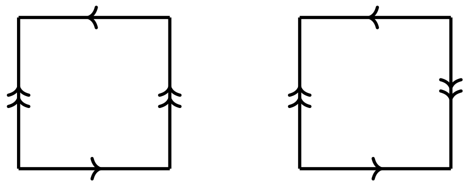
Exercises for Algebraic Topology I

November 26, 2020

1 *Introduction*

Exercise 1.1. We computed the homology of the torus by thinking of it as built from a square by gluing opposite edges, and triangulating this by cutting it into two triangles along the diagonal. Here are two other (non-orientable!) surfaces we can build by identifying opposite sides of a square, but now with a twist in either one or both directions:



The corresponding spaces are the Klein bottle and the real projective plane $\mathbb{RP}^2$, respectively. Triangulate these too by adding a diagonal and picking orientations, and compute the homology groups. [You should find that homology groups are not always *free* abelian groups.]

Exercise 1.2. The *Euler characteristic* of a triangulated surface is

$$\chi := V - E + F.$$

(i) Show that $\chi = h_0 - h_1 + h_2$ where h_i is the rank of the abelian group $H_i(\Sigma)$. Conclude that the Euler characteristic is a topological invariant. [Hint: For abelian groups $B \subseteq A$ the rank of A/B is given by $\operatorname{rk} A/B = \operatorname{rk} A - \operatorname{rk} B$. You will also need to write the boundary groups $B_i(\Sigma)$ as quotients.]

(ii) Conclude that for any way of covering the oriented surface of genus g by polygons we must have

$$V - E + F = 2 - 2g.$$

[Hint: Subdivide the polygons into triangles.]

(iii)* In particular, any convex polyhedron must satisfy *Euler's formula*,

$$V - E + F = 2.$$

Use this to classify the Platonic solids. [Hint: first observe that we have $pF = 2E = qV$ if the faces have p edges and q edges meet at each vertex, and show that $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$; since p and q are integers ≥ 3 there are not many possibilities.]

2 *Some Basic Topology and Category Theory*

Exercise 2.1. Let X be a topological space and S a set. Show that if we equip S with the discrete topology then any function $S \to X$ is continuous, and if we equip S with the indiscrete topology then any function $X \to S$ is continuous.

Exercise 2.2. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that a function $f\colon X \to Y$ is continuous if and only if for every $x \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \epsilon$. (Note that δ may depend on x.)

Exercise 2.3. Prove the following basic properties of isomorphisms in a category $\mathcal{C}$:

(i) If $f: x \to y$ and $g: y \to z$ are isomorphisms, so is $gf: x \to z$.

(ii) Given $f: x \to y$, if there exist $g, h: y \to x$ such that

$$gf = \mathrm{id}_x, \quad fh = \mathrm{id}_y,$$

then f is an isomorphism.

(iii) If f is an isomorphism, its inverse is unique.

(iv) If $F: \mathcal{C} \to \mathcal{D}$ is a functor and $f: x \to y$ is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.

(v) Being isomorphic is an equivalence relation on objects of $\mathcal{C}$.

Exercise 2.4. Let X be a topological space and $U \subseteq X$ a subset. Show that the subspace topology on U has the following universal property: if T is a topological space, then a continuous map from T to U is a map of sets $T \to U$ such that the composite $T \to U \hookrightarrow X$ is continuous.

Exercise 2.5. Show that there is a functor $\mathsf{Top} \to \mathsf{Set}$ that takes a topological space to its underlying set (the "forgetful" functor) and two functors $\mathsf{Set} \to \mathsf{Top}$ that take a set to itself equipped with the discrete and indiscrete topologies, respectively.

Exercise 2.6. Let x, y, z be objects of a category $\mathcal{C}$. Show that there is a canonical isomorphism

$$x \times (y \times z) \cong (x \times y) \times z,$$

provided these products exist.

Exercise 2.7. Show that the cartesian product of (abelian) groups is also the categorical product in Grp and Ab, when equipped with the canonical group structure.

Exercise 2.8 (∗)**.** Given a set I and a collection x_i $(i \in I)$ of objects of a category $\mathcal{C}$, their product (if it exists) is an object $\prod_{i \in I} x_i$ together with projections $\pi_i: \prod_{i \in I} x_i \to x_i$ satisfying the following universal property: given an object y and morphisms $f_i: y \to x_i$ for $i \in I$, there exists a unique morphism $f: y \to \prod_{i \in I} x_i$ such that $\pi_i f = f_i$. Show that I-indexed cartesian products are categorical products in the category Set, and also in the categories $\mathsf{Ab}, \mathsf{Grp}, \mathsf{Top}$ when equipped with canonical (abelian) group structures and topologies. (What is an I-indexed product when I is empty?)

Exercise 2.9. Show that the coproduct in Top of topological spaces X, Y is the disjoint union $X \amalg Y$ of sets, with a subset $U \subseteq X \amalg Y$ defined to be open if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y.

Exercise 2.10 (∗)**.** What is the coproduct of two copies of $\mathbb{Z}$ in Grp?

Exercise 2.11 (∗)**.** Define I-indexed coproducts for any indexing set I, as in Exercise 2.8. Describe these in the categories Set and Top.

Exercise 2.12 (∗)**.** If $\mathcal{C}$ is a category, we define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ to be the category with the same objects as $\mathcal{C}$, but with the direction of morphisms reversed — thus $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) := \mathrm{Hom}_{\mathcal{C}}(y, x)$. Check that a coproduct in $\mathcal{C}$ is the same thing as a product in $\mathcal{C}^{\mathrm{op}}$.

Exercise 2.13 ($*$). Suppose a topological space X can be written as a union of subsets X_i ($i \in I$) such that the subsets X_i are open and disjoint. Show that $X \cong \coprod_{i \in I} X_i$ (i.e. the topology on X is the coproduct topology).

Exercise 2.14. Suppose R is a relation on a set I. Show that the quotient $I/R = I/\overline{R}$ together with the quotient map

$$\pi \colon I \to I/R, \quad \pi(i) = [i]_{\overline{R}}$$

has the following universal property: any function $f \colon I \to J$ for which $i \sim_R j$ implies $f(i) = f(j)$ factors uniquely through π,

$$\begin{tikzcd} I \arrow[r, "\pi"] \arrow[dr, "f"'] & I/R \arrow[d, dashed, "\exists! \overline{f}"] \\ & J. \end{tikzcd}$$

Exercise 2.15. Let $D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the closed n-disk and

$$\partial D^n := S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$$

be the $(n-1)$-sphere, both equipped with the subspace topology from $\mathbb{R}^n$.

(i) Find explicit homeomorphisms $D^1/\partial D^1 \cong S^1$ and $D^2/\partial D^2 \cong S^2$. [Feel free to use that these are compact Hausdorff spaces, so that a continuous bijection is necessarily a homeomorphism.]

(ii) Show that the following three descriptions of the torus are homeomorphic:

$$\begin{aligned} T_1 &:= \{(x,y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}/((x,0) \sim (x,1), (0,y) \sim (1,y)) \\ T_2 &:= S^1 \times S^1 \\ T_3 &:= \{(R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)\} \subseteq \mathbb{R}^3 \quad (R > r) \end{aligned}$$

(iii)* Find an explicit homeomorphism $D^n/\partial D^n \cong S^n$.

Exercise 2.16. Show that being homotopic is an equivalence relation on the set $C(X, Y)$ of continuous maps $X \to Y$.

Exercise 2.17. Prove that $h\mathsf{Top}$ is a well-defined category, with composition and identities induced from Top (so that there is a functor $\mathsf{Top} \to h\mathsf{Top}$ that takes each continuous map to its equivalence class). What does Exercise 2.3 then tell you about homotopy equivalences?

Exercise 2.18. Let S be a set. Show that:

(i) if S is equipped with the discrete topology then S is contractible if and only if S has exactly one element,

(ii) if S is equipped with the indiscrete topology then S is contractible if and only if S is non-empty.

[Hint: Prove that with the discrete topology the only continuous paths are the constant ones, while any path is continuous for the indiscrete topology.]

Exercise 2.19 ($**$). (A topological proof that S^1 is not contractible.) View S^1 as $\{z \in \mathbb{C} : |z| = 1\}$ and let $\pi \colon \mathbb{R} \to S^1$ be the continuous map $x \mapsto e^{ix}$. We say that a continuous map $f \colon S^1 \to S^1$ *lifts to* $\mathbb{R}$ if there exists $\bar{f} \colon S^1 \to \mathbb{R}$ such that $f = \pi \bar{f}$.

(i) Show that if $g\colon S^1 \to S^1$ lifts to $\mathbb{R}$ and $f\colon S^1 \to S^1$ is another continuous map such that $f(x)/g(x) \neq -1$ for all $x \in S^1$ then f also lifts to $\mathbb{R}$.

(ii) Let $c_1\colon S^1 \to S^1$ be the constant map with value 1, and suppose f is homotopic to c_1, via a homotopy $H\colon S^1 \times [0,1] \to S^1$. Since $S^1 \times [0,1]$ is compact, we can choose $\delta > 0$ such if $|x-y| < \delta$ then $|H(x) - H(y)| < 2$ for all $x, y \in S^1 \times [0,1]$ (viewed as a subset of $\mathbb{R}^3$). Use this to show that f lifts to $\mathbb{R}$.

(iii) Use (ii) to prove that S^1 is not contractible (i.e. id_{S^1} is not homotopic to a constant map).

Exercise 2.20. Let X and Y be topological spaces.

(i) Show that any continuous map $f\colon X \to Y$ induces a function $\pi_0 f\colon \pi_0 X \to \pi_0 Y$, and that this makes π_0 a functor $\mathsf{Top} \to \mathsf{Set}$.

(ii) Show that if $f, g\colon X \to Y$ are homotopic, then $\pi_0 f = \pi_0 g$. [Hence π_0 is a functor $h\mathsf{Top} \to \mathsf{Set}$.]

(iii) Show that if $f\colon X \to Y$ is a homotopy equivalence, then $\pi_0 f$ is an isomorphism.

3 *Simplices and Singular Homology*

Exercise 3.1. If V is a vector space over a field k, consider the linear map $\eta_V\colon V \to V^{**}$ to the double dual, taking $v \in V$ to the linear functional

$$\eta_V(v)\colon V^* \to k, \quad \phi \mapsto \phi(v).$$

Prove that these maps are natural, i.e. they determine a natural transformation η of functors $\mathsf{Vect}_k \to \mathsf{Vect}_k$ from the identity to the double dual. Show that if we restrict to finite-dimensional vector spaces η becomes a natural isomorphism.

Exercise 3.2. Let $A_i, i \in I$ be a collection of abelian groups indexed by a set I, and define the inclusion $I_j\colon A_j \to \bigoplus_{i\in I} A_i$ by $I_j(a) = (a_i)_{i\in I}$ where $a_j = a$ and $a_i = 0$ otherwise. Show that the I_j's exhibit the direct sum $\bigoplus_{i\in I} A_i$ as the I-indexed coproduct in Ab, i.e. given homomorphisms $\phi_j\colon A_j \to B$ there exists a unique homomorphism $\phi\colon \bigoplus_{i\in I} A_i \to B$ with $\phi_j = \phi \circ I_j$.

Exercise 3.3. Given sets T_i, $i \in I$, show that there is a natural isomorphism

$$\mathbb{Z}\left(\coprod_i T_i\right) \cong \bigoplus_i \mathbb{Z}T_i.$$

[Hint: We can view the left-hand side as consisting of functions $f\colon \coprod_i T_i \to \mathbb{Z}$ that are 0 except at finitely many elements, while the right-hand side consists of a family of functions $f_i\colon T_i \to \mathbb{Z}$ that are all zero except at finitely many elements, and such that $f_i = 0$ except for finitely many indices i.]

Exercise 3.4 (Direct sums commute with quotients in Ab)**.** Show that given abelian groups A_i with subgroups $B_i \subseteq A_i$ for $i \in I$, there is a canonical isomorphism

$$\bigoplus_{i\in I} A_i/B_i \cong \left(\bigoplus_{i\in I} A_i\right) / \left(\bigoplus_{i\in I} B_i\right).$$

[Hint: Show that homomorphism $\bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} A_i/B_i$ (defined as the sum of the quotient maps) exhibits the target as the quotient by $\bigoplus_{i\in I} B_i$, by checking it satisfies the universal property of the quotient.]

4 Relative Homology and Long Exact Sequences

Exercise 4.1. Prove the 5-Lemma.

Exercise 4.2. Suppose we have an exact sequence

$$(\cdots)A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E(\cdots).$$

Show that there is a short exact sequence

$$0 \to \operatorname{coker} f \to C \to \ker i \to 0,$$

where the *cokernel* $\operatorname{coker} f$ is the quotient $B/\operatorname{im} f$. (Thus we can in a sense "decompose" a long exact sequence into a series of short exact sequences.)

Exercise 4.3. Let $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$ be a short exact sequence (SES). A *splitting* of the SES is a section $s\colon C \to B$, so that $qs = \mathrm{id}_C$. (The SES is *splittable* if a splitting exists, while a *split* SES is a SES together with a choice of splitting.)

(i) Show that a splitting s induces an isomorphism $A \oplus C \xrightarrow{\sim} B$. [Note that different splittings can give different isomorphisms.]

(ii) Show that if C is a free abelian group then the SES above is splittable. [Hint: Use the universal property of free abelian groups.]

(iii) Give an example of a SES that is not splittable.

Exercise 4.4. Given a commutative diagram of chain complexes and chain maps

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A_\bullet \arrow[r] \arrow[d] & B_\bullet \arrow[r] \arrow[d] & C_\bullet \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
0 \arrow[r] & A'_\bullet \arrow[r] & B'_\bullet \arrow[r] & C'_\bullet \arrow[r] & 0,
\end{tikzcd}$$

where the rows are exact, check that the boundary map on homology gives commutative squares

$$\begin{tikzcd}
H_n(C) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A) \arrow[d] \\
H_n(C') \arrow[r, "\partial"] & H_{n-1}(A').
\end{tikzcd}$$

Exercise 4.5. Suppose $A \subseteq B$ are subspaces of a topological space X, and the inclusion $i\colon A \hookrightarrow B$ induces isomorphisms $i_*\colon H_n(A) \xrightarrow{\sim} H_n(B)$ for all n. Prove that the natural homomorphism $H_n(X, A) \to H_n(X, B)$ is an isomorphism for all n. [Hint: Use the 5-Lemma and Exercise 4.4.]

Exercise 4.6. Let $0 \to A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{q_\bullet} C_\bullet \to 0$ be a short exact sequence of chain complexes. Show that the induced sequence of homology groups

$$\cdots \to H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(q)} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

is a long exact sequence.

Exercise 4.7. Suppose h_* is an ordinary homology theory satisfying the Eilenberg–Steenrod axioms.

(i) Suppose $X = \coprod_{i\in I} X_i$ is a coproduct and $A_i \subseteq X_i$ is a collection of subspaces. If $A := \coprod_{i\in I} A_i$, show that the inclusions $(X_i, A_i) \hookrightarrow (X, A)$ induce an isomorphism
$$\bigoplus_{i\in I} h_*(X_i, A_i) \cong h_*(X, A).$$
[Hint: Use the long exact sequence.]

(ii) If (X_i, x_i) is a collection of pointed spaces, their *wedge* is the quotient space
$$\bigvee_{i\in I} X_i := \left(\coprod_{i\in I} X_i\right) / \{x_i : i \in I\}$$
where we identify all the base points to a single point x. Show that if $(X_i, \{x_i\})$ is a good pair for every i then there is a canonical isomorphism
$$\bigoplus_{i\in I} \tilde{h}_*(X_i) \cong \tilde{h}_*(\bigvee_{i\in I} X_i),$$
where for a pointed space (X, x) we write $\tilde{h}_*(X) := h_*(X, x)$.

Exercise 4.8. Use the Mayer–Vietoris sequence to compute the homology of the orientable surface Σ_g of genus g. [Hint: Find a way to induct on g.]

Exercise 4.9. Think of $\mathbb{RP}^2$ as the quotient of D^2 where we identify x with $-x$ for $x \in \partial D^2$. Compute the homology of $\mathbb{RP}^2$ using the Mayer–Vietoris sequence with $A =$ a neighbourhood of the image of ∂D^2 and $B =$ the image of a smaller disc inside D^2. [Assume that the map $S^1 \to S^1$ that loops around twice is given on $H_1(S^1)$ by multiplication by 2.]

Exercise 4.10. The *cone* on a topological space X is the quotient $(X \times [0,1])/(X \times \{0\})$, and the *suspension* ΣX of X is the quotient of $(X \times [0,1])$ where we collapse $X \times \{0\}$ to a point and $X \times \{1\}$ to a different point.

(i) Show that CX is contractible for any X, and that ΣX is the union of two copies of CX with intersection X.

(ii) Use the Mayer–Vietoris sequence to show that $H_n(\Sigma X) \cong H_{n-1}(X)$ for $n > 1$.

(iii) By looking at what happens at the bottom of the Mayer–Vietoris sequence, show that $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$ for all n.

(iv) If $X = S^n$, convince yourself that ΣS^n is homeomorphic to S^{n+1}. [Think of the two cones as two "hemispheres" glued along the "equator".] Use (iii) to compute $\tilde{H}_*(S^n)$ again.

5 *Cellular Homology*

Exercise 5.1. Show that in the category Ab of abelian groups, the pushout of two homomorphisms $f\colon A \to B$, $g\colon A \to C$ is the cokernel of the homomorphism $(f, -g)\colon A \to B \oplus C$.

Exercise 5.2. Consider a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A'' \arrow[d] \\ B \arrow[r] & B' \arrow[r] & B'', \end{tikzcd}$$

in a category $\mathcal{C}$. If the left square is a pushout, then the right square is a pushout if and only if the outer (composite) square is a pushout.

Exercise 5.3 (Pushouts commute with coproducts)**.** Suppose we have pushout squares

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & B_i \arrow[d] \\ C_i \arrow[r] & D_i \end{tikzcd}$$

for $i \in I$ in some category $\mathcal{C}$. If I-indexed coproducts exist in $\mathcal{C}$, then the canonical square

$$\begin{tikzcd} \coprod_{i\in I} A_i \arrow[r] \arrow[d] & \coprod_{i\in I} B_i \arrow[d] \\ \coprod_{i\in I} C_i \arrow[r] & \coprod_{i\in I} D_i \end{tikzcd}$$

is also a pushout. [Hint: Use the universal properties.]

Exercise 5.4. Show that a morphism of Δ-sets $f\colon S \to T$ induces a canonical continuous map $|f|\colon |S| \to |T|$ between geometric realizations such that for every $\sigma \in S_n$ the triangle

$$\begin{tikzcd} & \Delta^n \arrow[dl, "e_\sigma"'] \arrow[dr, "e_{f(\sigma)}"] & \\ {|S|} \arrow[rr, "|f|"] & & {|T|} \end{tikzcd}$$

commutes. Check that this makes $|{-}|$ a functor $\mathsf{Set}_\Delta \to \mathsf{Top}$.

Exercise 5.5. The *combinatorial n-simplex* is the Δ-set Δ^n_{comb} with $(\Delta^n_{\text{comb}})_k$ being the set of subsets of $\{0, \dots, n\}$ of size $k+1$; it is convenient to label these as $[i_0 \cdots i_k]$ with $0 \le i_0 \le i_1 \le \cdots \le i_k \le n$. Then the face map $\partial_j\colon (\Delta^n_{\text{comb}})_k \to (\Delta^n_{\text{comb}})_{k-1}$ is given by

$$[i_0 \cdots i_k] \mapsto [i_0 \cdots i_{j-1} i_{j+1} \cdots i_k].$$

The *boundary* of Δ^n_{comb} is the Δ-set $\partial\Delta^n_{\text{comb}}$ obtained by removing the single n-simplex $[01 \cdots n]$, so that

$$(\partial\Delta^n_{\text{comb}})_k = \begin{cases} (\Delta^n_{\text{comb}})_k, & 0 \le k \le n-1, \\ \emptyset, & k \ge n, \end{cases}$$

with the same face maps in degrees $< n$.

(i) Convince yourself that $|\Delta^n_{\text{comb}}|$ is homeomorphic to Δ^n and $|\partial\Delta^n_{\text{comb}}|$ to the boundary of Δ^n.

(ii) Compute the simplicial homology of the Δ-sets Δ^3_{comb} and $\partial\Delta^3_{\text{comb}}$. [The space $|\partial\Delta^3_{\text{comb}}|$ is a tetrahedron, which is topologically a sphere, so the result should agree with the usual homology of the sphere.]

Exercise 5.6.

(i) Suppose we have subsets

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots .$$

Show that the union $\bigcup_{n=0}^\infty S_n$ is isomorphic to the sequential colimit of the inclusions $S_n \hookrightarrow S_{n+1}$.

(ii) Suppose we have subspaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots .$$

Show that the union $\bigcup_{n=0}^{\infty} X_n$ is homeomorphic to the sequential colimit of the continuous inclusions $X_n \hookrightarrow X_{n+1}$ if we give the union the topology where a subspace $U \subseteq \bigcup_{n=0}^{\infty} X_n$ is open if and only if $U \cap X_n$ is open in X_n for all n.

(iii) Also check the analogous statement for abelian groups.

Exercise 5.7. Suppose we have a sequence

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{\cdots}$$

of morphisms $f_i \colon X_i \to X_{i+1}$ in a category $\mathcal{C}$ such that f_i is an isomorphism for all $i \geq N$. Show that then X_N is a colimit of the sequence. [Hint: We have compatible isomorphisms $X_N \xrightarrow{\sim} X_i$ for $i > N$, inverting these we get compatible morphisms $X_i \to X_N$ for all N. Now check the universal property.]

Exercise 5.8. Show that the free abelian group functor $\mathbb{Z}(-) \colon \mathsf{Set} \to \mathsf{Ab}$ preserves sequential colimits.

Exercise 5.9. Given a diagram of abelian groups

$$A_0 \xrightarrow{f_0} A_1 \to \cdots ,$$

and subgroups $B_i \hookrightarrow A_i$ such that $f_i(B_i) \subseteq B_{i+1}$, show that there is a canonical isomorphism

$$\operatorname{colim}_i A_i/B_i \cong (\operatorname{colim}_i A_i) \,/\, (\operatorname{colim}_i B_i) .$$

Exercise 5.10. Suppose we have $C_\bullet \cong \operatorname{colim}_n C_{n,\bullet}$. Show that

$$Z_k(C) \cong \operatorname{colim}_n Z_k(C_n), \quad B_k(C) \cong \operatorname{colim}_n B_k(C_n),$$

and conclude using the previous exercise that

$$H_k(C) \cong \operatorname{colim}_n H_k(C_n).$$

Exercise 5.11. Compute the cellular homology of S^n using the cell structure with two cells in each dimension $\leq n$. [You need to keep track of the orientations of the generators. There are two ways to define this cell structure: either attach both cells using the identity, or attach one cell using the identity and one using an orientation-reversing map; it may be instructive to look at both.]

Exercise 5.12. Find a cell structure on the torus and compute the cellular homology.

Exercise 5.13. By the classification of finitely generated abelian groups, we can write any finitely generated abelian group A as a direct sum $\mathbb{Z}^r \oplus T$ where T is a torsion group (i.e. all its elements have finite order). The integer r is called the *rank* $\operatorname{rk} A$ of A. If $C_\bullet$ is a chain complex such that C_n is a finitely generated abelian group for all n, and vanishes except for finitely many n, then the *Euler characteristic* of $C_\bullet$ is

$$\chi(C_\bullet) = \sum_i (-1)^i \operatorname{rk} C_i.$$

(i) Show that
$$\chi(C_\bullet) = \sum_i (-1)^i \operatorname{rk} H_i(C).$$
[Assume that rank is additive in short exact sequences: if $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups, then $\operatorname{rk} B = \operatorname{rk} A + \operatorname{rk} C$.]

(ii) If X is a finite cell complex with Γ_n its set of of n-cells, the *Euler characteristic* of X is the alternating sum
$$\chi(X) = \sum_i (-1)^i |\Gamma_n|.$$
Prove that $\chi(X)$ is independent of the cell structure of X, and only depends on X up to homotopy equivalence.

6 Homotopy Invariance and Excision

Exercise 6.1. Show that the exterior product $\mu_{n,m}$ induces bilinear maps in homology $H_n(X) \times H_m(Y) \to H_{n+m}(X \times Y)$.

Exercise 6.2. Suppose the formula
$$\partial \iota_{k,l} = \sum_{i=0}^{k} (-1)^i (d^i \times \mathrm{id})_* \iota_{k-1,l} + (-1)^k \sum_{j=0}^{l} (-1)^j (\mathrm{id} \times d^j)_* \iota_{k,l-1}$$
holds for $\iota_{n-1,m}$ and $\iota_{n,m-1}$. Show that then the chain
$$\sum_{i=0}^{n} (-1)^i (d^i \times \mathrm{id})_* \iota_{n-1,m} + (-1)^n \sum_{j=0}^{m} (-1)^j (\mathrm{id} \times d^j)_* \iota_{n,m-1}$$
is a cycle.

Exercise 6.3.

(i) Show that being chain homotopic is an equivalence relation on the set of chain maps $A_\bullet \to B_\bullet$.

(ii) Show the chain homotopies are compatible with compositions: if $f, g \colon A_\bullet \to B_\bullet$ are chain homotopic, then so are ϕf and ϕg for any chain map $\phi \colon B_\bullet \to B'_\bullet$, and likewise for $f\psi$ and $g\psi$ for any $\psi \colon A'_\bullet \to A_\bullet$.

(iii) Define the *homotopy category* of chain complexes, where the objects are chain complexes and the set of morphisms from $A_\bullet$ to $B_\bullet$ is the set of chain homotopy classes of chain maps.

Exercise 6.4. Use the cone construction to show directly (without using homotopy invariance) that for any convex subset $K \subseteq \mathbb{R}^n$ we have $H_*(K) = 0$, $* > 0$.

Exercise 6.5. Suppose we have chains $R_n \in S_{n+1}(\Delta^n)$ with R_0 the unique simplex $\Delta^1 \to \Delta^0$, and define $\rho_n^X \colon S_n(X) \to S_{n+1}(X)$ by $\rho_n^X(\sigma) := \sigma_* R_n$ for $\sigma \colon \Delta^n \to X$ and extending linearly in σ. Show (by induction on n) that if the chains R_n satisfy
$$\partial R_n = -\rho_{n-1}(\partial \iota_n) + \mathrm{bs}_n^{\Delta^n} \iota_n - \iota_n$$
then the homomorphisms ρ_n^X are a natural chain homotopy between bs^X and id.

7 *Tensor Products and Homology with Coefficients*

Exercise 7.1. For integers n, m, show that $\mathbb{Z}/n \otimes \mathbb{Z}/m \cong \mathbb{Z}/r$ where $r = \gcd(n, m)$ is the greatest common divisor of n and m. In particular, if p and q are distinct primes, then $\mathbb{Z}/p \otimes \mathbb{Z}/q \cong 0$.

Exercise 7.2. For sets S, T, show that there is a canonical isomorphism

$$\mathbb{Z}S \otimes \mathbb{Z}T \cong \mathbb{Z}(S \times T).$$

Exercise 7.3. Prove the formal properties of $\otimes$ using the universal property.

Exercise 7.4. Show that $M \otimes_R N$ is the quotient of $M \otimes N$ by the subgroup generated by elements of the form $rm \otimes n - m \otimes rn$ for $r \in R$, $m \in M$, $n \in N$.

Exercise 7.5.

(i) Let R be an (associative, unital) ring. Show that an R-module is the same as an abelian group M and a homomorphism $\alpha\colon R \otimes M \to M$ such that the square

$$\begin{tikzcd} R \otimes R \otimes M \arrow[r, "\mathrm{id}_R \otimes \alpha"] \arrow[d, "\mu \otimes \mathrm{id}_M"] & R \otimes M \arrow[d, "\mu"] \\ R \otimes M \arrow[r, "\alpha"] & M \end{tikzcd}$$

and the triangle

$$\begin{tikzcd} M \arrow[r, "\sim"] \arrow[drr, equal] & \mathbb{Z} \otimes M \arrow[r, "\eta \otimes \mathrm{id}_M"] & R \otimes M \arrow[d, "\alpha"] \\ & & M \end{tikzcd}$$

commute, where the homomorphism $\mu\colon R \otimes R \to R$ is given by multiplication in R and $\eta\colon \mathbb{Z} \to R$ is given by the unit of R (i.e. $\eta(1) = 1$).

(ii) Show that an R-module homomorphism $\phi\colon M \to N$ is the same as a homomorphism of abelian groups such that the square

$$\begin{tikzcd} R \otimes M \arrow[r, "\mathrm{id} \otimes \phi"] \arrow[d] & R \otimes N \arrow[d] \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

commutes.

(iii) Show that if M is an abelian group and R is a ring, then $R \otimes M$ has a natural R-module structure. [Hint: Use the multiplication in R.]

(iv) Show that if M is an abelian group and N is an R-module, there is a natural correspondence between R-module homomorphisms $R \otimes M \to N$ and homomorphisms of abelian groups $M \to N$.

(v) If S is a set and R is a ring, show that $R \otimes \mathbb{Z}S$ has the universal property of the *free* R-module RS on S: R-module homomorphisms $RS \to M$ correspond to functions $S \to M$.

Exercise 7.6. If k is a field and V, W are k-vector spaces, with bases $\{x_i\}_{i \in I}$ and $\{y_j\}_{j \in J}$, respectively, show that $x_i \otimes y_j$ is a basis for $V \otimes_k W$. If V and W are finite-dimensional, conclude that

$$\dim(V \otimes_k W) = \dim V \cdot \dim W.$$

Exercise 7.7. Show that for $C_\bullet$ a chain complex, the natural map

$$H_k(C) \otimes \mathbb{Z} \to H_k(C \otimes \mathbb{Z}) \cong H_k C$$

is an isomorphism.

Exercise 7.8. Show that a chain homotopy h between chain maps $f, g\colon C_\bullet \to D_\bullet$ induces a chain homotopy between $f \otimes M, g \otimes M\colon C_\bullet \otimes M \to D_\bullet \otimes M$ for any abelian group M.

Exercise 7.9. For an integer n, let us write $n\colon \mathbb{Z} \to \mathbb{Z}$ for the homomorphism given by multiplication with n.

(i) For any abelian group M, use the universal property of $\otimes$ to show that under the natural isomorphism $M \cong M \otimes \mathbb{Z}$ the homomorphism $\mathrm{id} \otimes n\colon M \otimes \mathbb{Z} \to M \otimes \mathbb{Z}$ corresponds to the homomorphism $M \to M$ given by multiplication with n.

(ii) Use the natural isomorphism from Exercise 7.7 to show that the natural map $m_*\colon H_n(X, A; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$ induced by $m\colon \mathbb{Z} \to \mathbb{Z}$ on coefficients is again given by multiplication with m.

Exercise 7.10. Show that for integers n, m we have $\mathrm{Tor}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/r$, where $r = \gcd(n, m)$.

Exercise 7.11. For a continuous map $f\colon X \to Y$, the *mapping cone* $M(f)$ of M is defined as the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \\ CX \arrow[r] & M(f), \end{tikzcd}$$

where CX is the cone on X, as in exercise 5 from week 4.

(i) Show that $H_i(M(f), Y) \cong \tilde{H}_{i-1}(X)$ and the boundary map

$$H_i(M(f), Y) \to H_{i-1}(Y)$$

corresponds to $f_*\colon H_{i-1}(X) \to H_{i-1}(Y)$ (for $i > 1$). [Hint: $M(f)/Y \cong CX/X \cong \Sigma X$ plus Exercise 4.10; to identify the map use the naturality of the boundary map for $(CX, X) \to (M(f), Y)$.]

(ii) From (i) the long exact sequence for the pair $(M(f), Y)$ looks like

$$\cdots \to H_i(M(f)) \to H_{i-1}(X) \xrightarrow{f_*} H_{i-1}(Y) \to H_{i-1}(M(f)) \to \cdots$$

for $i > 1$. Use this to prove that $f_*\colon H_*(X) \to H_*(Y)$ is an isomorphism if and only if $\tilde{H}_*(M(f)) = 0$.

(iii) Complete the proof that f_* is an isomorphism in integral homology if it is an isomorphism in homology with $\mathbb{Q}$- and $\mathbb{F}_p$-coefficients for all primes p.

8 *Cohomology*

Exercise 8.1. For M an abelian group, show that $\mathrm{Hom}(\mathbb{Z}/n, M)$ is the group of n-torsion elements in M; in particular $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/r$ where $r = \gcd(n, m)$.

Exercise 8.2. If S is a set and M an abelian group, then we have canonical isomorphisms
$$\mathrm{Hom}(\mathbb{Z}S, M) \cong M^S \cong \prod_{s\in S} M,$$
of abelian groups.

Exercise 8.3. Prove the basic formal properties of Hom.

Exercise 8.4.

(i) Show that for abelian groups A, B, C there is a natural bijection between the sets of homomorphisms $A \otimes B \to C$ and $A \to \mathrm{Hom}(B, C)$.

(ii) Show that this bijection is moreover an isomorphism of abelian groups
$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C)).$$

(iii) Show that composition of homomorphisms of abelian groups gives a homomorphism
$$\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C).$$

Exercise 8.5. If $m\colon \mathbb{Z} \to \mathbb{Z}$ is the map given by multiplication with m, show that $m^*\colon \mathrm{Hom}(\mathbb{Z}, M) \to \mathrm{Hom}(\mathbb{Z}, M)$ corresponds under the isomorphism $\mathrm{Hom}(\mathbb{Z}, M) \cong M$ to the map $M \to M$ given by multiplication with m.

Exercise 8.6. Show that a chain homotopy between chain maps $f, g\colon C_\bullet \to D_\bullet$ induces a natural chain homotopy between $f^*, g^*\colon \mathrm{Hom}(D, M) \to \mathrm{Hom}(C, M)$ for any abelian group M.

Exercise 8.7. Show that for $A \subseteq X$ the relative cochains $S^n(X, A)$ can be identified with the subgroup of $S^n(X) \cong \mathbb{Z}^{\mathrm{Sing}_n(X)}$ consisting of functions $f\colon \mathrm{Sing}_n(X) \to \mathbb{Z}$ such that $f(\alpha) = 0$ for $\alpha \in \mathrm{Sing}_n(A) \subseteq \mathrm{Sing}_n(X)$.

Exercise 8.8. Use the Mayer-Vietoris sequence for cohomology to compute $\tilde{H}^*(S^n; M)$.

Exercise 8.9.

(i) Show that if we have short exact sequences $0 \to A_i \xrightarrow{j_i} B_i \xrightarrow{q_i} C_i \to 0$ for all $i \in I$, then there is a short exact sequence
$$0 \to \prod_{i\in I} A_i \xrightarrow{\prod_{i\in I} j_i} \prod_{i\in I} B_i \xrightarrow{\prod_{i\in I} q_i} \prod_{i\in I} C_i \to 0.$$

(ii) Use (i) to prove the additivity axiom for cohomology: for a disjoint union $X = \coprod_{i\in I} X_i$ the inclusions $X_i \hookrightarrow X$ induce an isomorphism
$$H^*(X) \xrightarrow{\sim} \prod_i H^*(X_i).$$

Exercise 8.10 (Change of coefficients in cohomology)**.** Let (X, A) be a subspace pair.

(i) Show that a homomorphism of abelian groups $\phi\colon M \to M'$ induces natural maps $\phi_*\colon H^*(X, A; M) \to H^*(X, A; M')$.

(ii) Let $0 \to M' \xrightarrow{i} M \xrightarrow{q} M'' \to 0$ be a short exact sequence of abelian groups. Show that this induces a long exact sequence in cohomology
$$\cdots \to H^n(X, A; M') \xrightarrow{i_*} H^n(X, A; M) \xrightarrow{q_*} H^n(X, A; M'') \to H^{n+1}(X, A; M') \to \cdots.$$

Exercise 8.11. Let (X, A) be a subspace pair and M an abelian group. Show that the natural maps $H^*(-;M) \to \mathrm{Hom}(H_*(-), M)$ fit in a commutative square

$$\begin{array}{ccc} H^n(A;M) & \xrightarrow{\quad\delta\quad} & H^{n+1}(X,A;M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(H_n(A);M) & \xrightarrow{\partial^*} & \mathrm{Hom}(H_{n+1}(X,A);M), \end{array}$$

where ∂ and δ denote the connecting maps in the homology and cohomology long exact sequences for (X, A), respectively.

9 *Homology of Products*

Exercise 9.1. Let $I_\bullet$ denote the chain complex with $I_1 = \mathbb{Z}$, $I_0 = \mathbb{Z}\{[0], [1]\}$ and $I_n = 0$ otherwise, with differential $\partial\colon I_1 \to I_0$ given by $\partial(1) = [1] - [0]$. Show that a chain homotopy between chain maps $C_\bullet \to D_\bullet$ is the same thing as a chain map $C_\bullet \otimes I_\bullet \to D_\bullet$. (Thus if we think of $I_\bullet$ as an "algebraic interval", chain homotopies are an algebraic version of homotopies between continuous maps.)

Exercise 9.2. Let $C_\bullet$ be a chain complex.

(i) Prove that the functor $C_\bullet \otimes -$ preserves chain homotopies and levelwise splittable short exact sequences of chain complexes. [Hint: For chain homotopies you can use Exercise 9.1.]

(ii) If $C_\bullet$ is levelwise free, show that $C_\bullet \otimes -$ preserves all short exact sequences of chain complexes.

Exercise 9.3. Show that if two chain complexes differ only by the signs of the boundary maps, then they are isomorphic.

Exercise 9.4.

(i) Show that for M an abelian group and $C_\bullet, D_\bullet$ chain complexes, there is a natural bijection between chain maps $C_\bullet \to \mathrm{Hom}(D, M)_\bullet$ and chain maps $C_\bullet \otimes D_\bullet \to M[0]$. [With our sign convention for the differential in $\mathrm{Hom}(D, M)_\bullet$ this bijection involves some signs. Alternatively, we can define the differential $\delta\phi$ for $\phi\colon D_{-n} \to M$ to be given by $(\delta\phi)(d) = (-1)^{n+1}\phi(\partial d)$ (without changing the homology, by Exercise 9.3).]

(ii)* For chain complexes $C_\bullet, D_\bullet$, define a chain complex $\mathrm{Hom}(C, D)_\bullet$ so that there is a natural bijection between chain maps $C_\bullet \otimes D_\bullet \to E_\bullet$ and chain maps $C_\bullet \to \mathrm{Hom}(D_\bullet, E_\bullet)$. [Hint: Do it first for graded abelian groups and then figure out the differential. This again involves some signs, and if you want a sign convention that recovers our previous definition of $\mathrm{Hom}(D, M)$ as $\mathrm{Hom}(D, M[0])$ then the bijection between maps also needs some signs.]

Exercise 9.5. Check that the properties of the exterior multiplication maps $\mu_{n,m}\colon S_n(X) \times S_m(Y) \to S_{n+m}(X \times Y)$ imply that these fit together into a natural chain map

$$\mu\colon S_\bullet(X) \otimes S_\bullet(Y) \to S_\bullet(X \times Y).$$

Exercise 9.6. Use the Künneth Theorem to compute the homology of $\mathbb{RP}^2 \times \mathbb{RP}^2$.

10 The Ring Structure on Cohomology

Exercise 10.1.

(i) Prove that there is a natural homomorphism of abelian groups

$$\mathrm{Hom}(A, M) \otimes \mathrm{Hom}(B, N) \to \mathrm{Hom}(A \otimes B, M \otimes N),$$

where A, B, M, N are abelian groups, given by tensoring homomorphisms.

(ii) Use this to define a natural chain map

$$\mathrm{Hom}(C, M)_\bullet \otimes \mathrm{Hom}(D, N)_\bullet \to \mathrm{Hom}(C \otimes D, M \otimes N)_\bullet,$$

where $C_\bullet, D_\bullet$ are chain complexes and M, N are abelian groups.

Exercise 10.2.

(i) Show that the cross product $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ can be expressed in terms of the cup product by the formula

$$\xi \times \eta = p_X^* \xi \smile p_Y^* \eta$$

where p_X, p_Y are the projections from $X \times Y$ to X and Y. [Hint: Use the explicit formula for the cup and cross products.]

(ii) If R, R' are commutative rings, we can equip the tensor product $R \otimes R'$ with a commutative ring structure with the multiplication defined on generators by

$$(r_1 \otimes r_1') \cdot (r_2 \otimes r_2') = r_1 r_1' \otimes r_2 r_2'.$$

Check that the analogous construction for graded rings also makes sense. [Note that to get commutativity in the graded cases we need to add a sign.]

(iii) Show that the cross product map $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ is a ring homomorphism with respect to the tensor product of the cup product on X and Y and the cup product on $X \times Y$. [Hint: This amounts to checking the relation

$$(\xi \smile_X \xi') \times (\eta \smile_Y \eta') = (\xi \times \eta) \smile_{X \times Y} (\xi' \times \eta'),$$

for which you can use part (i) and naturality of cup products.]

(iv) Prove that if X and Y are finite type cell complexes and the integral cohomology groups of X are all free abelian groups, then the cross product map

$$H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$

is an isomorphism of rings. [Hint: Use the Künneth Theorem for cohomology.]

(v) Compute the ring structure on $H^*(S^n \times S^m)$.

Exercise 10.3. Convince yourself that the diagrammatic definition of a (commutative) ring agrees with the (equational) one you have seen before.

Exercise 10.4. Show that if we define the cup product on the chain level using the Alexander–Whitney map, then $S^\bullet(X; R)$ is a (strictly) associative and unital dg-ring for any ring R.

Exercise 10.5.

1. Show that under the isomorphism $H^0(X;R) \cong R^{\pi_0 X}$, the cup product in degree 0 corresponds to the pointwise multiplication of functions $\pi_0 X \to R$, with unit the constant function with value $1 \in R$ and product $(f \cdot g)(x) = f(x) \cdot g(x)$. [Hint: Use the explicit formula from the Alexander–Whitney map.]

2. By additivity for cohomology we have an isomorphism $H^i(X;R) \cong \prod_{t\in\pi_0(X)} H^i(X_t;R)$ where X_t denotes the path-component of X corresponding to $t \in \pi_0 X$. Show that under this isomorphism the cup product

$$H^0(X;R) \times H^i(X;R) \to H^i(X;R)$$

for $i > 0$ takes $f\colon \pi_0 X \to R$ and $(\alpha_t)_{t\in\pi_0 X}$ to $(f(t)\alpha_t)_t$ (in terms of the natural R-module structure on $H^i(X_t;R)$).

Exercise 10.6.

(i) If R_i, $i \in I$, are rings, then the cartesian product $\prod_{i\in I} R_i$ can be given a commutative ring structure with pointwise multiplication (i.e. $(r_i)_{i\in I} \cdot (r'_i)_{i\in I} = (r_i r'_i)_{i\in I}$). Check that this has the universal property of the product in the category of rings (i.e. given ring homomorphisms $\phi_i\colon R' \to R_i$ for each i, there exists a unique ring homomorphism $R' \to \prod_{i\in I} R_i$ that projects to ϕ_i in the ith coordinate). Also check the analogous statement holds for graded rings (where the cartesian product is taken degreewise).

(ii) Show that for topological spaces X_i, $i \in I$, the map

$$H^*(\coprod_{i\in I} X_i) \to \prod_{i\in I} H^*(X_i),$$

induced by the inclusions $X_i \hookrightarrow \coprod_{i\in I} X_i$, is an isomorphism of rings.

(iii) Compute the ring structure on $H^*(S^n \vee S^m)$. [Hint: The canonical map $S^n \amalg S^m \to S^n \vee S^m$ induces a ring homomorphism $H^*(S^n \vee S^m) \to H^*(S^n \amalg S^m)$; check that this is an isomorphism in degrees $* > 0$.]

Exercise 10.7 ($*$)**.** Let Σ_g be the orientable closed surface of genus g. There is a continuous map from Σ_g to a wedge of g tori that pinches the "necks" between the g holes to points,

$$q\colon \Sigma_g \to \bigvee_g (S^1 \times S^1)$$

Using that the induced map in cohomology is a ring homomorphism, compute the ring structure on $H^*(\Sigma_g)$. [Recall that in Exercise 4.8 you computed that

$$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z}, & * = 0,2 \\ \mathbb{Z}^{2g}, & * = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and use Exercise 10.6 and Exercise 10.2 to compute the ring structure for the wedge of tori.]

Exercise 10.8 ($*$)**.** Show that $H^*(\mathbb{CP}^n \times \mathbb{CP}^m)$ is a truncated graded polynomial ring in two variables,

$$H^*(\mathbb{CP}^n \times \mathbb{CP}^m) \cong \mathbb{Z}[x,y]/(x^{n+1}, y^{m+1}),$$

where both generators are in degree 2. [Hint: First check that there is an isomorphism of (ungraded) polynomial rings $\mathbb{Z}[x] \otimes \mathbb{Z}[y] \cong \mathbb{Z}[x,y]$. Use Exercise 10.2.]

11 Manifolds and Poincaré Duality

Exercise 11.1. Use the explicit formula for the Alexander–Whitney map to get a formula for the chain-level cap product, and use this to prove the identities relating the cap product to the cup product and Kronecker pairing.

Exercise 11.2. Let M be a compact n-manifold whose homology groups are all finitely generated. (This is in fact true for all compact smooth manifolds.)

(i) Show that if M is orientable, then it is R-orientable for every commutative ring R. [Use the universal coefficient theorem.]

(ii) Show that if M is orientable, then $H_{n-1}(M)$ contains no torsion. [Apply Poincaré duality with $\mathbb{Z}/p$-coefficients and the universal coefficient theorem for every prime p.]

(iii) Suppose M is non-orientable and assume this implies M is also not $\mathbb{Z}/p$-orientable for any odd prime p, and that $H_n(M) = H_n(M; \mathbb{Z}/p) = 0$. Show that the torsion subgroup of $H_{n-1}(M)$ is $\mathbb{Z}/2$.

Exercise 11.3. Use Poincaré duality to show that $S^n \vee S^m$ is not homotopy equivalent to a compact manifold for $n, m > 0$. [In the case $m = 2n$ you need to use the cup product, which you computed in Exercise 10.6.]

Exercise 11.4. Use the Künneth theorem to compute the integral (co)homology of the n-torus

$$T^n := (S^1)^{\times n}.$$

Apply Poincaré duality to deduce the binomial coefficient identity

$$\binom{n}{k} = \binom{n}{n-k}.$$

What is the ring structure?

Exercise 11.5. For which n does there exist a compact connected oriented $2n$-manifold M such that $H_n(M) \cong \mathbb{Z}$?

Exercise 11.6. Show that if X is path-connected and non-compact, then $H^0_c(X) = 0$. [Hint: Use the definition of $S^\bullet_c(X)$ as a subcomplex of $S^\bullet(X)$.]