Algebraic Topology 1

27 & 28 Oct 2020

Singular Cohomology

$A, B \in Ab \rightsquigarrow \mathrm{Hom}(A,B)$ homomor. s $A \to B$

$C_\bullet \in Ch,\ M \in Ab \rightsquigarrow \mathrm{Hom}(C,M)_\bullet$ w/ $\mathrm{Hom}(C,M)_n = \mathrm{Hom}(C_{-n}, M)$

$\partial : C_n \to C_{n-1}$ $\qquad \partial^* : \mathrm{Hom}(C_{n-1}, M) \to \mathrm{Hom}(C_n, M)$

$$\begin{matrix} \partial^* : \mathrm{Hom}(C_{n-1}, M) & \to & \mathrm{Hom}(C_n, M) \\ \| & & \| \\ \mathrm{Hom}(C,M)_{1-n} & & \mathrm{Hom}(C,M)_{-n} \end{matrix}$$

$$(\partial^*)^2 = (\partial^2)^* = 0^* = 0$$

$\varphi : C_\bullet \to D_\bullet$ chain map $\rightsquigarrow \varphi^* : \mathrm{Hom}(D,M)_\bullet \to \mathrm{Hom}(C,M)_\bullet$
chain map

Get functor $\mathrm{Hom}(-,M) : Ch^{op} \to Ch$

Alternatively: Work w/ cochain complexes (differential goes $C^n \to C^{n+1}$)

Equivalent, via chain cx. $C_\bullet$ $\leftrightarrow$ cochain cx. $C^\bullet$ w/ $C^n = C_{-n}$

Exercise: $\mathrm{Hom}(-, M)$ preserves chain homotopies

$f, g : C_\bullet \to D_\bullet$ chain htpc. $\rightsquigarrow$ f^*, g^* chain htpc.

$\mathrm{Hom}(D, M) \to \mathrm{Hom}(C, M)$

Singular cochains:

$A \subset X$ top. sp.s $\quad \underbrace{S^\bullet(X, A; M)}_{\text{singular cochains on } (X,A) \text{ w/ coefficients in } M} := \mathrm{Hom}(S_\bullet(X, A), M)$

$$S^n(X, A; M) = \mathrm{Hom}(S_n(X, A), M) = S^\bullet(X, A; M)_{-n}$$

<u>Singular cohomology</u>: $H^i(X,A;M) = H_{-i}(S^\bullet(X,A;M))$

$S^\bullet : \mathrm{Pair}^{op} \longrightarrow \mathrm{Ch}$

$H^* : \mathrm{Pair}^{op} \longrightarrow \mathrm{grAb}$

$\delta : S^n(X,A;M) \longrightarrow S^{n+1}(X,A;M)$ induced by $\partial : S_{n+1} \to S_n$

- coboundary map

Coboundaries: $B^n(X,A;M) =$ image of $\delta : S^{n-1}(X,A;M) \to S^n(X,A;M)$

Cocycles: $Z^n(X,A;M) =$ ker of $\delta : S^n(X,A;M) \to S^{n+1}(X,A;M)$

More explicitly:

$$S^n(X;M) = \mathrm{Hom}(S_n(X), M)$$
$$= \mathrm{Hom}(\mathbb{Z}\,\mathrm{Sing}_n X, M)$$
$$\cong M^{\mathrm{Sing}_n X}$$

$$\partial_i : \mathrm{Sing}_n X \to \mathrm{Sing}_{n-1} X \rightsquigarrow \partial_i^* : M^{\mathrm{Sing}_{n-1} X} \to M^{\mathrm{Sing}_n X}$$

$\delta : S^{n-1}(X;M) \to S^n(X;M)$ is given by

$$(\varphi : \mathrm{Sing}_{n-1} X \to M) \longmapsto \sum_{i=0}^{n} (-1)^i \underbrace{\partial_i^* \varphi}_{\varphi \circ \partial_i}$$

$\varphi : S_n X \to M$ is a cocycle iff $\varphi \circ \partial = 0$ i.e. φ vanishes on $B_n X$

Why cohomology?

- on cohomology groups, get a graded commutative ring
- interplay between homology & cohomology (Poincaré duality)

<u>Example:</u> $H^0 X$?

$$S^0(X;M) = M^{\mathrm{Sing}_0 X} = M^X$$

$$\delta : S^0 \longrightarrow S^1 = M^{\mathrm{Sing}_1 X}$$

$$\varphi : X \to M \longmapsto (\delta\varphi)(p) = \varphi(\partial_0 p) - \varphi(\partial_1 p) = \varphi(p(1)) - \varphi(p(0))$$

$$p : I \to X \in \mathrm{Sing}_1 X$$

φ cocycle means $\delta\varphi = 0 \iff \delta\varphi(p) = 0 \ \forall\ p$

$\iff \varphi(p(1)) = \varphi(p(0)) \ \forall$ path p in X

I.e. φ is constant on path components

$$H^0(X;M) = Z^0(X;M) = M^{\pi_0 X} \cong \prod_{i \in \pi_0 X} M$$

If $\pi_0 X$ is finite, this is same as $\bigoplus_{i \in \pi_0 X} M \cong M \otimes \mathbb{Z}\pi_0 X = H_0(X;M)$

– but not isom.c to H_0 if $\pi_0 X$ is infinite

E.g. if $\pi_0 X$ is countable, so is $H_0(X)$
but $H^0(X)$ is uncountable.

<u>Example:</u> $H^*(*)$

$\mathrm{Sing}_n(*) = \{\Delta^n \xrightarrow{c_n} *\}$

function $\mathrm{Sing}_n(*) \xrightarrow{\gamma_n} \mathbb{Z}$, $c_n \longmapsto 1$

$S^n(X) = \mathbb{Z}^{\mathrm{Sing}_n(*)} \cong \mathbb{Z}\gamma_n$

$$(\delta \gamma_n)(c_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \underbrace{\gamma_n(\underbrace{\partial_i c_{n+1}}_{c_n})}_{=1} = \sum_{i=0}^{n+1} (-1)^i$$

$\uparrow$
$S^{n+1}(X)$

n+2 terms
- cancel for n+2 even

$$= \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$\cdots 0 \xrightarrow{1} S^0 X \xrightarrow{0} S^1 X \xrightarrow{-1} S^2 X \xrightarrow{-2} \cdots$$

$$\cdots 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \cdots$$

$$Z^n = \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \qquad H^n(*) = \begin{cases} 0, & n > 0 \\ \mathbb{Z}, & n = 0 \end{cases}$$

$$B^n = \begin{cases} 0, & n \text{ odd} \\ \mathbb{Z}, & n \text{ even} > 0 \end{cases}$$

$A \subset X$

$$0 \to S_\bullet A \to S_\bullet X \to S_\bullet(X,A) \to 0$$

levelwise free chain cx
∴ levelwise splittable

$\mathrm{Hom}(-,M)$ preserves split SESs

– get SES

$$0 \to S^\bullet(X,A;M) \to S^\bullet(X;M) \to S^\bullet(A;M) \to 0$$

$$S^\bullet(X,A;M) = \mathrm{Hom}(S_\bullet(X,A);M)$$

$\cong$ kernel of $S^\bullet(X;M) \to S^\bullet(A;M)$

$\Rightarrow$ $S^n(X,A;M) \cong$ subgroup of $S^n(X;M) \cong M^{\mathrm{Sing}_n X}$ containing functions $\mathrm{Sing}_n X \to M$ that vanish on $\mathrm{Sing}_n A$

Get LES in homology:

$$\cdots H^n(X,A;M) \longrightarrow H^n(X;M) \to H^n(A;M) \xrightarrow{\delta} H^{n+1}(X,A;M) \cdots$$

$$\Big(\overset{\shortparallel}{H_{-n}}(S^\bullet(X,A;M)) \qquad\qquad \Big(H_{-n}(S^\bullet(A;M)) \to H_{-n-1}(S^\bullet(X,A;M))$$

We also have a Mayer-Vietoris sequence in cohomology:

$A, B \subset X$, $X = A^\circ \cup B^\circ$, have LES

$$\cdots H^n X \xrightarrow{(i^*, i'^*)} H^n A \oplus H^n B \xrightarrow{j^* - j'^*} H^n(A \cap B) \xrightarrow{\Delta} H^{n+1} X \cdots$$

$$\begin{tikzcd} A \cap B \arrow[r, hook, "j"] \arrow[d, hook, "j'"'] & A \arrow[d, hook, "i"] \\ B \arrow[r, hook, "i'"] & X \end{tikzcd}$$

Defn.: An (ordinary) <u>cohomology theory</u> consists

- functors $h^n : \mathrm{Pair}^{op} \longrightarrow Ab$, $n \in \mathbb{Z}$, $h^n(X) = h^n(X, \emptyset)$
  $f^* = h^n(f)$
- natural coboundary maps $\delta : h^n(A) \to h^{n+1}(X, A)$

s.t.

(LES) $$\cdots\; h^n(X,A) \underset{\text{from } (X,\emptyset)\to(X,A)}{\longrightarrow} h^n(X) \underset{\text{from } A \hookrightarrow X}{\longrightarrow} h^n(A) \xrightarrow{\delta} h^{n+1}(X,A) \cdots$$ is a LES

(Homotopy invariance): $f, g : (X,A) \to (Y,B)$ homotopic

$$\Rightarrow f^* = g^*$$

(Excision) $U \subset A \subset X$, $\bar{U} \subset A^\circ$ then

$$h^n(X,A) \to h^n(X \setminus U, A \setminus U) \text{ is isomor.}$$

(Additivity) $X = \coprod_{i \in I} X_i$, $\quad h^n(X) \xrightarrow{\sim} \prod_{i \in I} h^n(X_i)$

(Dimension) $h^n(*) = 0, \quad n \neq 0$

<u>Propn.</u>: $H^*(-; M)$ is a cohlgy theory.

<u>Proof</u>:

(LES) Above

(Ht.py inv.ce) $\mathrm{Hom}(-, M)$ preserves chain ht.pies, so follows from ht.py $f \simeq g \Rightarrow f_*, g_* : S_\bullet(X,A) \to S_\bullet(Y,B)$ chain htpc.

(Excision) Induced by proof for H_* via $\mathrm{Hom}(-, M)$

(Dimension) Above

(Additivity) Exercise. $\square$

$\exists$ natural map $H^n(X; M) \to \mathrm{Hom}(H_n X, M)$ but not always isomor-
Instead have cohomology version of UCT.

<u>Lemma</u>: $C_\bullet$ ch. cx., $M \in \mathrm{Ab}$, $\exists$ a nat. map

$$H_{-k}\mathrm{Hom}(C, M) \to \mathrm{Hom}(H_k C, M)$$

$[\varphi]$ rep. by $\varphi : C_k \to M$ cocycle $\longmapsto$ $[c]$ rep. by $c \in C_k$ $\longmapsto$ $\varphi(c)$

Proof:

$$Z_{-k}\operatorname{Hom}(C,M) \xrightarrow{\circledast} \operatorname{Hom}(H_kC, M)$$

$$\varphi \longmapsto ([c] \mapsto \varphi(c))$$

$\varphi \circ \partial = 0$

$\Rightarrow \varphi(c + \partial c') = \varphi(c)$ so $\varphi([c]) = \varphi(c)$ is well-defined

$B_{-k}\operatorname{Hom}(C,M)$ is in kernel:

$\partial^*\psi \quad \Rightarrow \quad \partial^*\psi(c) = \psi(\partial c) = 0$ if c is cycle

$\psi : C_{k+1} \to M$

$\circledast$ factors through the quotient $H_{-k}\operatorname{Hom}(C,M)$. $\square$

Ext

$0 \to F_1 \xrightarrow{i} F_0 \to A \to 0$ free res. of $A \in Ab$

$M \in Ab$

$$\mathrm{Ext}(A,M) = \mathrm{coker}\left(i^* : \mathrm{Hom}(F_0, M) \to \mathrm{Hom}(F_1, M)\right)$$

Think of $F_\bullet = \begin{pmatrix} F_1 \\ \downarrow i \\ F_0 \end{pmatrix}$ as two-term chain complex,

then $H_* F_\bullet = \begin{cases} A, & * = 0 \\ 0, & * \neq 0 \end{cases}$

Hom$(-,M)$ left exact,

$0 \to \mathrm{Hom}(A,M) \to \mathrm{Hom}(F_0,M) \to \mathrm{Hom}(F_1,M)$

$$H_* \mathrm{Hom}(F_\bullet, M) = \begin{cases} \mathrm{Hom}(A,M), & * = 0 \\ \mathrm{Ext}(A,M), & * = -1 \end{cases}$$

$F_\bullet, F'_\bullet$ free resolutions of A $\Rightarrow$ chain ht.py equivalent

& $\mathrm{Hom}(-, M)$ preserves chain ht.pies

$\Rightarrow$ $H_* \mathrm{Hom}(F, M) \cong H_* \mathrm{Hom}(F', M)$ $\Rightarrow$ $\mathrm{Ext}(A,M)$ is independent of choice of $F_\bullet$

Example:

- F free, $\begin{array}{c} 0 \\ \downarrow \\ F \end{array}$ is a free res. $\rightsquigarrow$ $\mathrm{Ext}(F,M) = 0$

- $\mathbb{Z}/m$, $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0$

$$\begin{array}{ccc} m^*: \mathrm{Hom}(\mathbb{Z},M) & \longrightarrow & \mathrm{Hom}(\mathbb{Z},M) \\ \wr\wr & & \wr\wr \\ M & \xrightarrow{m} & M \end{array}$$

so $\mathrm{Ext}(\mathbb{Z}/m, M) = \mathrm{coker}\left(M \xrightarrow{m} M\right)$
$= M/mM$

- D divisible abelian group (such as $\mathbb{Q}$), then $\mathrm{Hom}(-, D)$ preserves SES, hence $\mathrm{Ext}(A, D) = 0$

<u>Propn.</u>: SES $0 \to A' \to A \to A'' \to 0$, get LES

$$0 \to \mathrm{Hom}(A'', B) \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B) \to \mathrm{Ext}(A'', B) \to \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B) \to 0$$

Universal coefficient theorem for cohomology

Thm.: $A \subset X$, $M \in Ab$, have natural SESs

$$0 \to \mathrm{Ext}(H_{n-1}(X,A), M) \to H^n(X,A;M) \to \mathrm{Hom}(H_n(X,A), M) \to 0$$

Propn.: $C_\bullet$ levelwise free chain cx., $M \in Ab$, have natural SESs

$$0 \to \mathrm{Ext}(H_{n-1}C, M) \to H_{-n}\mathrm{Hom}(C, M) \to \mathrm{Hom}(H_n C, M) \to 0$$

Proof: $B_n = B_n C$, $Z_n = Z_n C$, $H_n = H_n C$

$0 \to B_n \xrightarrow{j_n} Z_n \to H_n \to 0$ SES – a free resolution of $H_n C$

$\subseteq C_n$ free

$\Rightarrow$ free

$$0 \to Z_n \to C_n \xrightarrow{\partial'} B_{n-1} \to 0 \quad \text{SES}$$

boundary so map to B_{n-1}

$B'_n = B_{n-1}$

$Z_\bullet, B'_\bullet$ chain w/ differential 0

$$0 \to Z_\bullet \to C_\bullet \xrightarrow{\partial'} \underbrace{B'_\bullet}_{} \to 0 \quad \text{SES of chain cx's}$$

levelwise free

$\Rightarrow$ levelwise splittable SES

$\Rightarrow$ get SES

$$0 \to \mathrm{Hom}(B', M)_\bullet \to \mathrm{Hom}(C, M)_\bullet \to \mathrm{Hom}(Z, M)_\bullet \to 0$$

Get LES

$$\cdots \operatorname{Hom}(\underbrace{B_{-n-1}}_{B'_{-n}}, M) \to H_n \operatorname{Hom}(C, M) \to \operatorname{Hom}(Z_{-n}, M) \xrightarrow{\Delta_n} \operatorname{Hom}(\underbrace{B_{-n}}_{B'_{1-n}}, M) \cdots$$

As in UCT for $H_*(-;M)$, unwinding defn. of bd.ry map in LES gives

$$\Delta_n = j_{-n}^* : \operatorname{Hom}(Z_{-n}, M) \to \operatorname{Hom}(B_{-n}, M)$$

$$\ker \Delta_n \cong \operatorname{Hom}(H_{-n}, M)$$

$$\operatorname{coker} \Delta_n \cong \operatorname{Ext}(H_{-n}, M)$$

Have SES around $H_n \operatorname{Hom}(C, M)$:

$$0 \to \underset{\operatorname{Ext}(H_{-n-1}, M)}{\overset{\shortparallel}{\operatorname{coker} \Delta_{n+1}}} \to H_n \operatorname{Hom}(C, M) \to \underset{\operatorname{Hom}(H_{-n}, M)}{\overset{\shortparallel}{\ker \Delta_n}} \to 0$$

$\square$

- As in UCT for $H_*(-;M)$ these SESs are non-canonically splittable – $\exists$ non-natural isomor.

$$H^n(X;M) \cong \mathrm{Hom}(H_n X, M) \oplus \mathrm{Ext}(H_{n-1}X, M)$$

<u>Cor.</u>: If $f:(X,A)\to(Y,B)$ gives isomors $H_i(X,A) \xrightarrow[f_*]{\sim} H_i(Y,B)$ $\forall i$ then $f^*: H^i(Y,B;M) \to H^i(X,A;M)$ is isomor.

<u>Proof</u>: 5-Lemma

<u>Cor.</u>: If D is divisible (e.g. $\mathbb{Q}$) then $H^n(X,A;D) \cong \mathrm{Hom}(H_n(X,A), D)$

Thm.: $A \subset X$, $M \in Ab$, have natural SESs

$$0 \to \mathrm{Ext}(H_{n-1}(X,A), M) \to H^n(X,A;M) \to \mathrm{Hom}(H_n(X,A), M) \to 0$$

Example: If $H_n X$ is free $\forall n$ then Ext's $= 0$ & get

isomor. $H^n(X;M) \cong \mathrm{Hom}(H_n X, M)$

If $M = \mathbb{Z}$ & $H_n X$ is f.g. & free then

$$\mathrm{Hom}(\underbrace{H_n X}_{\mathbb{Z}^r}, \mathbb{Z}) \cong \mathbb{Z}^r \cong H_n X$$

here we have isomor.s between

$H_n X \cong H^n X$

Example: $H^*(\mathbb{RP}^n)$

$$H_* \mathbb{RP}^n = \begin{cases} \mathbb{Z}/2, & * \text{ odd}, < n \\ \mathbb{Z}, & * = 0, \quad * = n \text{ odd} \\ 0, & * \text{ even} > 0, \quad * > n \end{cases}$$

$$\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}) = 0$$

$$\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}) = \mathrm{coker}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) = \mathbb{Z}/2$$

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$$

$$\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$$

$$0 \to \mathrm{Ext}(H_{i-1}\mathbb{RP}^n, \mathbb{Z}) \longrightarrow H^i \mathbb{RP}^n \longrightarrow \mathrm{Hom}(H_i \mathbb{RP}^n, \mathbb{Z}) \to 0$$

i even, $0 < i \leq n$: $\quad \mathbb{Z}/2 \longrightarrow H^i \mathbb{RP}^n \longrightarrow 0$

i odd, $0 < i < n$: $\quad 0 \longrightarrow H^i \mathbb{RP}^n \longrightarrow 0$

$i = n$ odd

$$0 \to 0 \to H^n \mathbb{RP}^n \to \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0$$
$$\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$

$$H^* \mathbb{RP}^n = \begin{cases} \mathbb{Z}, & * = 0, \ * = n \text{ odd} \\ \mathbb{Z}/2, & * \text{ even}, \ 0 < * \leq n \\ 0, & \text{otherwise} \end{cases}$$

<u>Example:</u> Suppose $H_* X$ is finitely generated $\forall *$

Write $H_n X \cong F_n \oplus T_n$

F_n: free, $\mathbb{Z}^k$

T_n: torsion, $\bigoplus_{i=0}^{N} \mathbb{Z}/m_i$

$\exists$ isomor. $\operatorname{Hom}(F_n, \mathbb{Z}) \cong F_n$

$\operatorname{Ext}(F_n, \mathbb{Z}) = 0$

$\operatorname{Hom}(T_n, \mathbb{Z}) = 0$

$\operatorname{Ext}(T_n, \mathbb{Z}) \cong T_n$

UCT gives SES

$$0 \to T_{n-1} \to H^n X \to F_n \to 0$$

$\Rightarrow H^n X \cong F_n \oplus T_{n-1}$ – free part stays in place, torsion moves up by 1

Cellular cohomology

X cell complex, Γ_n = n-dim'l cell, X_k – k-dim'l subcomplex

$$H^*(X_k, X_{k-1}; M) \cong \tilde{H}^*(\underbrace{\bigvee_{\Gamma_k} S^k}_{\simeq X_k / X_{k-1}}; M) \cong \prod_{\Gamma_k} H^*(S^k; M) \cong \begin{cases} M^{\Gamma_k}, & * = k \\ 0, & * \neq k \end{cases}$$

Can define $C^k_{cell}(X;M) := H^k(X_k, X_{k-1}; M)$ (in deg. $-k$)

Cellular differential:

$$\delta : H^k(X_k, X_{k-1}; M) \longrightarrow H^k(X_k; M) \longrightarrow H^{k+1}(X_{k+1}, X_k; M)$$

$(X_k, X_{k-1}) \leftarrow (X_k, \emptyset)$

bdry in LES for (X_{k+1}, X_k)

Then $\delta^2 = 0$ & $H_{-k}(C^\bullet_{cell}(X;M)) \cong H^k(X;M)$

<u>Propn.</u>: $C^\bullet_{cell}(X;M) \cong \mathrm{Hom}(C^{cell}_\bullet(X), M)$.

$\mathrm{Hom}(C^{cell}_k X, M)$

$$\text{Note UCT gives } C^k_{cell}(X;M) = H^k(X_k, X_{k-1}; M) \xrightarrow{\sim} \mathrm{Hom}(H_k(X_k, X_{k-1}), M)$$

Exercise: $A \subset X$, have commutative square relating bd.ry maps in LES

$$\begin{tikzcd}
H^n(A;M) \arrow[r, "\delta"] \arrow[d] & H^{n+1}(X,A;M) \arrow[d] \\
\mathrm{Hom}(H_nA, M) \arrow[r, "\partial^*"] & \mathrm{Hom}(H_{n+1}(X,A), M)
\end{tikzcd}$$

$\partial : H_{n+1}(X,A) \to H_nA$

Proof of Prop.: We have commutative diagrams

$$\begin{tikzcd}
H^k(X_k, X_{k-1}; M) \arrow[r] \arrow[d, "\wr"'] & H^k(X_k; M) \arrow[r] \arrow[d] & H^{k+1}(X_{k+1}, X_k; M) \arrow[d, "\wr"] \\
\mathrm{Hom}(H_k(X_k, X_{k-1}); M) \arrow[r] & \mathrm{Hom}(H_kX_k, M) \arrow[r] & \mathrm{Hom}(H_{k+1}(X_{k+1}, X_k), M)
\end{tikzcd}$$

$(X_k, X_{k-1}) \leftarrow (X_k, \emptyset)$

$\square$

Cor.: $\alpha \in \Gamma_{k+1}$, $f_\alpha : S^k \to X_k$ attaching map

$\beta \in \Gamma_k$, $q_\beta : X_k \to X_k / X_{k-1} \cong \bigvee_{\Gamma_k} S^k \longrightarrow S^k$ projection to sphere indexed by β

$$\begin{array}{cccc} \delta : & C^k_{\mathrm{cell}}(X;M) & \longrightarrow & C^{k+1}_{\mathrm{cell}}(X;M) \\ & \cong & & \cong \\ & M^{\Gamma_k} & \longrightarrow & M^{\Gamma_{k+1}} \\ & \in & & \\ & \varphi & \longmapsto & \delta\varphi(\alpha) = \sum_{\beta \in \Gamma_k} \underbrace{\deg(q_\beta f_\alpha)}_{\text{non-zero only finitely many times by compactness of } S^k} f(\beta) \end{array}$$

<u>Example:</u> $\mathbb{RP}^n$: cell str. w/ one cell in each deg. $\leq n$

$C_\bullet^{cell}$:

$$\cdots 0 \to \overset{n}{\mathbb{Z}} \to \cdots \xrightarrow{0} \overset{2}{\mathbb{Z}} \xrightarrow{2} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}} \to 0$$

$C^\bullet_{cell}$: $\mathbb{Z}$ in deg.s $0, -1, \dots, -n$

$$\overset{1}{0} \to \overset{0}{\mathbb{Z}} \xrightarrow{0} \overset{-1}{\mathbb{Z}} \xrightarrow{2} \overset{-2}{\mathbb{Z}} \xrightarrow{0} \overset{-3}{\mathbb{Z}} \xrightarrow{2} \cdots \longrightarrow \overset{-n}{\mathbb{Z}} \to 0$$

$$Z_i(C^\bullet_{cell}) = \begin{cases} \mathbb{Z}, & i \text{ even}, \ 0 \geq i \geq -n \ \& \ -n \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

$$B_i(C^\bullet_{cell}) = \begin{cases} 0, & i = 0 \ \& \ i \text{ odd} \\ 2\mathbb{Z}, & i \text{ even}, \ 0 > i \geq -n \end{cases}$$

$$H^k \mathbb{RP}^n = \begin{cases} \mathbb{Z}, & k = 0, \ n \text{ odd} \\ \mathbb{Z}/2, & k \text{ even}, 0 < k \leq n \\ 0, & \text{otherwise} \end{cases}$$

Homology of products

$X, Y \in \mathrm{Top}$ want to understand $H_*(X \times Y)$ in terms of $H_* X$ & $H_* Y$

Step 1: Define $\otimes$ of chain cxs & give alg'c description $H_*(C_\bullet \otimes D_\bullet)$ in terms of $H_* C$, $H_* D$ if $C_\bullet$ is levelwise free

Step 2: $\exists$ natural chain ht.py eq.ce between $S_\bullet(X \times Y)$ and $S_\bullet X \otimes S_\bullet Y$ (Eilenberg-Zilber thm.)

(This is also the main input to get products in H^*.)

Tensor Products of Chain Complexes

A_*, B_* graded abelian groups

(abelian groups A_n, $n \in \mathbb{Z}$)

Want $A_* \otimes B_*$ where $a \in A_p$, $b \in B_q$ $\rightsquigarrow$ $a \otimes b$ in degree $p+q$

$\rightsquigarrow$ $$(A_* \otimes B_*)_n = \bigoplus_{p+q=n} A_p \otimes B_q$$

Basic properties of $\otimes$ carry over

$C_\cdot, D_\cdot$ chain complexes

Define $C_\cdot \otimes D_\cdot$ by $(C_\cdot \otimes D_\cdot)_n := \bigoplus\limits_{p+q=n} C_p \otimes D_q$

<u>Note:</u> An element in $(C \otimes D)_n$ can be (non-uniquely) written as a finite sum $\sum\limits_{i=1}^{N} c_i \otimes d_i$

$c_i \in C_{p_i}$, $d_i \in D_{q_i}$

$p_i + q_i = n$

$\partial : (C_\cdot \otimes D_\cdot)_n \longrightarrow (C_\cdot \otimes D_\cdot)_{n-1}$

on $c \otimes d$ by $\partial(c \otimes d) = \partial_C c \otimes d + (-1)^p c \otimes \partial_D d$

$c \in C_p$ $d \in D_q$

$$\partial^2(c \otimes d) = \partial\left(\partial c \otimes d + (-1)^p c \otimes \partial d\right)$$

$$= \underbrace{\cancel{\partial^2 c}}_{0} \otimes d + \underbrace{(-1)^{p-1} \partial c \otimes \partial d + (-1)^p \partial c \otimes \partial d}_{\text{cancel}} + (-1)^p (-1)^p c \otimes \underbrace{\cancel{\partial^2 d}}_{0}$$

$$= 0$$

cancel — need sign to get chain cx.

<u>Notation:</u> $M \in \mathrm{Ab}$ write $M[n]$ for the gr. ab. group / ch. cx.

$$
\begin{array}{ll}
\vdots & \\
0 & \\
\downarrow & \\
M & - \text{deg. } n \\
\downarrow & \\
0 & \\
\vdots &
\end{array}
$$

Have associativity: $C_\bullet \otimes (D_\bullet \otimes E_\bullet) \cong (C_\bullet \otimes D_\bullet) \otimes E_\bullet$

unit: $C_\bullet \otimes \mathbb{Z}[0]$ in deg. n: $\displaystyle\bigoplus_{p+q=n} C_p \otimes (\mathbb{Z}[0])_q = C_n \otimes \mathbb{Z}$ $(q=0)$

$\otimes$ on Ab was also symmetric: $A \otimes B \cong B \otimes A$

Symmetry for Ch: $\tau : C_\bullet \otimes D_\bullet \xrightarrow{\sim} D_\bullet \otimes C_\bullet$ by

$$\tau(c \otimes d) = (-1)^{pq}\, d \otimes c$$

$c \in C_p,\ d \in D_q$

Need sign for τ to be a chain map:

$$\tau(\partial(c \otimes d)) = \tau(\partial c \otimes d + (-1)^p c \otimes \partial d)$$

$$= \underbrace{(-1)^{(p-1)q}}_{\overset{\shortparallel}{(-1)^{pq-q}}} d \otimes \partial c + \underbrace{(-1)^p (-1)^{p(q-1)}}_{(-1)^{pq}} \partial d \otimes c$$

$$\partial \tau(c \otimes d) = (-1)^{pq} \partial(d \otimes c) = \underline{(-1)^{pq}}\, \partial d \otimes c + \underbrace{(-1)^{pq}(-1)^{q}}_{(-1)^{pq+q}} d \otimes \partial c$$

— would not agree w/o some sign to compensate for sign from ∂

Convention: We use same sign as symmetry for $\otimes$ of grAb so that taking underlying abelian groups is compatible w/ symmetry

Exercise: $-\otimes C_\bullet : Ch \to Ch$ preserves chain ht.pies & levelwise splittable SESs.

If $C_\bullet$ is levelwise free it preserves all SESs.

Exercise: $I_\bullet = $

$$\begin{array}{cc} \vdots & \\ 0 & \\ \downarrow & \\ \mathbb{Z} & 1 \\ \partial \downarrow & \\ \mathbb{Z}\{[0],[1]\} & 0 \\ \downarrow & \\ 0 & \\ \vdots & \end{array} \qquad \partial(1) = [1] - [0]$$

Chain map $C_\bullet \otimes I_\bullet \to D_\bullet \longleftrightarrow$ two chain maps $C_\bullet \to D_\bullet$ and chain ht.py between them

Lemma: $\exists$ natural map $H_*C \otimes H_*D \longrightarrow H_*(C \otimes D)$

$\otimes$ of gr Ab

compatible w/ associativity, unit, symmetry isomorphisms

Proof: $[c] \in H_p C$, $[d] \in H_q D$, $[c] \otimes [d] \longmapsto [c \otimes d]$

rep. by cycle $c \in C_p$ rep. by cycle $d \in D_q$

$$\partial(c \otimes d) = \partial c \otimes d + (-1)^p c \otimes \partial d = 0 \quad \text{so cycle}$$

$$(c + \partial c') \otimes d = c \otimes d + \partial c' \otimes d = c \otimes d + \partial(c' \otimes d)$$

$$\partial(c' \otimes d) = \partial c' \otimes d \quad \text{as } \partial d = 0$$

So well-defined, independent of choice of representatives. $\square$

Defn: $A_*, B_* \in grAb$, $\quad \mathrm{Tor}(A,B)_n = \bigoplus_{p+q=n} \mathrm{Tor}(A_p, B_q)$

Propn: $C_\bullet, D_\bullet \in Ch$, $\quad C_\bullet$ is levelwise free.

$\exists$ natural SESs

$$0 \to (H_*C \otimes H_*D)_n \to H_n(C \otimes D) \to \mathrm{Tor}(H_*C, H_*D)_{n-1} \to 0$$

Note: Specializes alg.c UCT if $D_\bullet = M[0]$, $M \in Ab$

Lemma: $C_\bullet$ levelwise free chain cx. w/ zero differentials.

Then $C_* \otimes H_* D \cong H_* C \otimes H_* D \longrightarrow H_*(C \otimes D)$ is an isomor.

Proof: For $c \in C_p$, $d \in D_q$ then $\partial(c \otimes d) = (-1)^p c \otimes \partial d$

$C_\bullet$ levelwise free $\Rightarrow C_n \otimes -$ preserves SESs,

so $$Z_n(C \otimes D) \cong \bigoplus_{p+q=n} C_p \otimes Z_q D$$

$$B_n(C \otimes D) \cong \bigoplus_{p+q=n} C_p \otimes B_q D$$

$$H_n(C \otimes D) \cong \bigoplus_{p+q} C_p \otimes H_q D = (C_* \otimes H_* D)_n. \quad \square$$

Proof of Propn.:

$Z_k = Z_k C$, $B_k = B_k C$, $H_k = H_k C$

Hence SES $0 \to B_k \xrightarrow{j_k} Z_k \to H_k \to 0$ free resolution (as C levelwise free)

$$0 \to Z_k \to C_k \xrightarrow{\partial'} B_{k-1} \to 0$$
$$B_{k-1} = B'_k$$

$Z_\bullet, B'_\bullet$ as ch. cx.s w/ diff. 0 — & levelwise free

$0 \to Z_\bullet \to C_\bullet \to B'_\bullet \to 0$ SES of ch. cx.s, levelwise splittable.

Get SES of ch. cx.s

$$0 \to Z_\bullet \otimes D_\bullet \to C_\bullet \otimes D_\bullet \to B'_\bullet \otimes D_\bullet \to 0$$

apply Lemma to compute H_*

LES in homology is

$$\cdots \xrightarrow{\Delta_n} (Z_* \otimes H_* D)_n \longrightarrow H_n(C \otimes D) \to (B'_* \otimes H_* D)_n \xrightarrow{\Delta_{n-1}} (Z_* \otimes H_* D)_{n-1} \to \cdots$$

Have SESs centred at $H_n(C \otimes D)$

$$0 \to \operatorname{coker} \Delta_n \longrightarrow H_n(C \otimes D) \longrightarrow \ker \Delta_{n-1} \to 0$$

Unwinding defn of Δ_n we get:

$$\Delta_n : \bigoplus_{p+q=n} B_p \otimes H_q D \longrightarrow \bigoplus_{p+q=n} Z_p \otimes H_q D \text{ is } (j \otimes id)_n = \bigoplus_{p+q=n} j_p \otimes id_{H_q D}$$

$\Rightarrow$ coker $\Delta_n = (H_*C \otimes H_*D)_n$

ker $\Delta_n = \mathrm{Tor}(H_*C, H_*D)_n$

Get SES

$$0 \to (H_*C \otimes H_*D)_n \to H_n(C \otimes D) \to \mathrm{Tor}(H_*C, H_*D)_{n-1} \to 0.$$

$\square$

<u>Remark:</u> Can show these SESs are non-canonically splittable

– have (non-natural) isomorphisms

$$H_n(C \otimes D) \cong (H_*C \otimes H_*D)_n \oplus \mathrm{Tor}(H_*C, H_*D)_{n-1}.$$

<u>Remark:</u> Works for ch. cxs of R-modules for any PID R

e.g. if k is a field get $H_*(C \otimes_k D) \cong H_*C \otimes_k H_*D$.