Hand in exercise (test examination) for FY2450 Astrophysics, spring 2010

Hand in for correction is not obligatory, but is recommended.
It may be done at lectures, in my office (Jan Myrheim, room E5-102), or in my mailbox at the Department of Physics, 3rd floor.
You may give partial answers, and do so several times.

A good general advice at the examination (and elsewhere) is that it pays off to do good and conscientious work. Explain clearly how you solve the problems, and check your units.
Correct answers give full score, but a wrong answer with a good explanation is rewarded better than a wrong answer without explanation.
Think about the precision of numerical answers, a rule of thumb is to give just enough digits that the last digit is uncertain.

1) As of March 22, 2010, 442 planets have been observed outside our own solar system. Some of them, and the stars they are orbiting, are listed in tables below.
$R_\odot$ is the solar radius and $M_\odot$ the solar mass. The astronomical unit is the semimajor axis of the orbit of the Earth around the Sun. The surface temperature of a star is the effective temperature, assuming that the star radiates like a black body.

Star	Distance parsec	Spectral class, surface temperature		Radius $R_\odot$	Mass $M_\odot$
Fomalhaut	7.7	A3 V	8540 K	1.82	2.06
HD 160691	15.3	G3 IV-V	5700 K	1.25	1.08
61 Vir	8.52	G5 V	5531 K	0.94	0.95
Gliese 581	6.26	M3 V	3480 K	0.38	0.31
Gliese 876	4.7	M4 V	3350 K	0.36	0.33

Star	Planet	Mass Jupiter masses	Semimajor axis Astronomical Units	Eccentricity
Fomalhaut	b	< 3	115	0.11
HD 160691	c	0.03321	0.09094	0.17
HD 160691	d	0.5219	0.921	0.067
HD 160691	b	1.676	1.50	0.128
HD 160691	e	1.814	5.235	0.10
61 Vir	b	0.016	0.050201	0.12
61 Vir	c	0.057	0.2175	0.14
61 Vir	d	0.072	0.476	0.35
Gliese 581	e	0.00610	0.03	0
Gliese 581	b	0.0492	0.041	0
Gliese 581	c	0.01686	0.07	0.17
Gliese 581	d	0.02231	0.22	0.38
Gliese 876	d	0.02	0.021	0.14
Gliese 876	c	0.83	0.132	0.266
Gliese 876	b	2.64	0.211	0.029

Explain briefly the principle for how to detect the presence of planets in orbit around a star by studying the spectrum of the star and using the Doppler effect.
Derive the following formula for the temperature T_p on a planet at a distance d from a star, when the star has surface temperature T_s and radius R_s,

$$T_p = T_s \sqrt{\frac{R_s}{2d}} \sqrt[4]{\frac{1 - a_v}{1 - a_{\mathrm{ir}}}} \; . \tag{1}$$

The planet has albedo a_v for visible light, i.e. for the light from the star, and albedo a_{ir} for infrared radiation, which is radiated from the planet.
Assume (for lack of a better assumption) that $a_v = a_{\mathrm{ir}}$, and use the formula for calculating the surface temperature on the planets listed in the table.
Is it necessary to take into account that some of the planets have rather eccentric elliptical orbits?
Would you choose some of these planets as destinations for a holiday trip?
Compute both absolute and apparent magnitudes for the five stars in the table, given that the Sun has an absolute magnitude of 4.8 and a surface temperatur of 5762 K.
Which of the stars can be seen without a telescope?

2) Normal atmospheric pressure at sea level is (per definition) one atmosphere.
A pressure of one atmosphere is almost exactly 10^5 pascal = 10^5 N/m^2 (normal atmospheric pressure is defined as 101 325 pascal).
What is the mass of the column of air above 1 m^2 of the surface of the Earth?
What is the total mass of the Earth's atmosphere?
How large a fraction is that of the mass of the Earth?

3) Let us try to use the equations that are used for computing the internal structure of the Sun on the atmosphere of the Earth.
What is the pressure at the summit of Galdhøpiggen (2469 meter above sea level)?
At the summit of Mont Blanc (4807 m)?
At the summit of Chomolungma (better known as Mount Everest, 8848 m)?
Assume that air is an ideal gas, that the temperature T is constant, and that the acceleration of gravity g is constant and equal to 9.8 m/s^2.
Is it reasonable to assume that the temperature is constant?
In problem 5) below a different way of computing is proposed.
The equation of state for an ideal gas is

$$P = \frac{\rho k_B T}{\overline{m}} \; , \tag{2}$$

where P is the pressure, ρ is the mass density, $\overline{m}$ is the average mass of the gas particles, and k_B is Boltzmann's constant.
The condition for hydrostatic equilibrium is the equation

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\rho g \; , \tag{3}$$

where z is the altitude (measured from the sea level, or from the centre of the Earth).
Solve the problem e.g. by deducing a relation between P and z of the form

$$P = P_0 \, \mathrm{e}^{-\alpha(z - z_0)} \; , \tag{4}$$

where α, P_0 and z_0 are constants.

4) The atmospheric pressure on Mars is variable, typically 0.6 % of the atmospheric pressure on the Earth.
If this is the pressure at the lowest point on the surface of Mars, what is then the pressure at the highest point, the summit of the volcano Olympus Mons, which is 31 km higher?

5) Foehn wind (a warm wind) occurs e.g. in Trondheim as a strong wind from the south. If moist air comes from the south and rises to pass the mountain of Dovre, the water vapour in the air may fall down as rain or snow south of Dovre, and the heat liberated when the vapour condenses, heats the air (or causes it to cool less).
What is the temperature difference between a point on Dovre which is 1100 meter above sea level, and a point in Trondheim 100 meter above sea level, if the southern wind is so strong that the air has not time enough to cool, and the adiabatic equation of state

$$\frac{P}{\rho^\gamma} = \text{constant} \tag{5}$$

holds? The adiabatic index γ is 7/5 for air, which is a diatomic gas, N_2 and O_2.
Assume as before that air is an ideal gas, and that hydrostatic equilibrium holds (approximately), so that you may use the equations (2) and (3), with the only difference that the temperature is no longer constant.
It is well known that local climate is strongly dependent on the altitude above sea level, and a rule of thumb says that the mean temperature decreases by one degree Celsius for every 120 meter above sea level.
Compare with your answer to this problem. Comment?

6a) Denote by $\vec{g}$ the acceleration of gravity in a gravitational field. A (small) mass m falls in the gravitational field with the acceleration $\vec{g}$, which does not depend on the mass m, but may vary with time t and position $\vec{r}$. Thus, $\vec{g} = \vec{g}(\vec{r}, t)$ in general. The acceleration of gravity is minus the gradient of the gravitational potential $\phi = \phi(\vec{r}, t)$,

$$\vec{g} = -\nabla\phi \; . \tag{6}$$

The gravitational potential is determined from the mass density $\rho = \rho(\vec{r}, t)$ by Poisson's equation,

$$\nabla^2\phi = -\nabla \cdot \vec{g} = 4\pi G\rho \; , \tag{7}$$

where G is Newton's gravitational constant. We usually reqire that ϕ should satisfy the boundary condition that $\phi(\vec{r}, t) \to 0$ when $|\vec{r}| \to \infty$.

Show that if the gravitational field is rotationally symmetric about the origin, that is, if $\phi(\vec{r}, t) = \phi(r, t)$ where $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$, then

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) . \tag{8}$$

One way to prove this is as follows. Show that

$$\frac{\partial r}{\partial x} = \frac{x}{r} \; , \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \; , \qquad \frac{\partial r}{\partial z} = \frac{z}{r} \; . \tag{9}$$

Then compute

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{10}$$

by using the chain rule for differentiation, for example that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial x} \ . \tag{11}$$

6b) Assume that a spherical planet has radius R, constant mass density $\rho = \rho_0$, and total mass

$$M = \rho_0 \, \frac{4\pi R^3}{3} \ . \tag{12}$$

Choose a coordinate system with its origin at the centre of the planet. We assume that the gravitational potential is time independent and depends only on the radius r, that is, $\phi(\vec{r}, t) = \phi(r)$.

Solve Poisson's equation $\nabla^2 \phi = 4\pi G \rho$ with the boundary condition $\phi(r) \to 0$ when $r \to \infty$.

Proceed, for example, as follows.
The equation to be solved is the second order ordinary differential equation

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}\phi}{\mathrm{d}r} \right) = \begin{cases} 4\pi G \rho_0 & \text{for} \quad 0 < r < R \ , \\ 0 & \text{for} \quad r > R \ . \end{cases} \tag{13}$$

Find first the general solution, with two integration constants, of the equation

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}\phi}{\mathrm{d}r} \right) = C \qquad \text{where } C \text{ is constant.} \tag{14}$$

In the region $r < R$ there are singular solutions, with $\phi(r) \to \pm\infty$ when $r \to 0$, we reject those because they correspond to a point mass at the origin, and point masses exist only in textbooks on theoretical physics.
In the region $r > R$ we reject those solutions that do not go to zero at infinity.
Then we have to join the solutions for $r < R$ and for $r > R$ in such a way that both $\phi(r)$ and the derivative $\phi'(r)$ are continuous at $r = R$. Why must $\phi(r)$ and $\phi'(r)$ be continuous at $r = R$?

6c) The mass inside a radius r is

$$M_r(r) = \int_{|\vec{r}\,'| \le r} \mathrm{d}^3\vec{r}\,' \ \rho(\vec{r}\,') = \begin{cases} \rho_0 \, \dfrac{4\pi r^3}{3} & \text{for} \quad r \le R \ , \\ \rho_0 \, \dfrac{4\pi R^3}{3} = M & \text{for} \quad r \ge R \ . \end{cases} \tag{15}$$

Show that the acceleration of gravity $g = |\vec{g}| = |\nabla \phi|$ at a distance r from the centre of the planet is

$$g(r) = \frac{G M_r(r)}{r^2} \ . \tag{16}$$

Note that $g(r)$, both inside and outside the planet, depends only on the mass $M_r(r)$ inside the radius r, and is the same as if all this mass was located at the centre.

6d) The condition for hydrostatic equilibrium in a gas or a liquid is that the pressure P varies with position such that

$$\nabla P = \rho \vec{g} = -\rho \nabla \phi \,, \tag{17}$$

where ρ is the mass density and $\vec{g} = -\nabla \phi$ the acceleration of gravity. Compare with equation (3).

If the temperature T is the same everywhere, then we may write the equation of state for an ideal gas, equation (2), as

$$P = K\rho \tag{18}$$

where $K = k_B T / \overline{m}$ is constant. We then get the equation

$$\nabla \phi = -\frac{\nabla P}{\rho} = -\frac{K}{\rho} \nabla \rho \,, \tag{19}$$

which may be integrated once to give that

$$\phi_0 - \phi = K \ln\left(\frac{\rho}{\rho_0}\right) , \tag{20}$$

where ϕ_0 and ρ_0 are integration constants.
Poisson's equation $\nabla^2 \phi = 4\pi G \rho$ then becomes an equation for the gravitational potential ϕ alone,

$$\nabla^2 \phi = 4\pi G \rho_0 \, \mathrm{e}^{\frac{\phi_0 - \phi}{K}} \,. \tag{21}$$

This equation has a solution of the form

$$\phi(r) = \phi_0 + A \ln\left(\frac{r}{r_0}\right) \,, \tag{22}$$

with constants A and r_0. Which values must these constants take?
Is this solution meaningful, from a physical point of view? Do there exist other solutions?

6e) The adiabatic equation of state for an ideal gas,

$$P = K\rho^\gamma \tag{23}$$

where K and γ are constants, holds in a region where the heat transport takes place by convection, for example inside a star, or in the atmosphere of a planet. Convection regions where the equation holds are for example from the surface of the Sun and down to 70 % of the solar radius, or everywhere inside a main sequence star of mass less than 0.3 solar masses. The same equation of state holds also inside a white dvarf star, where the pressure is due to a degenerate electron gas.

The adiabatic index γ is the ratio between the heat capacities at constant pressure and at constant volume. For a non-relativistic monatomic gas we have

$$\gamma = \frac{5}{3} = 1 + \frac{1}{\nu} \qquad \text{with} \qquad \nu = \frac{3}{2} \; . \tag{24}$$

The quantity ν is called the *polytropic index*.
The equation for hydrostatic equilibrium together with the adiabatic equation of state gives that

$$\nabla\phi = -\frac{\nabla P}{\rho} = -K\gamma\rho^{\gamma-2}\nabla\rho \; . \tag{25}$$

This equation may be integrated once, to give the equation

$$\phi_0 - \phi = K\frac{\gamma}{\gamma-1}\,\rho^{\gamma-1} = K(\nu+1)\,\rho^{\frac{1}{\nu}} \; . \tag{26}$$

The integration constant ϕ_0 here is the gravitational potential at the surface of the star, where $\rho \to 0$.
We now write P_c for the pressure and ρ_c for the density at the centre of the star, and we introduce a dimensionless variable Θ such that

$$\rho = \rho_c\Theta^\nu \; , \qquad P = K\rho^\gamma = K{\rho_c}^\gamma\Theta^{\nu\gamma} = P_c\Theta^{\nu+1} \; . \tag{27}$$

Then we get

$$\phi_0 - \phi = K(\nu+1)\,{\rho_c}^{\frac{1}{\nu}}\,\Theta = K(\nu+1)\,{\rho_c}^{\gamma-1}\,\Theta = \frac{(\nu+1)P_c}{\rho_c}\,\Theta \; . \tag{28}$$

Poisson's equation $\nabla^2\phi = 4\pi G\rho$ now gives the *Lane–Emden equation*,

$$\nabla^2\Theta = -\frac{\rho_c}{(\nu+1)P_c}\,\nabla^2\phi = -\frac{4\pi G{\rho_c}^2}{(\nu+1)P_c}\,\Theta^\nu \; . \tag{29}$$

When we assume rotational symmetry and time independenc, so that $\Theta = \Theta(r)$ and

$$\nabla^2\Theta = \frac{\mathrm{d}^2\Theta}{\mathrm{d}r^2} + \frac{2}{r}\,\frac{\mathrm{d}\Theta}{\mathrm{d}r} = \frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Theta}{\mathrm{d}r}\right) \; , \tag{30}$$

and when we at the same time introduce the dimensionless radial coordinate

$$\xi = \frac{r}{r_0} \qquad \text{where} \qquad r_0 = \sqrt{\frac{(\nu+1)P_c}{4\pi G{\rho_c}^2}} \; , \tag{31}$$

then we get the Lane–Emden equation on the dimensionless form

$$\frac{1}{\xi^2}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\,\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right) + \Theta^\nu = 0 \; . \tag{32}$$

The solution has a Taylor series expansion around $\xi = 0$. When the polytropic index ν is $3/2$, which is the interesting case to us here, the expansion gives that

$$\Theta = 1 - \frac{\xi^2}{6} + \frac{\xi^4}{80} - \frac{\xi^6}{1440} + \frac{\xi^8}{31\,104} - \frac{19\,\xi^{10}}{14\,256\,000} + \cdots \; . \tag{33}$$

We defined Θ such that $\Theta(0) = 1$. Table 1 gives the numerical solution of the equation for $\nu = 3/2$. We see from the table that $\Theta(\xi) = 0$ for $\xi = \xi_1 = 3.653\,753\,74\ldots$. Unfortunately, the series expansion becomes too imprecise as we approach the zero point ξ_1, so that we have to use for example numerical integration in order to solve the equation.
We have now found the radius of the star as a function of the central density ρ_c and the central pressure P_c,

$$R = \xi_1 r_0 = 3.654\,\sqrt{\frac{5P_c}{8\pi G\rho_c^{\,2}}}\ . \tag{34}$$

In order to compute the mass M we may compute the acceleration of gravity at the surface of the star, $g(R) = GM/R^2$. In general we have for $r \leq R$ that

$$g = g(r) = |\nabla\phi| = \frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{(\nu+1)P_c}{\rho_c}\frac{\mathrm{d}\Theta}{\mathrm{d}r} = -\frac{(\nu+1)P_c}{\rho_c\, r_0}\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} = -\frac{(\nu+1)P_c}{\rho_c\, r_0}\,\Theta'(\xi)\ . \tag{35}$$

Show that the mass may be expressed as

$$M = 4\pi\rho_c r_0^{\,3}(-\xi_1^{\,2}\Theta'(\xi_1))\ . \tag{36}$$

According to Table 1 we have $-\xi_1^{\,2}\Theta'(\xi_1) = 2.714\,055$.

6f) The Sun has radius $R = 7.0 \times 10^5$ km and mass $M = 2.0 \times 10^{30}$ kg.
The equations (34) and (36) are not necessarily valid for the Sun. Use them anyhow to estimate the central pressure P_c and the central density ρ_c of the Sun.
Compare with the mean density

$$\overline{\rho} = \frac{3M}{4\pi R^3}\ . \tag{37}$$

Compare also with the following results obtained from detailed solar models,
$P_c = 2.342 \times 10^{16}$ N/m^2 and $\rho_c = 1.527 \times 10^5$ kg/m^3. Comment?

6g) Use the above values for P_c and ρ_c for the Sun in order to compute the central temperature T_c, under the assumption that the equation of state for an ideal gas holds.
At the centre of the Sun, 34 % of the mass is hydrogen (about half the original value of 71 %), and 64 % of the mass is helium (the original helium fraction was 27 %).

Sphere of polytropic index n = 1.5

ξ	θ	θ'	θ^{n+1}	$\xi^2\theta'$
0.000E+00	1.000000E+00	0.000000E+00	1.000000E+00	0.000000E+00
1.000E-01	9.983346E-01	-3.328337E-02	9.958417E-01	-3.328337E-04
2.000E-01	9.933533E-01	-6.626800E-02	9.834660E-01	-2.650720E-03
3.000E-01	9.851007E-01	-9.866007E-02	9.631671E-01	-8.879406E-03
4.000E-01	9.736505E-01	-1.301756E-01	9.354223E-01	-2.082809E-02
5.000E-01	9.591039E-01	-1.605449E-01	9.008741E-01	-4.013622E-02
6.000E-01	9.415881E-01	-1.895169E-01	8.603050E-01	-6.822610E-02
7.000E-01	9.212547E-01	-2.168630E-01	8.146092E-01	-1.062629E-01
8.000E-01	8.982765E-01	-2.423798E-01	7.647600E-01	-1.551231E-01
9.000E-01	8.728456E-01	-2.658923E-01	7.117763E-01	-2.153728E-01
1.000E+00	8.451698E-01	-2.872555E-01	6.566892E-01	-2.872555E-01
1.100E+00	8.154699E-01	-3.063557E-01	6.005095E-01	-3.706904E-01
1.200E+00	7.839768E-01	-3.231109E-01	5.441994E-01	-4.652797E-01
1.300E+00	7.509276E-01	-3.374711E-01	4.886470E-01	-5.703261E-01
1.400E+00	7.165631E-01	-3.494173E-01	4.346464E-01	-6.848578E-01
1.500E+00	6.811243E-01	-3.589602E-01	3.828829E-01	-8.076604E-01
1.600E+00	6.448499E-01	-3.661387E-01	3.339232E-01	-9.373150E-01
1.700E+00	6.079733E-01	-3.710173E-01	2.882115E-01	-1.072240E+00
1.800E+00	5.707202E-01	-3.736839E-01	2.460697E-01	-1.210736E+00
1.900E+00	5.333066E-01	-3.742469E-01	2.077029E-01	-1.351031E+00
2.000E+00	4.959368E-01	-3.728321E-01	1.732071E-01	-1.491329E+00
2.100E+00	4.588015E-01	-3.695799E-01	1.425811E-01	-1.629847E+00
2.200E+00	4.220770E-01	-3.646419E-01	1.157389E-01	-1.764867E+00
2.300E+00	3.859239E-01	-3.581784E-01	9.252398E-02	-1.894764E+00
2.400E+00	3.504866E-01	-3.503553E-01	7.272415E-02	-2.018046E+00
2.500E+00	3.158926E-01	-3.413414E-01	5.608524E-02	-2.133383E+00
2.600E+00	2.822524E-01	-3.313061E-01	4.232472E-02	-2.239629E+00
2.700E+00	2.496598E-01	-3.204174E-01	3.114380E-02	-2.335843E+00
2.800E+00	2.181919E-01	-3.088400E-01	2.223804E-02	-2.421306E+00
2.900E+00	1.879094E-01	-2.967337E-01	1.530634E-02	-2.495531E+00
3.000E+00	1.588576E-01	-2.842527E-01	1.005820E-02	-2.558275E+00
3.100E+00	1.310664E-01	-2.715447E-01	6.219118E-03	-2.609544E+00
3.200E+00	1.045515E-01	-2.587507E-01	3.534484E-03	-2.649608E+00
3.300E+00	7.931464E-02	-2.460063E-01	1.771672E-03	-2.679009E+00
3.400E+00	5.534424E-02	-2.334426E-01	7.205782E-04	-2.698597E+00
3.500E+00	3.261573E-02	-2.211909E-01	1.921179E-04	-2.709588E+00
3.600E+00	1.109099E-02	-2.093927E-01	1.295467E-05	-2.713729E+00
3.620E+00	6.926088E-03	-2.071024E-01	3.992270E-06	-2.713953E+00
3.630E+00	4.860744E-03	-2.059675E-01	1.647240E-06	-2.714013E+00
3.640E+00	2.806714E-03	-2.048397E-01	4.173453E-07	-2.714044E+00
3.650E+00	7.639242E-04	-2.037196E-01	1.612968E-08	-2.714055E+00
3.652E+00	3.567080E-04	-2.034966E-01	2.403157E-09	-2.714055E+00
3.653E+00	1.532672E-04	-2.033852E-01	2.908190E-10	-2.714055E+00
3.65375374E+00	0.000000E+00	-2.033013E-01	0.000000E+00	-2.714055E+00

Table 1: Spherically symmetric solution of the Lane–Emden equation for polytropic index $\nu = 3/2$.