\times Y) \underset{\mu}{\overset{\varphi}{\rightleftarrows}} S_\cdot X \otimes S_\cdot Y$

given in deg. 0 by specified isomor.

So $\varphi\mu$ & $\mu\varphi$ are chain maps given by id in deg. 0

$\Rightarrow$ chain htpc. to respective id's. $\square$

Cor.: $H_*(X\times Y) \cong H_*(S_\cdot X \otimes S_\cdot Y)$.

$S_\cdot X$ & $S_\cdot Y$ are levelwise free, so we get:

Cor. (Künneth Thm.): $\exists$ natural SESs

$$0 \to (H_*X \otimes H_*Y)_n \to H_n(X\times Y) \to \mathrm{Tor}(H_*X, H_*Y)_{n-1} \to 0$$

(non-canonically splittable)

Example: $(n \neq m,\ n, m > 0)$

$$H_k(S^n \times S^m) \cong (H_* S^n \otimes H_* S^m)_k$$

$$= \bigoplus_{p+q=k} H_p S^n \otimes H_q S^m$$

$$= \begin{cases} \mathbb{Z}, & k = 0\ (p=q=0), \\ & k = n\ (p=n,\ q=0) \\ & k = m\ (p=0,\ q=m) \\ & k = n+m\ (p=n,\ q=m), \\ 0, & \text{otherwise} \end{cases}$$

$n = m$:

$$H_k(S^n \times S^n) = \begin{cases} \mathbb{Z}, & k = 0,\ 2n \\ \mathbb{Z} \oplus \mathbb{Z}, & k = n \\ 0, & \text{otherwise} \end{cases}$$

Exercise: Compute $H_*(\mathbb{RP}^2 \times \mathbb{RP}^2)$

Remark: Künneth Thm. works w/ coefficients in any PID.
If k is a field then $\mathrm{Tor} = 0$ &

$$H_*(X\times Y; k) \cong H_*(X;k) \otimes_k H_*(Y;k)$$

The Alexander-Whitney map

One explicit choice of $S_\cdot(X\times Y) \to S_\cdot X \otimes S_\cdot Y$ is the Alexander-Whitney map.

Notation: $\alpha^n_p : \Delta^p \hookrightarrow \Delta^n$ = p-face w/ vertices $0, 1, \dots, p$

$\omega^n_q : \Delta^q \hookrightarrow \Delta^n$ = q-face w/ vertices $n-q, \dots, n-1, n$

$aw_n^{X,Y} : S_n(X \times Y) \to (S_\bullet X \otimes S_\bullet Y)_n$ is given on $(\sigma, \tau) : \Delta^n \to X \times Y$ by

$$aw_n(\sigma, \tau) = \sum_{p+q=n} \underbrace{\sigma \circ \alpha_p^n}_{\sigma|_{0,1,\dots,p}} \otimes \underbrace{\tau \circ \omega_q^n}_{\tau|_{p,p+1,\dots,n}} = (\sigma_* \otimes \tau_*) \underbrace{\left(\sum_{p+q=n} \alpha_p^n \otimes \omega_q^n \right)}_{aw(\delta_n)}$$

Propn: aw is a natural family of chains maps, i.e. for $f : X \to X'$, $g : Y \to Y'$ cts. maps,

$$\begin{tikzcd} S_\cdot(X \times Y) \arrow[r, "aw"] \arrow[d, "(f \times g)_*"'] & S_\cdot(X) \otimes S_\cdot(Y) \arrow[d, "f_* \otimes g_*"] \\ S_\cdot(X' \times Y') \arrow[r, "aw"] & S_\cdot(X') \otimes S_\cdot(Y') \end{tikzcd}$$ commutes

Proof: Reduce to checking $\partial\, aw(\delta_n) = aw(\partial \delta_n)$ by explicit computation. (Notes/Exercise)

Note: $\exists$ Eilenberg-Zilber & Künneth thm.s for cohl.gy & for relative hl.gy, but require some technical hypotheses. (See notes.)

Cross and Cup Products

We can use $S_\bullet(X \times Y) \to S_\bullet X \otimes S_\bullet Y$ to define products in cohl.gy.

Exercise:

(i) $\exists$ natural homomor.s of abelian groups

$$\mathrm{Hom}(A,M) \otimes \mathrm{Hom}(B,N) \longrightarrow \mathrm{Hom}(A \otimes B, M \otimes N)$$
$$(\varphi : A \to M) \otimes (\psi : B \to N) \longmapsto (\varphi \otimes \psi : A \otimes B \to M \otimes N)$$

(ii) $\exists$ natural chain maps ($C, D \in \mathrm{Ch}$, $M, N \in \mathrm{Ab}$)

$$\mathrm{Hom}(C_\bullet, M)_\bullet \otimes \mathrm{Hom}(D, N)_\bullet \longrightarrow \mathrm{Hom}(C \otimes D, M \otimes N)_\bullet$$

[LHS in deg. n is $\bigoplus_{p+q=n} \mathrm{Hom}(C_{-p}, M) \otimes \mathrm{Hom}(D_{-q}, N)$

RHS : $\mathrm{Hom}\left(\bigoplus_{p+q=n} C_{-p} \otimes D_{-q}, M \otimes N\right)$]

<u>Step 1</u>: From exercise, have natural maps

$$S^\bullet(X; M) \otimes S^\bullet(Y; N) \longrightarrow \mathrm{Hom}(S_\bullet X \otimes S_\bullet Y, M \otimes N)$$

<u>Step 2</u>: Compose w/ $S_\bullet(X \times Y) \to S_\bullet X \otimes S_\bullet Y$ to get

$$S^\bullet(X; M) \otimes S^\bullet(Y; N) \to \mathrm{Hom}(S_\bullet X \otimes S_\bullet Y, M \otimes N) \longrightarrow \mathrm{Hom}(S_\bullet(X \times Y), M \otimes N) = S^\bullet(X \times Y; M \otimes N)$$

<u>Step 3</u>: R a ring, mult. gives a homomor. $R \otimes R \xrightarrow{\mu} R$, and so

$$S^\bullet(X; R) \otimes S^\bullet(Y; R) \longrightarrow S^\bullet(X \times Y; R \otimes R) \xrightarrow{\mu_*} S^\bullet(X \times Y; R)$$ (chains-level cross product)

Step 4: $\Delta : X \to X \times X$ diagonal, gives $\Delta^* : S^\bullet(X\times X; R) \to S^\bullet(X;R)$

Get

$$S^\bullet(X;R) \otimes S^\bullet(X;R) \xrightarrow{\text{(step 3)}} S^\bullet(X\times X;R) \xrightarrow{\Delta^*} S^\bullet(X;R)$$ (chain level cup product)

Step 5: In cohomology Steps 4 & 5 give

$$H^*(X;R) \otimes H^*(Y;R) \to H_*\big(S^\bullet(X;R) \otimes S^\bullet(Y;R)\big) \xrightarrow{(3)} H^*(X\times Y;R)$$

$$\xi \otimes \eta \longmapsto \xi \times \eta$$ (cross product)

$$H^*(X;R) \otimes H^*(X;R) \longrightarrow H^*(X\times X;R) \xrightarrow{\Delta^*} H^*(X;R)$$

$$\xi \otimes \eta \longmapsto \xi \smile \eta$$ (cup product)

$$\xi \smile \eta = \Delta^*(\xi \times \eta)$$

So given

$$\xi \in H^p(X;R),\ \eta \in H^q(Y;R) \rightsquigarrow \xi \times \eta \in H^{p+q}(X\times Y;R)$$

$$\xi \in H^p(X;R),\ \eta \in H^q(X;R) \rightsquigarrow \xi \cup \eta \in H^{p+q}(X;R)$$

Note that since any two choices of maps in Step 2 are chain htpc, these products in cohomology are independent of this choice.

<u>Example:</u> $H^*(S^2 \vee S^4) \cong H^*(\mathbb{C}P^2)$ as abelian groups (both $\mathbb{Z}$ in deg. 0, 2, 4, 0 o/w)

x gen. in deg. 2 then $x^2 = x \cup x = \begin{cases} 0 & \text{in } S^2 \vee S^4 \\ \text{gen.} & \text{in } \mathbb{C}P^2 \end{cases}$

Eilenberg-Zilber Thm.: $S_\cdot(X\times Y) \to S_\cdot X \otimes S_\cdot Y$ chain ht.py eq.ce, unique up to chain ht.py eq.ce

One particular choice: Alexander-Whitney map:

$(\sigma,\tau): \Delta^n \to X\times Y$, $\quad aw(\sigma,\tau) = \sum_{p+q=n} \sigma\circ\alpha^n_p \otimes \tau\circ\omega^n_q$

$$= (\sigma_*\otimes\tau_*)\Big(\sum_{p+q=n}\alpha^n_p\otimes\omega^n_q\Big)$$

$\alpha^n_p : \Delta^p \hookrightarrow \Delta^n$ – vertices $0,1,\dots,p$

$\omega^n_q : \Delta^q \hookrightarrow \Delta^n$ – vertices $n-q,\dots,n-1,n$

Cross and cup product

Cross product: (R a ring)

$$S^\bullet(X;R)\otimes S^\bullet(Y;R) \to \mathrm{Hom}(S_\cdot X\otimes S_\cdot Y, R\otimes R) \xrightarrow{\text{compose w/ E-Z \& mult. in } R} \mathrm{Hom}(S_\cdot(X\times Y), R)$$

$S^\bullet(X;R) = \mathrm{Hom}(SX,R)$, $\quad S^\bullet(Y;R) = \mathrm{Hom}(SY,R)$, $\quad \mathrm{Hom}(S_\cdot(X\times Y),R) = S^\bullet(X\times Y;R)$

$\bigoplus_{p+q=n} S^p(X;R)\otimes S^q(Y;R)$ $\qquad \mathrm{Hom}\big(\bigoplus_{p+q=n} S_pX\otimes S_qY, R\otimes R\big)$

Cup product: $S^\cdot(X;R)\otimes S^\cdot(X;R) \to S^\cdot(X\times X;R) \xrightarrow{\Delta^*} S^\cdot(X;R)$

More explicitly:

$\varphi \in S^p(X;R) = R^{\mathrm{Sing}_p X}$ $\rightsquigarrow$ $n = p+q$

$\psi \in S^q(Y;R) = R^{\mathrm{Sing}_q Y}$ $\quad \varphi \times \psi \in S^n(X \times Y; R) = R^{\mathrm{Sing}_n(X\times Y)}$

$$\varphi \times \psi\,(\sigma,\tau) = \varphi(\underbrace{\sigma \circ \alpha^n_p}_{\sigma|_{0,1,\dots,p}}) \cdot \psi(\underbrace{\tau \circ \omega^n_q}_{\tau|_{p,p+1,\dots,n}})$$

multiply in R

$X = Y$:

$$(\varphi \cup \psi)(\sigma) = \varphi \times \psi\,(\Delta \circ \sigma) = \varphi \times \psi\,(\sigma,\sigma)$$

$$= \varphi(\underbrace{\sigma \circ \alpha^n_p}_{\sigma|_{0,1,\dots,p}}) \cdot \psi(\underbrace{\sigma \circ \omega^n_q}_{\sigma|_{p,p+1,\dots,n}})$$

This induces $H^p(X;R) \times H^q(Y;R) \xrightarrow{\times} H^{p+q}(X \times Y; R)$

$$H^p(X;R) \times H^q(X;R) \xrightarrow{\cup} H^{p+q}(X;R)$$

These maps are well-defined, indep. of choice of E-Z map (as chain htpc. maps agree on H_*)

<u>Note</u>: These products are preserved by maps in cohomology from cts. maps, e.g.

$$f: X \to X' \rightsquigarrow f^*([a] \cup [b]) = f^*[a] \cup f^*[b]$$

in $H^* X'$ in $H^* X$

<u>Remark</u>: $\exists$ a relative version of products, but this some technical assumptions on subspaces.

Example: In $H^* S^n$, cup products are trivial for degree reasons

- If $x \in H^n S^n$ $x^2 = x \cup x = 0$ because $H^{2n} S^n = 0$

In $H^*(S^n \times S^m)$ there is a non-trivial $\cup$:

$x \in H^n$, $y \in H^m$ generators then $x \cup y$ is a generator in H^{n+m}.

Exercise: On $H^0(X;R) \cong R^{\pi_0 X}$, $\cup$ is given by pointwise mult. in R.

Also $H^0(X;R) \times H^n(X;R) \xrightarrow{\cup} H^n(X;R)$

$$X = \coprod_{\alpha \in \pi_0 X} X_\alpha \qquad H^n(X;R) \cong \prod_\alpha H^n(X_\alpha;R)$$

and $(f \cup \xi)_\alpha = f(\alpha) \cdot \xi_\alpha$

Graded rings

We'll see: $\smile$ makes $H^*(X;R)$ a graded (comm.) ring when R is a (comm.) ring.

Start w/ diagrammatic descr. of rings:

$R \in Ab$, $\quad m: R\otimes R \to R$, $\quad u: \mathbb{Z} \to R$

multiplication $\qquad$ $u(1)$ = unit in R

s.t. the following diagrams commute

(A)
$$\begin{tikzcd}
R\otimes R\otimes R \arrow[r, "m\otimes id"', "(*)"] \arrow[d, "id\otimes m"] & R\otimes R \arrow[d, "m"] \\
R\otimes R \arrow[r, "m"] & R
\end{tikzcd}$$

(U)
$$\begin{tikzcd}
\mathbb{Z}\otimes R \arrow[d, "u\otimes id"'] \arrow[dr, "\sim"] & & R\otimes \mathbb{Z} \arrow[dl, "\sim"'] \arrow[d, "id\otimes u"] \\
R\otimes R \arrow[r, "m"] & R & R\otimes R \arrow[l, "m"']
\end{tikzcd}$$

(*) really uses $R\otimes(R\otimes R) \cong (R\otimes R)\otimes R$

commutative if also the following commutes:

(C)
$$\begin{tikzcd}
R\otimes R \arrow[d, "\tau"', "\wr"] \arrow[dr, "m"] & \\
R\otimes R \arrow[r, "m"'] & R
\end{tikzcd}$$

isomor. that swaps factors in $\otimes$

Same defn. applies in any cat. w/ $\otimes$ & a symmetry

In grAb: A <u>graded ring</u> is $R_* \in \mathrm{grAb}$ w/ $m: R_*\otimes R_* \to R_*$,
$\eta: \mathbb{Z}[0] \to R_*$ s.t. analogues of (A), (U) commute

- R_* is commutative if also (C) commutes.

Explicitly we have: bilinear maps $R_p \times R_q \to R_{p+q}$,

$e \in R_0$ $(e = u(1))$ s.t.

$$e \cdot x = x = x \cdot e$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

Symmetry isomor. on grAb has a sign: $A_* \otimes B_* \xrightarrow{\sim} B_* \otimes A_*$

$$a \otimes b \longmapsto (-1)^{pq} b \otimes a$$

$a \in A_p$, $b \in B_q$

$\Rightarrow$ Graded commutativity means: $x \in R_p$, $y \in R_q$, $x \cdot y = (-1)^{pq} y \cdot x$

In Ch:

A <u>differential graded ring</u> (or dg-ring) is $R_\bullet \in Ch$,

chain maps $m: R_\bullet \otimes R_\bullet \to R_\bullet$, $u: \mathbb{Z}[0] \to R_\bullet$

s.t. (A), (U) commute, <u>commutative</u> if (C) commutes

Explicitly: graded ring str. s.t. e is a cycle ($\partial e = 0$)

($e = u(1)$, $\partial e = u(\partial 1) = 0$)

and $\partial(a \cdot b) = (\partial a) \cdot b + (-1)^p a \cdot \partial b$ (Leibniz formula)

$a \in R_p$, $b \in R_q$

A (comm.) dg. ring str. on $R_\bullet$ $\rightsquigarrow$ (comm.) graded ring str. on $H_* R$

w/ mult. $H_* R \otimes H_* R \to H_*(R \otimes R) \longrightarrow H_* R$

or $[x], [y] \rightsquigarrow [x] \cdot [y] = [x \cdot y]$

$x \in R_p$, $y \in R_q$ cycles

- well-defined since $(x+\partial x')\cdot y = x\cdot y + \underbrace{\partial x'\cdot y}_{\shortparallel}$

$\partial(x'\cdot y)$ as $\partial y = 0$

To get a ring str. on $H_* R$ something weaker than a dg-ring str. is sufficient:

<u>Defn.</u>: A <u>homotopy (comm) dg-ring</u> is a ch. cx. $R_\cdot$ + $m: R_\cdot \otimes R_\cdot \to R_\cdot$ + unit $u: \mathbb{Z}[0] \to R$ s.t. (A), (U) ((C)) commute up to chain ht.pys, i.e. $\exists$ chain ht.pies between composites.

Since chain ht.pic maps are equal on H_*, a homotopy (comm.) dg-ring $R_\cdot$ $\rightsquigarrow$ graded (comm.) ring $H_* R$.

Homotopy Ring Structures on Cochains

Two approaches

- for any E-Z we can show (via acyclic models) that there is a natural homotopy (comm.) dg-ring str. on $S^{\bullet}(X;R)$, R (comm.) ring

- use A-W map, then $S^{\bullet}(X;R)$ is a dg-ring (in the strict sense), but if R is comm. we only have commutativity up to chain ht.py

In fact, it's impossible to find a natural comm. dg-ring str. on $S^{\bullet}(X)$

- can use the "failure" of commutativity to get more str. on $H^{*}(X)$
- cohomology operations (see AlgTop 2).

<u>Propn.</u>: For E–Z maps $\varphi^{X,Y} : S_\bullet(X\times Y) \to S_\bullet X \otimes S_\bullet Y$ we have

(i) $\forall\ X, Y, Z \in \mathrm{Top}$,

$$\begin{tikzcd}
S_\bullet(X\times Y\times Z) \arrow[r] \arrow[d] & S_\bullet(X\times Y)\otimes S_\bullet Z \arrow[d] \\
S_\bullet X\otimes S_\bullet(Y\times Z) \arrow[r] & S_\bullet X\otimes S_\bullet Y\otimes S_\bullet Z
\end{tikzcd}$$

commutes up to a natural chain ht.py

$u : S_\bullet(*) \to \mathbb{Z}[0]$

(id in deg. 0, 0 in deg. >0)

(ii)
$$\begin{tikzcd}
S_\bullet(X\times *) \arrow[r] \arrow[dr, "\sim"'] & S_\bullet X\otimes S_\bullet(*) \arrow[r, "\mathrm{id}\otimes u"] & S_\bullet X\otimes \mathbb{Z}[0] \arrow[dl, "\sim"] \\
 & S_\bullet X &
\end{tikzcd}$$

commutes up to a natural chain ht.py (same w/ $*\times X$)

(iii)

$$\begin{tikzcd} S_\bullet(X\times Y) \arrow[r] \arrow[d, "t_*"'] & S_\bullet X\otimes S_\bullet Y \arrow[d, "\tau"] \\ S_\bullet(Y\times X) \arrow[r] & S_\bullet Y\otimes S_\bullet X \end{tikzcd}$$

$t : X\times Y \xrightarrow{\sim} Y\times X$

commutes up to a natural chain ht.py

<u>Proof:</u> Acyclic models. ... $\square$

Apply $\mathrm{Hom}(-,R)$ & combine w/ $S^\bullet(X;R)\otimes S^\bullet(Y;R)\to \mathrm{Hom}(S_\bullet X\otimes S_\bullet Y, R)$:

<u>Cor.</u> (Take $R=\mathbb{Z}$ to simplify notation)

(i)

$$\begin{tikzcd} S^\bullet X\otimes S^\bullet Y\otimes S^\bullet Z \arrow[r] \arrow[d] & S^\bullet(X\times Y)\otimes S^\bullet Z \arrow[d] \\ S^\bullet X\otimes S^\bullet(Y\times Z) \arrow[r] & S^\bullet(X\times Y\times Z) \end{tikzcd}$$

commutes up to a natural chain ht.py

(Assoc. for cross product)

(ii)

$$\begin{tikzcd}
S^\bullet X \otimes \mathbb{Z}[0] \arrow[d] \arrow[drr, "\sim"] & & \\
S^\bullet X \otimes S^\bullet(*) \arrow[r] & S^\bullet(X\times *) \arrow[r, "\sim"] & S^\bullet X
\end{tikzcd}$$

commutes up to a natural chain htpy (and same for $\mathbb{Z}[0]\otimes S^\bullet X$)

(iii)

$$\begin{tikzcd}
S^\bullet X \otimes S^\bullet Y \arrow[r] \arrow[d, "\tau"] & S^\bullet(X\times Y) \arrow[d, "t^*"] \\
S^\bullet Y \otimes S^\bullet X \arrow[r] & S^\bullet(Y\times X)
\end{tikzcd}$$

commutes up to a natural chain htpy

(In cohomology, get $\xi\in H^pX$, $\eta\in H^qY$, $\zeta\in H^rZ$, $1\in H^0(*)=\mathbb{Z}$

$\xi\times(\eta\times\zeta) = (\xi\times\eta)\times\zeta$ in $H^{p+q+r}(X\times Y\times Z)$

$\xi\times 1 = \xi = 1\times\xi$ in H^pX, $t^*(\xi\times\eta) = (-1)^{pq}\,\eta\times\xi$ in $H^{p+q}(Y\times X)$

Cor.: $S^\bullet X$ has a natural homotopy comm. dg-ring str.

Proof:

(A)

$$
\begin{tikzcd}
S^\bullet X \otimes S^\bullet X \otimes S^\bullet X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(From previous Cor.)}"] & S^\bullet(X\times X) \otimes S^\bullet X \arrow[r, "\Delta^* \otimes id"] \arrow[d] \arrow[dr, phantom, "\text{(E-2 map natural)}"] & S^\bullet X \otimes S^\bullet X \arrow[d] \\
S^\bullet X \otimes S^\bullet(X\times X) \arrow[r] \arrow[d, "id \otimes \Delta^*"] \arrow[dr, phantom, "\text{(E-2 map natural)}"] & S^\bullet(X\times X\times X) \arrow[r, "(\Delta\times id)^*"] \arrow[d, "(id\times\Delta)^*"] & S^\bullet(X\times X) \arrow[d, "\Delta^*"] \\
S^\bullet X \otimes S^\bullet X \arrow[r] & S^\bullet(X\times X) \arrow[r, "\Delta^*"'] & S^\bullet X
\end{tikzcd}
$$

from

$$
\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "="] & X\times X \arrow[d, "id\times\Delta"] \\
X\times X \arrow[r, "\Delta\times id"'] & X\times X\times X
\end{tikzcd}
$$

(v)

$$\begin{tikzcd}
S^\bullet X \otimes \mathbb{Z}[0] \arrow[d] \arrow[dr, "\sim"] & & \\
S^\bullet X \otimes S^\bullet(*) \arrow[r] \arrow[d] & S^\bullet(X \times *) \arrow[d] \arrow[dr, "\sim"] & \\
S^\bullet X \otimes S^\bullet X \arrow[r] & S^\bullet(X \times X) \arrow[r] & S^\bullet X
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[dr, "\sim"'] & X \times X \arrow[d] \\
& X \times *
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
S^\bullet X \otimes S^\bullet X \arrow[r] \arrow[d, "\tau"'] & S^\bullet(X \times X) \arrow[d, "t^*"] \arrow[dr, "\Delta^*"] & \\
S^\bullet X \otimes S^\bullet X \arrow[r] & S^\bullet(X \times X) \arrow[r, "\Delta^*"'] & S^\bullet X
\end{tikzcd}$$

$$\begin{tikzcd}
& X \times X \arrow[d, "t"] \\
X \arrow[ur, "\Delta"] \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

$\square$

Cor.: R a (comm.) ring, then $H^*(X;R)$ has a natural graded (comm.) ring str. given by $\cup$.

Defn.: R comm. ring. The graded polynomial ring $R[x_1, \dots, x_n]$ w/ x_i in deg. d_i has in deg m

$$R[x_1, \dots, x_n]_m = \text{free } R\text{-module on gen.s } x_1^{i_1} \cdots x_n^{i_n}$$

w/ $i_1 d_1 + \dots + i_n d_n = m$

mult. "as expected" but $x_i x_j = (-1)^{d_i d_j} x_j x_i$

If all gen.s are in even degrees, or $R = \mathbb{F}_2$ $(\mathbb{Z}/2)$ this is just a regular polynomial ring if we forget grading.

Thm.: We have isomors of graded comm. rings

$$H^*(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2[x]/x^{n+1}, \quad H^*(\mathbb{RP}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[x]$$

deg. $x = 1$

$$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[y]/y^{n+1} \quad H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[y]$$

deg. $y = 2$

Remark: For $\mathbb{RP}^n$, $H^i(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{Z}/2$ for $0 \leq i \leq n$
$= \{0, x_i\}$

so ring str. amounts to $x_i \smile x_j \neq 0$ if $i+j \leq n$
(so x_{i+j})

For $\mathbb{CP}^n$: $y_i \in H^{2i}(\mathbb{CP}^n)$ generator, $y_i \smile y_j$ is a generator in H^{2i+2j} for $i+j \leq n$.