Notes on astrophysics

For the course FY 2450

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Preface

These notes supplement the text book ("Astronomy: A Physical Perspective", by Marc L. Kutner) and the lecture notes by Michael Kachelrieß for the basic astrophysics course FY 2450. They are not required reading, it should be enough to read the text book in order to pass the examination.

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Chapter 1

Laws of motion

1.1 Newton's laws

Newton's first law

Newton's laws of motion hold in special refence frames called *inertial frames.* By definition, our meter sticks and clocks are located in an inertial frame if we observe that Newton's first law holds:

> *A body remains at rest, or moves in a straight line at constant speed, unless acted upon by a net outside force.*

The first law may be regarded as a special case of the second law: if there is no net force, there is also no acceleration.

In a terrestrial laboratory at rest on the ground we observe that bodies tend to fall to the ground. This does not necessarily mean that the laboratory is not an inertial frame, since we explain the observed downwards acceleration as the effect of an outside force, the gravitational pull of the Earth. We may compensate this force by pulling or pushing in the opposite direction, but we can not eliminate it, without removing the whole Earth, or putting our laboratory in free fall. Nevertheless, when due account is taken of the gravitational force, our laboratory on the ground is a reasonably good inertial frame.

It is not a perfect inertial frame, however, because of the rotation of the Earth. If we want to use Newton's law of motion in the laboratory *as if* it were an inertial frame, we may have to introduce two extra forces, the centrifugal force and the Coriolis force, in addition to the gravitational force from the Earth, to account for the observed deviations from straight line motion. The centrifugal and Coriolis forces are not accepted as forces in Newton's sense of the word, because they do not satisfy Newton's third law. They are called *fictitious forces*, and a reference frame in which fictitious forces appear is not an inertial frame.

We obtain a more nearly perfect inertial frame by doing experiments in free fall, for example inside a space station orbiting the Earth, making sure that the space station does not rotate. Even in this laboratory we might have to take into account residual gravitational forces, so called *tidal forces*, if we do extremely precise experiments. They are due to the fact that the gravitational field of the Earth is not perfectly homogeneous: there is a tiny variation in the size and direction of the gravitational force, even over the small distance from one point to another inside a space station.

Newton's second law

Newton's second law is the equation of motion for a point-like particle,

$$\boldsymbol{F} = m\boldsymbol{a} \; . \tag{1.1}$$

Here m is the mass (the *inertial mass*) of the particle, and $\boldsymbol{F}$ is the force acting on it. Let $\boldsymbol{r} = \boldsymbol{r}(t)$ denote the time dependent position of the particle, then $\boldsymbol{v} = \dot{\boldsymbol{r}} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ is the velocity, and $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}$ is the acceleration. Each dot over a symbol denotes a differentiation with respect to the time t.

Note that Newton's second law is a vector equation, it consists of three separate equations, one for each of the x, y, z components,

$$F_x = ma_x = m\ddot{x} \; , \qquad F_y = ma_y = m\ddot{y} \; , \qquad F_z = ma_z = m\ddot{z} \; . \tag{1.2}$$

Newton's third law

Newton's third law is the law of action and reaction:

> *Whenever one body exerts a force on a second body, the second body exerts an equal and opposite force on the first body.*

If the force on the second body from the first one is $\boldsymbol{F}_{21}$, and the force on the first body from the second one is $\boldsymbol{F}_{12}$, Newton's third law is the relation

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21} \; . \tag{1.3}$$

An important point which is not always clearly stated is that all forces observed in nature are two-body forces. If there are more than two bodies present, then the force between two of them does not depend on the presence of the others, and the force on one body is the sum of the two-body forces from all the others. Hence, Newton's third law implies that the sum of all *internal forces* in a physical system, i.e. all the forces between the particles in the system, is zero.

Fictitious forces

It is sometimes convenient to work in an accelerated coordinate system, for example following the rotation of the Earth. Then we are no longer in an inertial system, and we have to modify Newton's second law by including what we call non-Newtonian, or fictitious, forces. A fictitious force has no reaction force of equal magnitude and opposite direction.

The fictitious forces in a rotating coordinate system are the centrifugal and Coriolis forces. Let $\boldsymbol{\Omega}$ be a vector along the rotation axis, for example the rotation axis of the Earth, such that $\Omega = |\boldsymbol{\Omega}|$ is the angular velocity (rotation angle divided by time). Also let $\boldsymbol{r}_0$ be a point on the rotation axis, for example the centre of mass of the Earth. Assume that a pointlike particle of mass m is located at the position $\boldsymbol{r}$ and moving with the velocity $\boldsymbol{v}$. Then the centrifugal force is

$$\boldsymbol{F}_{\mathrm{cf}} = -m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\boldsymbol{r} - \boldsymbol{r}_0)) \; . \tag{1.4}$$

The Coriolis force,

$$\boldsymbol{F}_{\mathrm{C}} = -2m\,\boldsymbol{\Omega} \times \boldsymbol{v}\ , \tag{1.5}$$

is proportional to the velocity. These expressions are derived in Appendix B.

Introducing the Coriolis force is one possible way to explain the observation made by Foucault in 1851, that when a pendulum oscillates during several hours, its plane of oscillation rotates. The oscillation plane of a pendulum on the North Pole would be fixed relative to the distant stars, so that relative to the Earth it would rotate by 360 degrees during 24 hours (or more precisely 23 hours and 56 minutes, which is the period of rotation of the Earth relative to the fixed stars).

Einstein based his general theory of relativity on the postulate that the fictitious forces present in non-inertial reference frames are in principle no different from the more "respectable" gravitational force. Thus, the special status of inertial frames is to some degree a matter of convention. In the general theory of relativity physical laws have to be formulated mathematically in such a way that they have the same mathematical form in all reference frames, not only in inertial frames.

Newton's law of universal gravitation

The gravitational force on a pointlike particle of mass m_1 from another pointlike particle of mass m_2 is

$$\boldsymbol{F}_{12} = -\frac{Gm_1m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2)\ . \tag{1.6}$$

Here $G = 6.6726 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ is Newton's gravitational constant.

This law agrees with Newton's third law, since it predicts that the force on the second particle from the first one is

$$\boldsymbol{F}_{21} = -\frac{Gm_2m_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}\,(\boldsymbol{r}_2 - \boldsymbol{r}_1) = -\boldsymbol{F}_{12}\ . \tag{1.7}$$

The gravitational force is a *central force*, that is, its direction is precisely towards the attracting particle. The absolute value (the size) of the gravitational force between the two point masses is

$$F = |\boldsymbol{F}_{12}| = |\boldsymbol{F}_{21}| = \frac{Gm_1m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}\,|\boldsymbol{r}_1 - \boldsymbol{r}_2| = \frac{Gm_1m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}\ . \tag{1.8}$$

It is proportional to each of the two masses, and it is inversely proportional to the square of the distance between the masses, in other words, it is an "inverse square" law.

According to Newton's law of gravitation the gravitational force acts *instantaneously* over a finite distance. This is a fundamental flaw in the theory, which Newton himself recognized. The problem became even more serious after Einstein developed the special theory of relativity, which postulates that nothing, at least no signal transmitting information, can propagate with a speed larger than the vacuum speed of light. Einstein replaced Newton's theory with a new gravitational theory, the general theory of relativity, in order to repair this basic flaw. We will say very little here about Einstein's gravitational theory.

1.2 Momentum and kinetic energy

Introducing the momentum

$$\boldsymbol{p} = m\boldsymbol{v} \,, \tag{1.9}$$

and using that the mass m is constant, we may write Newton's second law in the following way,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F} \,. \tag{1.10}$$

That is, the force $\boldsymbol{F}$ acting during the infinitesimal time interval $\mathrm{d}t$ changes the momentum by the infinitesimal amount

$$\mathrm{d}\boldsymbol{p} = \boldsymbol{F}\,\mathrm{d}t \,. \tag{1.11}$$

When the time dependent force $\boldsymbol{F} = \boldsymbol{F}(t)$ acts during a finite time interval from t_1 to t_2, the momentum changes from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$, where

$$\boldsymbol{p}_2 - \boldsymbol{p}_1 = \int_{\boldsymbol{p}_1}^{\boldsymbol{p}_2} \mathrm{d}\boldsymbol{p} = \int_{t_1}^{t_2} \boldsymbol{F}\,\mathrm{d}t \,. \tag{1.12}$$

A force acting on a moving particle performs a *work*, and as a result the *energy* of the particle increases by an amount equal to the work. The energy is increased by a positive work of an external force, and reduced by a negative work.

When the force $\boldsymbol{F}$ acts during an infinitesimal displacement $\mathrm{d}\boldsymbol{r}$, it performs an infinitesimal work $\mathrm{d}W$ equal to the scalar product of the force and the displacement,

$$\mathrm{d}W = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z \,. \tag{1.13}$$

Introducing the velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, we may write the displacement as $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$. During a finite time interval from t_1 to t_2, when the particle moves from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$, the work is

$$W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_{t_1}^{t_2} \boldsymbol{F}\cdot\boldsymbol{v}\,\mathrm{d}t \,. \tag{1.14}$$

If $\boldsymbol{F}$ is the total force (the vector sum of all forces) on the particle, then $\boldsymbol{F} = m\boldsymbol{a}$, by Newton's second law, and hence the work is

$$W = \int_{t_1}^{t_2} m\boldsymbol{a}\cdot\boldsymbol{v}\,\mathrm{d}t = \int_{t_1}^{t_2} m\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\boldsymbol{v}\,\mathrm{d}t = \frac{1}{2}\,m\boldsymbol{v}\cdot\boldsymbol{v}\Big|_{t_1}^{t_2} \,. \tag{1.15}$$

We define the *kinetic energy* of the particle as

$$E_K = \frac{1}{2}\,m\boldsymbol{v}\cdot\boldsymbol{v} = \frac{1}{2}\,m\boldsymbol{v}^2 = \frac{1}{2}\,mv^2 = \frac{1}{2}\,m(v_x^2 + v_y^2 + v_z^2) \,. \tag{1.16}$$

With this definition, the change in the kinetic energy of the particle equals the work performed by the total force.

In particular, for a free particle, which is not acted upon by any force, the mass m and the velocity $\boldsymbol{v}$ are both constants of motion. To be more precise, each of the three velocity components v_x, v_y, v_z is a constant of motion. It follows that both the momentum and the kinetic energy of a free particle are constants of motion.

1.3 Relativistic momentum and energy

The above expressions for momentum and energy are valid for a non-relativistic particle. We say that a particle is relativistic when the absolute value of its velocity, $v = |\boldsymbol{v}|$, approaches the speed of light, $c = 299\,792\,458$ m/s, which is the absolute speed limit.

The formulation of Newton's second law in terms of momentum, Equation (1.10), has the advantage that it is valid even for relativistic particles, after we modify the definition of momentum. The definition valid for relativistic as well as for non-relativistic particles, is

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = m\boldsymbol{v}\left(1 + \frac{v^2}{2c^2} + \cdots\right) . \tag{1.17}$$

Expanding to lowest order in the ratio v/c, we get back the non-relativistic definition $\boldsymbol{p} = m\boldsymbol{v}$.

One rather common interpretation of the formula for the relativistic momentum is that $\boldsymbol{p} = m'\boldsymbol{v}$, and that the mass m' depends on the velocity,

$$m' = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} . \tag{1.18}$$

Then m is called the rest mass (the mass of the particle at rest). This is of course a possible definition, but it is not recommended, because it seems (to some people, like me) more confusing than useful to distinguish between "rest mass" and "mass in motion".

The relativistic version of Newton's second law is very well tested experimentally. For example, it was used for computing the orbits of electrons in the electric and magnetic fields inside the LEP accelerator ("Large Electron Positron collider", now closed down) at CERN, outside Geneva. The electrons in LEP reached a velocity of $0.999\,999\,999\,995\,c$, which means that in one second they would lose 5 mm on a photon.

The above non-relativistic expression for the kinetic energy E_K may be derived in the following way, which leads to the relativistic expression for energy when we introduce the relativistic momentum. By Newton's second law, we have that

$$\boldsymbol{F} \cdot \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \cdot \boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p} \cdot \boldsymbol{v}) - \boldsymbol{p} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} . \tag{1.19}$$

This formula holds both with the non-relativistic and the relativistic formula for the momentum. With the non-relativistic formula we have that

$$\boldsymbol{p} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{v} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{v} \cdot \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v} \cdot \boldsymbol{F} = \boldsymbol{F} \cdot \boldsymbol{v} , \tag{1.20}$$

and hence, as before,

$$\boldsymbol{F} \cdot \boldsymbol{v} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p} \cdot \boldsymbol{v}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\boldsymbol{v}^2\right) . \tag{1.21}$$

With the relativistic formula we have that

$$\boldsymbol{p} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}\left(\boldsymbol{v} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right) = -mc^2\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{1 - \frac{v^2}{c^2}} , \tag{1.22}$$

and hence,

$$\boldsymbol{F}\cdot\boldsymbol{v} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}\cdot\boldsymbol{v}+mc^2\sqrt{1-\frac{v^2}{c^2}}\right) = \frac{\mathrm{d}E}{\mathrm{d}t}\;, \tag{1.23}$$

when we introduce the famous relativistic formula for the energy E,

$$E=\boldsymbol{p}\cdot\boldsymbol{v}+mc^2\sqrt{1-\frac{v^2}{c^2}}=\frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}}+mc^2\sqrt{1-\frac{v^2}{c^2}}=\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\;. \tag{1.24}$$

Expansion in powers of the ratio v/c gives that

$$E=mc^2\left(1+\frac{v^2}{2c^2}+\cdots\right)=mc^2+\frac{1}{2}\,mv^2+\frac{3}{8}\,\frac{mv^4}{c^2}+\cdots\;. \tag{1.25}$$

The kinetic energy is the energy E minus the rest energy mc^2,

$$E_K=E-mc^2=\frac{1}{2}\,mv^2+\cdots\;. \tag{1.26}$$

Thus, the formula for the kinetic energy E_K is different in the non-relativistic and the relativistic cases, but the formula for the work takes the same form,

$$W=\int_{t_1}^{t_2}\boldsymbol{F}\cdot\boldsymbol{v}\,\mathrm{d}t=E_K(t_2)-E_K(t_1)\;. \tag{1.27}$$

A useful formula relating relativistic energy and momentum is the following,

$$E^2=m^2c^4+\boldsymbol{p}^2c^2\;. \tag{1.28}$$

It holds even for particles of zero mass, when $m=0$ we get the simple relation

$$E=|\boldsymbol{p}|\,c\;. \tag{1.29}$$

Photons, the quanta of the quantized electromagnetic field, are the only known particles of mass zero. In the future, there will presumably be a quantum theory of gravitation, with field quanta, gravitons, that will also have mass zero. The neutrinos are other particles having masses close to, but not exactly equal to zero.

Chapter 2

The one-particle Kepler problem

We want to study the motion of particles interacting by gravitation, and it is natural to start with one single particle in a gravitational field. The one-particle problem may be thought of as the limiting case of the two-particle problem when there are two pointlike masses, one small mass m and one large mass M. We write $m \ll M$ to tell that m is much smaller than M. This is a good approximation to the physical problem of one planet, for example the Earth, moving around the Sun.

Later on, we will see that the general two-particle problem may be reduced to this special one-particle problem, even when the two masses are comparable. The deeper reason that the reduction is possible, is that the total momentum is conserved.

We assume that only the small mass m is moving, while the large mass M is lying at rest all the time. This assumption is consistent with the laws of motion, since the forces on the two masses are equal and opposite, and therefore the large mass will have a much smaller acceleration.

It is natural to choose a coordinate system having its origin at the position of the stationary mass. Thus, when $\boldsymbol{r} = \boldsymbol{r}(t)$ is the position of the small mass m at time t, and $r = |\boldsymbol{r}|$, the gravitational force on this particle is

$$\boldsymbol{F} = -\frac{GMm}{r^3}\,\boldsymbol{r}\ . \tag{2.1}$$

In combination with Newton's second law,

$$\boldsymbol{F} = m\boldsymbol{a} = m\,\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\ , \tag{2.2}$$

this gives the following equation of motion, which is a second order ordinary differential equation,

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{GM}{r^3}\,\boldsymbol{r}\ . \tag{2.3}$$

We see that the small mass m cancels out of the equation of motion: the motion of a small mass m in the gravitational field from a very much larger mass M is independent of m. This is a non-trivial result. In fact, the role of the mass m in Equation (2.1) is that the gravitational force is proportional to it. This type of mass may be called *gravitational mass*, or *heavy mass*, it is the mass we feel when we hold a stone in our hand. The mass m in

Equation (2.2) plays a very different role, it determines how hard it is to change the state of motion of the particle. This type of mass is called *inertial mass*, it is the mass we feel at the moment when we throw the same stone.

It was one of Newton's great discoveries that these two types of mass are proportional, so that he was able to define them to be the same, by introducing the proportionality constant G in the formula for the gravitational force. He tested this prediction of his theory in the gravitational field of the Earth by comparing the oscillation periods of two pendulums of identical length and shape, made of different substances and having different masses. His experiment was a null experiment, there should be no difference if his hypothesis was correct, and indeed he observed no difference.

The right hand side of Equation (2.3) is what we call the *gravitational field*, or equivalently, the *acceleration of gravity*, due to the point mass M placed at the origin. It is a vector field,

$$\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{r}) = -\frac{GM}{r^3}\,\boldsymbol{r}\ . \tag{2.4}$$

In general, the procedure for measuring a gravitational field is to measure the force on a so called *test particle*, which is supposed to be pointlike, and to have a sufficiently small mass m, so that it has a negligible influence on the motion of the masses giving rise to the gravitational field. It follows from Newton's law of universal gravitation that the gravitational force $\boldsymbol{F}$ on such a test particle is proportional to its mass m, therefore it is natural to define the gravitational field at a given point as the gravitational force divided by the mass,

$$\boldsymbol{g} = \frac{\boldsymbol{F}}{m}\ . \tag{2.5}$$

The acceleration $\boldsymbol{a}$ of the test mass is given by Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$, thus we see that $\boldsymbol{g} = \boldsymbol{a}$. The gravitational field at the given point is simply the acceleration of any test mass placed there.

> *All particles of sufficiently small mass are subject to the same acceleration in a gravitational field.*

2.1 Circular motion

We will now demonstrate that circular motion in any plane through the origin, with any given constant radius r, and a constant angular velocity ω depending on r, is a possible solution of the equation of motion. We choose our coordinate system with the z axis orthogonal to the plane, so that the orbital plane is the (x, y) plane. Thus, the position at time t will be

$$\boldsymbol{r}(t) = x(t)\,\boldsymbol{i} + y(t)\,\boldsymbol{j} = r\cos(\omega t)\,\boldsymbol{i} + r\sin(\omega t)\,\boldsymbol{j}\ , \tag{2.6}$$

if we choose the zero point of time in such a way that $\boldsymbol{r}(t) = r\,\boldsymbol{i}$ at $t = 0$. Differentiating once with respect to time we get the velocity $\boldsymbol{v}$, and differentiating once more we get the acceleration $\boldsymbol{a}$,

$$\begin{aligned} \boldsymbol{v} &= \dot{\boldsymbol{r}} = -\omega r\cos(\omega t)\,\boldsymbol{i} + \omega r\cos(\omega t)\,\boldsymbol{j}\ , \\ \boldsymbol{a} &= \dot{\boldsymbol{v}} = -\omega^2 r\cos(\omega t)\,\boldsymbol{i} - \omega^2 r\sin(\omega t)\,\boldsymbol{j} = -\omega^2\,\boldsymbol{r}\ . \end{aligned} \tag{2.7}$$

Now Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$ takes the form

$$-\frac{GMm}{r^3}\,\boldsymbol{r} = -m\omega^2\,\boldsymbol{r}\ . \tag{2.8}$$

We see that this equation of motion is solved if we take

$$\omega^2 = \frac{GM}{r^3}\ . \tag{2.9}$$

This relation is one form of Kepler's third law, valid in the special case of circular motion. The period P of the orbit is the time interval for which

$$\omega P = 2\pi\ . \tag{2.10}$$

Thus we get Kepler's third law in its usual form, as a relation between the period P and the semimajor axis of an ellipse, in this particular case the radius r of a circle,

$$P^2 = \frac{4\pi^2}{\omega^2} = \frac{4\pi^2}{GM}\,r^3\ . \tag{2.11}$$

2.2 Conservation of energy

The gravitational field of the point mass M,

$$\boldsymbol{g} = -\frac{GM}{r^3}\,\boldsymbol{r}\ , \tag{2.12}$$

is a vector field which is minus the gradient of a scalar field ϕ, called the *gravitational potential.*

To see this, we compute first the gradient of the distance to the origin, which is the scalar function

$$r = \sqrt{x^2 + y^2 + z^2}\ . \tag{2.13}$$

We have that

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}}\,2x = \frac{x}{r}\ , \tag{2.14}$$

with similar results for $\partial r/\partial y$ and $\partial r \partial z$. Hence,

$$\nabla r = \boldsymbol{i}\,\frac{\partial r}{\partial x} + \boldsymbol{j}\,\frac{\partial r}{\partial y} + \boldsymbol{k}\,\frac{\partial r}{\partial z} = \boldsymbol{i}\,\frac{x}{r} + \boldsymbol{j}\,\frac{y}{r} + \boldsymbol{k}\,\frac{z}{r} = \frac{\boldsymbol{r}}{r} = \boldsymbol{e}_r\ . \tag{2.15}$$

At any given point, $\boldsymbol{e}_r$ is the unit vector pointing away from the origin. Using the chain rule, we get that

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial r}{\partial x}\,\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{x}{r}\,\frac{1}{r^2} = -\frac{x}{r^3}\ . \tag{2.16}$$

This is the x component of the vector equation

$$\nabla\left(\frac{1}{r}\right) = -\frac{\boldsymbol{r}}{r^3}\ . \tag{2.17}$$

It follows that

$$\boldsymbol{g} = -\nabla\phi \tag{2.18}$$

when we define the *gravitational potential* from the point mass M at the origin as

$$\phi = -\frac{GM}{r} \; . \tag{2.19}$$

It follows further that the gravitational force on the small mass m is

$$\boldsymbol{F} = m\boldsymbol{g} = -m\nabla\phi = -\nabla V \; , \tag{2.20}$$

when we define the *potential energy* of the mass m in the gravitational field as

$$V = m\phi = -\frac{GMm}{r} \; . \tag{2.21}$$

The motivation for introducing the potential energy V is that the sum of the kinetic and potential energy, the *total mechanical energy*

$$E = E_K + V = \frac{1}{2}m\boldsymbol{v}^2 - \frac{GMm}{r} \; , \tag{2.22}$$

is a constant of motion.

To prove this, we have to prove that the time derivative of E vanishes,

$$\dot{E} = \dot{E}_K + \dot{V} = 0 \; . \tag{2.23}$$

(We write $\dot{E} = \mathrm{d}E/\mathrm{d}t$ for the time derivative of E, and so on.) We calculate

$$\dot{E}_K = \frac{1}{2}m(\dot{\boldsymbol{v}}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\dot{\boldsymbol{v}}) = m\dot{\boldsymbol{v}}\cdot\boldsymbol{v} = m\boldsymbol{a}\cdot\boldsymbol{v} = \boldsymbol{F}\cdot\boldsymbol{v} = -\frac{GMm}{r^3}\,\boldsymbol{r}\cdot\boldsymbol{v} \; , \tag{2.24}$$

and

$$\dot{V} = \frac{GMm}{r^2}\,\dot{r} = \frac{GMm}{r^2}\,\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r} \; , \tag{2.25}$$

which proves Equation (2.23). In the calculation of $\dot{V}$ we may use the following shortcut to show that $\dot{r} = \boldsymbol{r}\cdot\boldsymbol{v}/r$. We differentiate both sides of the identity $r^2 = \boldsymbol{r}\cdot\boldsymbol{r}$, this gives that

$$2r\dot{r} = \dot{\boldsymbol{r}}\cdot\boldsymbol{r} + \boldsymbol{r}\cdot\dot{\boldsymbol{r}} = 2\boldsymbol{r}\cdot\dot{\boldsymbol{r}} = 2\boldsymbol{r}\cdot\boldsymbol{v} \; . \tag{2.26}$$

We may also calculate $\dot{E}$ in the following way, which gives the same result, but shows even more clearly what is really going on. Here we use Newton's second law in the form $m\boldsymbol{a} = -\nabla V$, and we use the chain rule to calculate $\dot{V}$,

$$\dot{E} = \dot{E}_K + \dot{V} = m\dot{\boldsymbol{v}}\cdot\boldsymbol{v} + (\nabla V)\cdot\dot{\boldsymbol{r}} = (m\boldsymbol{a} + \nabla V)\cdot\boldsymbol{v} = 0 \; . \tag{2.27}$$

We see that we have proved a very general result: the total (mechanical) energy $E = E_K + V$ is a conserved quantity (a constant of motion) whenever the force field $\boldsymbol{F}$ is minus the gradient of a potential energy function V, that is, when Newton's second law holds in the form $m\boldsymbol{a} = -\nabla V$, and the potential energy function has no explicit time dependence, that is, we have that $V = V(\boldsymbol{r})$ and not $V = V(\boldsymbol{r}, t)$. We have proved this result here for one particle, but it is easily generalized to any number of particles.

2.3 Conservation of angular momentum

We define the *angular momentum* of the particle as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times (m\boldsymbol{v}) \; . \tag{2.28}$$

It is easy to show that the angular momentum of a point particle is a constant of motion when the particle is moving in a central force field, that is, when the force $\boldsymbol{F}$ always points along $\boldsymbol{r}$. We just compute the time derivative of $\boldsymbol{L}$ and find that it vanishes,

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{r} \times \boldsymbol{F} = 0 + 0 = 0 \; . \tag{2.29}$$

The conservation law for the vector $\boldsymbol{L}$ is actually a combination of three conservation laws, one for each of the three components

$$\begin{aligned} L_x &= yp_z - zp_y = m(yv_z - zv_y) \; , \\ L_y &= zp_x - xp_z = m(zv_x - xv_z) \; , \\ L_z &= xp_y - yp_x = m(xv_y - yv_x) \; . \end{aligned} \tag{2.30}$$

Note that

$$\boldsymbol{L} = m \, \frac{\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \; . \tag{2.31}$$

The length $|\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}|$ of the infinitesimal vector $\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}$ can be interpreted geometrically as an area, it is twice the area swept out by the radius vector $\boldsymbol{r}$ during the infinitesimal time interval $\mathrm{d}t$. This shows that Kepler's second law is a consequence of the conservation law for angular momentum.

2.4 The general Kepler orbit

In order to find the general solution of the equation of motion for our one-particle problem, we use the conservation laws for energy and angular momentum. We assume that $\boldsymbol{L} \neq 0$, because if $\boldsymbol{L} = 0$, it means that the velocity $\boldsymbol{v}$ is along the radius vector $\boldsymbol{r}$, so that the particle is going to hit the point mass at the origin. A planet with zero angular momentum would crash into the Sun.

Reduction to two dimensions

Since $\boldsymbol{r} \cdot \boldsymbol{L} = \boldsymbol{r} \cdot (\boldsymbol{r} \times \boldsymbol{p}) = 0$, and since $\boldsymbol{L}$ is a constant vector, we conclude that the particle moves all the time in the plane which goes through the origin and is orthogonal to $\boldsymbol{L}$. Therefore we choose an (x, y, z) coordinate system with its z axis along $\boldsymbol{L}$. In this coordinate system, the particle moves in the (x, y) plane, it has always $z = 0$, and its angular momentum components are $L_x = L_y = 0$, $L_z = L = |\boldsymbol{L}| > 0$.

The position of the particle at time t is

$$\boldsymbol{r} = x \, \boldsymbol{i} + y \, \boldsymbol{j} = r \cos\varphi \, \boldsymbol{i} + r \sin\varphi \, \boldsymbol{j} = r \, \boldsymbol{e}_r \; . \tag{2.32}$$

The Cartesian coordinates (x, y) and the polar coordinates (r, φ) are all time dependent: $x = x(t)$, $y = y(t)$, $r = r(t)$, and $\varphi = \varphi(t)$. We introduce the unit vector in the direction along $\boldsymbol{r}$,

$$\boldsymbol{e}_r = \cos\varphi\,\boldsymbol{i} + \sin\varphi\,\boldsymbol{j}\ , \tag{2.33}$$

which is also time dependent, since $\varphi = \varphi(t)$. The unit vector orthogonal to $\boldsymbol{e}_r$, in the same plane, is

$$\boldsymbol{e}_\varphi = -\sin\varphi\,\boldsymbol{i} + \cos\varphi\,\boldsymbol{j}\ . \tag{2.34}$$

As usual we denote time derivatives by dots,

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}\ , \qquad \dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\ . \tag{2.35}$$

We have that

$$\dot{\boldsymbol{e}}_r = -(\sin\varphi)\dot{\varphi}\,\boldsymbol{i} + (\cos\varphi)\dot{\varphi}\,\boldsymbol{j} = \dot{\varphi}\,\boldsymbol{e}_\varphi\ . \tag{2.36}$$

With this notation the velocity is

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\,\boldsymbol{e}_r + r\,\dot{\boldsymbol{e}}_r = \dot{r}\,\boldsymbol{e}_r + r\dot{\varphi}\,\boldsymbol{e}_\varphi\ . \tag{2.37}$$

Since $\boldsymbol{e}_r$ and $\boldsymbol{e}_\varphi$ are orthogonal unit vectors, we have that

$$\boldsymbol{v}\cdot\boldsymbol{v} = \dot{r}^2 + r^2\dot{\varphi}^2\ , \tag{2.38}$$

and the energy is

$$E = \frac{1}{2}\,m(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{GMm}{r}\ . \tag{2.39}$$

Since

$$\boldsymbol{e}_r \times \boldsymbol{e}_r = 0\ , \qquad \boldsymbol{e}_r \times \boldsymbol{e}_\varphi = \boldsymbol{k}\ , \tag{2.40}$$

the angular momentum is

$$\boldsymbol{L} = m(\boldsymbol{r}\times\boldsymbol{v}) = mr^2\dot{\varphi}\,\boldsymbol{k}\ . \tag{2.41}$$

And since we have chosen our coordinate system in such a way that $\boldsymbol{L} = L_z\,\boldsymbol{k} = L\,\boldsymbol{k}$, where $L = |\boldsymbol{L}|$, we have that

$$L = mr^2\dot{\varphi}\ . \tag{2.42}$$

Reduction to one dimension

Instead of using Newton's second law directly, we use the conservation laws for the energy E and the angular momentum L. A conservation law is a partial solution of the equation of motion (sometimes even a complete solution), we say that it is a *first integral* of the equation of motion.

First we use the conservation law for angular momentum to express the time derivative of the polar angle φ in the following way,

$$\dot{\varphi} = \frac{L}{mr^2} \; . \tag{2.43}$$

Next we eliminate $\dot{\varphi}$ in the conservation law for energy, to obtain the equation

$$E = \frac{1}{2}\,m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \; . \tag{2.44}$$

Remember that E, L, G, M, and m are constants.

This is precisely the equation of motion for a particle in one dimension, having an "effective potential energy"

$$V_{1d} = V_{1d}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \; . \tag{2.45}$$

In addition to the gravitational potential energy $V = -GMm/r$, inversely proportional to r, there appears the so called "centrifugal potential energy" $L^2/(2mr^2)$, inversely proportional to r^2.

We see that $V_{1d} \to 0$ as $r \to \infty$, that $V_{1d} < 0$ for $r > L^2/(2GMm^2)$, and that $V_{1d} \to +\infty$ as $r \to 0$. For a given angular momentum L the one dimensional potential energy V_{1d} has a minimum value at a distance $r = r_0$ which is the solution of the equation

$$\frac{\mathrm{d}V_{1d}}{\mathrm{d}r} = -\frac{L^2}{mr^3} + \frac{GMm}{r^2} = 0 \; . \tag{2.46}$$

Thus,

$$r_0 = \frac{L^2}{GMm^2} \; . \tag{2.47}$$

At the distance $r = r_0$ the gravitational potential energy is

$$V_0 = V(r_0) = -\frac{GMm}{r_0} = -\frac{G^2M^2m^3}{L^2} \; . \tag{2.48}$$

Adding the centrifugal energy we get the minimum value of V_{1d},

$$V_{1d0} = V_{1d}(r_0) = \frac{L^2}{2mr_0^2} - \frac{GMm}{r_0} = \frac{L^2}{2mr_0}\,\frac{GMm^2}{L^2} - \frac{GMm}{r_0} = -\frac{GMm}{2r_0} = \frac{V_0}{2} \; . \tag{2.49}$$

The special case of circular motion

Since the one dimensional kinetic energy $(1/2)m\dot{r}^2$ is never negative, $V_{1d0} = V_0/2$ is also the lower limit to the total energy E, given the angular momentum L. For any given value of the energy E (with $E \geq V_0/2$) the distance to the origin, r, has a positive lower limit. If $E < 0$ there is also an upper limit for r, which means that the particle is bound and can never escape to infinity.

Clearly $r = r_0 =$ constant is a solution of the equation of motion such that the particle moves in a circle. In this circular orbit the total energy is

$$E_0 = V_{1d0} = \frac{V_0}{2} \; . \tag{2.50}$$

This is an interesting result. In a circular orbit the total energy E is constant, and is exactly half of the potential energy V, which is also constant,

$$E = E_K + V = \frac{V}{2} \; . \tag{2.51}$$

Another formulation of the same result is that the kinetic energy E_K is half the absolute value of the potential energy,

$$E_K = E - V = -\frac{V}{2} = \frac{|V|}{2} \; . \tag{2.52}$$

It is actually a special case of a much more general result, called the virial theorem, which we will prove later.

The general motion

Equation (2.44) is a first order ordinary differential equation for r as a function of t, it is separable and can therefore be solved explicitly.

However, we get a simpler equation by means of two special tricks. The first trick is to solve for r as a function of φ instead of t. By the chain rule for differentiation we get that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\,\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\,\dot{\varphi} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\,\frac{L}{mr^2} \; . \tag{2.53}$$

Hence the energy is

$$E = \frac{L^2}{2m}\left(\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{1}{r^2}\right) - \frac{GMm}{r} \; . \tag{2.54}$$

The second trick is to introduce a new variable

$$u = \frac{1}{r} \; . \tag{2.55}$$

Since

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = -\frac{1}{r^2}\,\frac{\mathrm{d}r}{\mathrm{d}\varphi} \; , \tag{2.56}$$

the energy is

$$E = \frac{L^2}{2m}\left(\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2\right) - GMmu \; . \tag{2.57}$$

Instead of solving this equation directly, we differentiate it with respect to φ, and get that

$$0 = \frac{L^2}{m}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\,\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u\,\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right) - GMm\,\frac{\mathrm{d}u}{\mathrm{d}\varphi} \; . \tag{2.58}$$

To satisfy this equation we must have either $\mathrm{d}u/\mathrm{d}\varphi = 0$, which is once more the case of circular motion, or else

$$0 = \frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u - \frac{GMm^2}{L^2} \; . \tag{2.59}$$

The most general solution of the last equation is

$$u = u_0 - A\cos(\varphi - \varphi_0) \ , \tag{2.60}$$

where A and φ_0 are arbitrary integration constants, and u_0 is a constant which is *not* arbitrary,

$$u_0 = \frac{1}{r_0} = \frac{GMm^2}{L^2} \ . \tag{2.61}$$

We may always choose $A \geq 0$, since we get the same solution by switching the sign of A while adding π to φ_0. We should check explicitly that Equation (2.60) is a solution of Equation (2.57), and we find that it is, when

$$E = \frac{L^2}{2m}\left(A^2 + u_0^2\right) - GMmu_0 = \frac{L^2}{2m}\left(A^2 - u_0^2\right) \ . \tag{2.62}$$

There are many other ways to write this expression for the energy, for example,

$$E = \frac{L^2}{2m}\left(A^2 - \frac{1}{r_0^2}\right) = \frac{L^2A^2}{2m} - \frac{GMm}{2r_0} \ . \tag{2.63}$$

Equation (2.60) is the same as the equation

$$r = \frac{1}{u_0 - A\cos(\varphi - \varphi_0)} = \frac{r_0}{1 - r_0A\cos(\varphi - \varphi_0)} \ . \tag{2.64}$$

We see that if $0 \leq A < u_0$, or equivalently $0 \leq r_0A < 1$, then r oscillates between a minimum value, for $\cos(\varphi - \varphi_0) = -1$, and a maximum value, for $\cos(\varphi - \varphi_0) = 1$,

$$r_{\mathrm{min}} = \frac{r_0}{1 + r_0A} \ , \qquad r_{\mathrm{max}} = \frac{r_0}{1 - r_0A} \ . \tag{2.65}$$

This means that the particle is bound, and by Equation (2.62) the energy of a bound orbit is negative, $E < 0$.

In the case of a planet orbiting the Sun, the point of minimum distance is called *perihelion*, and the point of maximum distance is called *aphelion*. Assuming that the orbit is an ellipse, the semi-major axis of the ellipse must be

$$a = \frac{r_{\mathrm{min}} + r_{\mathrm{max}}}{2} = \frac{u_0}{u_0^2 - A^2} \ . \tag{2.66}$$

The special case $A = 0$ is the same circular solution that we found earlier,

$$r = \frac{1}{u_0} = r_0 = \frac{L^2}{GMm^2} \ . \tag{2.67}$$

The integration constant φ_0 is an angle which determines the orientation of the orbit in the (x, y) plane. From now on we will assume that $\varphi_0 = 0$, this means simply that we choose suitable coordinate axes in the plane. Since $x = r\cos\varphi$, our solution (2.60) with $\varphi_0 = 0$ may be written as

$$r - r_0Ax = r_0 \ . \tag{2.68}$$

We will verify now that this is the equation of an ellipse when $0 < r_0 A < 1$.

Let us derive the equation for an ellipse with one focus at the origin and the other focus at $y = 0$, $x = 2ea$, where a is the semi-major axis and e the *eccentricity* of the ellipse. The definition of an ellipse is that the sum of the distances to the two foci is constant, and the constant is equal to $2a$, twice the semi-major axis. That is, a point (x, y) on the ellipse satisfies the equation

$$r + \sqrt{(x - 2ea)^2 + y^2} = 2a \; , \tag{2.69}$$

with $r = \sqrt{x^2 + y^2}$, or

$$\sqrt{x^2 - 4eax + 4e^2a^2 + y^2} = 2a - r \; . \tag{2.70}$$

Squaring this equation, then subtracting the identity $x^2 + y^2 = r^2$ and dividing by $4a$, gives that

$$r - ex = (1 - e^2)a \; . \tag{2.71}$$

We see that the orbit we found, Equation (2.68), with $0 \leq A < u_0$, is an ellipse with eccentricity

$$e = r_0 A = \frac{L^2 A}{GMm^2} \tag{2.72}$$

and semi-major axis

$$a = \frac{r_0}{1 - e^2} = \frac{r_0}{1 - (r_0 A)^2} = \frac{u_0}{u_0^2 - A^2} \; . \tag{2.73}$$

The energy and period of an elliptical orbit

The energy may now be written as a function of a alone,

$$E = \frac{L^2}{2m}\left(A^2 - u_0^2\right) = -\frac{L^2}{2m}\frac{u_0}{a} = -\frac{GMm}{2a} = \frac{V(a)}{2} \; . \tag{2.74}$$

Here $V(a) = -GMm/a$ is the potential energy at the distance a.

Incidentally, $V(a)/2$ is equal to half the time average $\langle V \rangle$ of the potential energy in the elliptical orbit. That the energy in a bound orbit is half of the time average of the potential energy, is again an example of the virial theorem.

The period of the elliptical orbit is

$$P = \int_0^P \mathrm{d}t = \int_0^P \frac{mr^2}{L}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\,\mathrm{d}t = \frac{m}{L}\int_0^{2\pi} r^2\,\mathrm{d}\varphi = \frac{2m\mathcal{A}}{L} \; , \tag{2.75}$$

where $\mathcal{A}$ is the area of the ellipse. The integral equals $2\mathcal{A}$ because the infinitesimal quantity $r^2\,\mathrm{d}\varphi$ is twice the area of an infinitesimal triangle, and all the infinitesimal triangles together make up the ellipse.

An ellipse with semi-major axis a and semi-minor axis $b = a\sqrt{1 - e^2}$ may be regarded as a circle of radius a which has been squeezed in one direction by a factor b/a. Therefore the area of the ellipse is

$$\mathcal{A} = \pi ab = \pi a^2 \sqrt{1 - e^2} \; . \tag{2.76}$$

It follows that

$$P^2 = \frac{4\pi^2 m^2 a^4 (1-e^2)}{L^2} = \frac{4\pi^2 m^2 a^3 r_0}{L^2} = \frac{4\pi^2 a^3}{GM} , \tag{2.77}$$

and this is Kepler's third law.

The average potential energy

The time average of the potential energy of the particle in an elliptical orbit is

$$\begin{aligned} \langle V \rangle &= \frac{1}{P}\int_0^P \left(-\frac{GMm}{r}\right) \mathrm{d}t = -\frac{1}{P}\int_0^P \frac{GMm}{r}\frac{mr^2}{L}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\,\mathrm{d}t = -\frac{1}{P}\int_0^{2\pi} \frac{GMm}{r}\frac{mr^2}{L}\,\mathrm{d}\varphi \\ &= -\frac{GMm^2}{PL}\int_0^{2\pi} r\,\mathrm{d}\varphi = -\frac{2GMm^2}{PL}\int_0^{\pi} r\,\mathrm{d}\varphi = -\frac{2GMm^2}{PL}\int_0^{\pi}\frac{\mathrm{d}\varphi}{u_0 - A\cos\varphi} . \end{aligned} \tag{2.78}$$

The standard trick for solving integrals like this is to introduce a new variable $w = \tan(\varphi/2)$. This gives that

$$\mathrm{d}w = \frac{1}{\cos^2\frac{\varphi}{2}}\frac{\mathrm{d}\varphi}{2} = (1+w^2)\frac{\mathrm{d}\varphi}{2} , \tag{2.79}$$

and

$$\cos\varphi = 2\cos^2\frac{\varphi}{2} - 1 = \frac{2}{1+w^2} - 1 = \frac{1-w^2}{1+w^2} , \tag{2.80}$$

Hence,

$$\begin{aligned} \int_0^{\pi}\frac{\mathrm{d}\varphi}{u_0 - A\cos\varphi} &= \int_0^{\infty}\frac{1}{\left(u_0 - A\,\frac{1-w^2}{1+w^2}\right)}\frac{2\,\mathrm{d}w}{1+w^2} = 2\int_0^{\infty}\frac{\mathrm{d}w}{u_0 - A + (u_0+A)w^2} \\ &= \frac{2}{u_0 - A}\sqrt{\frac{u_0 - A}{u_0 + A}}\int_0^{\infty}\frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{\sqrt{u_0^2 - A^2}} , \end{aligned} \tag{2.81}$$

where we have introduced the second new variable

$$x = w\sqrt{\frac{u_0 + A}{u_0 - A}} . \tag{2.82}$$

Thus,

$$\langle V \rangle = -\frac{2\pi GMm^2}{PL\sqrt{u_0^2 - A^2}} = -\frac{2\pi GMm^2\sqrt{a}}{PL\sqrt{u_0}} = -\frac{2\pi GMm^2\sqrt{a}}{2\pi m\sqrt{a^3}} = -\frac{GMm}{a} . \tag{2.83}$$

2.5 The virial theorem for one particle

The virial theorem for one small pointlike mass m in the gravitational field of a large pointlike mass M at rest at the origin is proved by computing the quantity

$$U = \boldsymbol{r}\cdot\boldsymbol{p} = \boldsymbol{r}\cdot(m\boldsymbol{v}) , \tag{2.84}$$

which is the time derivative of the *moment of inertia*, defined as

$$I = \frac{1}{2}\,m\boldsymbol{r}^2 = \frac{1}{2}\,m(x^2+y^2+z^2)\ . \tag{2.85}$$

Note that I is the moment of inertia about the origin, which is a point, and not about an axis. The moment of inertia about the z axis, for example, is defined as

$$I_z = \frac{1}{2}\,m(x^2+y^2)\ . \tag{2.86}$$

First we compute the time derivative of U,

$$\dot U = \ddot I = \dot{\boldsymbol{r}}\cdot\boldsymbol{p} + \boldsymbol{r}\cdot\dot{\boldsymbol{p}} = \boldsymbol{v}\cdot\boldsymbol{p} + \boldsymbol{r}\cdot\boldsymbol{F} = \boldsymbol{v}\cdot\boldsymbol{p} - \boldsymbol{r}\cdot(\nabla V) = 2E_K + V\ . \tag{2.87}$$

Here $E_K = (1/2)m\boldsymbol{v}^2 = (1/2)\boldsymbol{p}\cdot\boldsymbol{v}$ is the kinetic energy, and the quantity $\boldsymbol{r}\cdot\boldsymbol{F} = -\boldsymbol{r}\cdot(\nabla V)$ is called the *virial*, hence the name of the theorem. Because the potential energy is of the form $V = -GMm/r$, we have that

$$\boldsymbol{r}\cdot(\nabla V) = \boldsymbol{r}\cdot\left(\frac{GMm}{r^3}\,\boldsymbol{r}\right) = \frac{GMm}{r^3}\,\boldsymbol{r}\cdot\boldsymbol{r} = \frac{GMm}{r} = -V\ . \tag{2.88}$$

The relation $\boldsymbol{r}\cdot(\nabla V) = -V$ may be derived not only by direct calculation, as we just did, but also by the following very general argument. We observe that the gravitational potential energy

$$V(\boldsymbol{r}) = V(x,y,z) = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{x^2+y^2+z^2}} \tag{2.89}$$

is a *homogeneous function of degree* -1 of its arguments x, y, z. That is, if we scale all three arguments by a common factor $\lambda > 0$, we have that

$$V(\lambda\boldsymbol{r}) = V(\lambda x, \lambda y, \lambda z) = -\frac{GMm}{\lambda r} = \lambda^{-1}\,V(\boldsymbol{r})\ . \tag{2.90}$$

Taking $\lambda = 1+\epsilon$, where ϵ is infinitesimal, we get first that, to first order in ϵ,

$$\begin{aligned} V(\lambda\boldsymbol{r}) &= V(x+\epsilon x, y+\epsilon y, z+\epsilon z) \\ &= V(x,y,z) + \epsilon x\,\frac{\partial V(x,y,z)}{\partial x} + \epsilon y\,\frac{\partial V(x,y,z)}{\partial y} + \epsilon z\,\frac{\partial V(x,y,z)}{\partial z} \qquad (2.91) \\ &= V(\boldsymbol{r}) + \epsilon\,\boldsymbol{r}\cdot\nabla V(\boldsymbol{r})\ . \qquad (2.92) \end{aligned}$$

And second, because V is a homogeneous function of degree -1,

$$V(\lambda\boldsymbol{r}) = (1+\epsilon)^{-1}\,V(\boldsymbol{r}) = (1-\epsilon)\,V(\boldsymbol{r})\ . \tag{2.93}$$

Comparing these two expressions, we conclude that $\boldsymbol{r}\cdot\nabla V = -V$.

The second step in deriving the virial theorem is to take the time average of the equation $\dot U = 2E_K + V$, by integrating over a time interval from t_1 to t_2 and dividing by $t_2 - t_1$. Denoting the time average by brackets $\langle\ \rangle$ we get that

$$\frac{U(t_2) - U(t_1)}{t_2 - t_1} = 2\langle E_K\rangle + \langle V\rangle\ . \tag{2.94}$$

The left hand side of this equation vanishes in the case where we have a periodic orbit, and we integrate over one or more complete periods. More generally, it vanishes in the limit of an infinitely long time interval, if the system is bound, so that neither the position vector $\boldsymbol{r}$ nor the momentum vector $\boldsymbol{p}$ go to infinity. Note that if we do not want $\boldsymbol{r}$ to go to infinity, it is essential that we choose carefully our inertial system, in fact we have to choose a reference system in which the particle does not move far away.

The virial theorem, which we have now proved for one particle, is the statement that in a system which is gravitationally bound, the time average (over a long time interval) of twice the kinetic energy plus the potential energy is zero,

$$2\langle E_K\rangle + \langle V\rangle = 0\ . \tag{2.95}$$

Since the total energy $E = E_K + V$ is constant, it follows that

$$E = \langle E\rangle = \langle E_K\rangle + \langle V\rangle = -\langle E_K\rangle = \frac{\langle V\rangle}{2}\ . \tag{2.96}$$

We will see that the virial theorem holds for a system consisting of any number of particles, when the whole system is gravitationally bound, and gravitation is the dominating interaction. It has many applications in astrophysics, for example in understanding the stability of a star, or in measuring the total mass of a star cluster or a cluster of galaxies.

Chapter 3

The two-particle problem

Now that we have solved the gravitational one-particle problem, which is the special two-particle problem with very unequal masses, it turns out to be easy to solve the general two-particle problem with arbitrary masses, by reducing it to the one-particle case. Thus, we assume now that there are two pointlike masses m_1 and m_2, which we allow to be comparable.

3.1 The relative motion

The time dependent positions of the two particles are $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. It is convenient to introduce the relative position, which we define as

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \; . \tag{3.1}$$

The gravitational force on particle 1 from particle 2 is

$$\boldsymbol{F}_{12} = -\frac{Gm_1m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2) = -\frac{Gm_1m_2}{r^3}\,\boldsymbol{r} \; , \tag{3.2}$$

and the force on particle 2 from particle 1 is

$$\boldsymbol{F}_{21} = -\frac{Gm_2m_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}\,(\boldsymbol{r}_2 - \boldsymbol{r}_1) = \frac{Gm_2m_1}{r^3}\,\boldsymbol{r} = -\boldsymbol{F}_{12} \; . \tag{3.3}$$

We asume that there are no other forces acting on the particles. Then Newton's second law applied to each of them gives that

$$\boldsymbol{F}_{12} = m_1\boldsymbol{a}_1 = m_1\,\frac{\mathrm{d}^2\boldsymbol{r}_1}{\mathrm{d}t^2} \; , \qquad \boldsymbol{F}_{21} = m_2\boldsymbol{a}_2 = m_2\,\frac{\mathrm{d}^2\boldsymbol{r}_2}{\mathrm{d}t^2} \; . \tag{3.4}$$

These equations of motion are second order ordinary differential equations, one vector equation for each of the particles,

$$\frac{\mathrm{d}^2\boldsymbol{r}_1}{\mathrm{d}t^2} = -\frac{Gm_2}{r^3}\,\boldsymbol{r} \; , \qquad \frac{\mathrm{d}^2\boldsymbol{r}_2}{\mathrm{d}t^2} = \frac{Gm_1}{r^3}\,\boldsymbol{r} \; . \tag{3.5}$$

By subtracting them we get immediately an equation of motion for the relative position,

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\boldsymbol{r}_1}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\boldsymbol{r}_2}{\mathrm{d}t^2} = -\frac{G(m_1+m_2)}{r^3}\,\boldsymbol{r} = -\frac{GM}{r^3}\,\boldsymbol{r} \; , \tag{3.6}$$

where we have introduced the total mass

$$M = m_1 + m_2 \ . \tag{3.7}$$

We have chosen our notation in such a clever way that the equation we obtain is formally exactly the same as the one-particle equation we solved above. Thus, we know already how to solve it.

3.2 The centre of mass and conservation of momentum

By definition, the centre of mass of the two particles has the position

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{M} \ . \tag{3.8}$$

It moves with the velocity

$$\boldsymbol{V} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{M} \ . \tag{3.9}$$

The total momentum, which we define as

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = M\boldsymbol{V} \ , \tag{3.10}$$

is conserved, as usual, because of Newton's second and third laws,

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{p}_1}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{p}_2}{\mathrm{d}t} = \boldsymbol{F}_{12} + \boldsymbol{F}_{21} = 0 \ . \tag{3.11}$$

The conservation of total momentum means that the centre of mass moves with a constant velocity.

3.3 The reduced mass

Knowing the centre of mass position $\boldsymbol{R}$ and the relative position $\boldsymbol{r}$ is the same as knowing the two particle positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. In fact, we find easily that

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{M}\,\boldsymbol{r} \ , \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{M}\,\boldsymbol{r} \ . \tag{3.12}$$

The same relations hold for the velocities, as we find by taking the time derivatives,

$$\boldsymbol{v}_1 = \boldsymbol{V} + \frac{m_2}{M}\,\boldsymbol{v} \ , \qquad \boldsymbol{v}_2 = \boldsymbol{V} - \frac{m_1}{M}\,\boldsymbol{v} \ . \tag{3.13}$$

We may now compute the total kinetic energy,

$$\begin{aligned} E_K &= \frac{1}{2}\,m_1\boldsymbol{v}_1^{\,2} + \frac{1}{2}\,m_2\boldsymbol{v}_2^{\,2} = \frac{1}{2}\,(m_1 + m_2)\boldsymbol{V}^2 + \frac{1}{2}\,\frac{m_1 m_2^2 + m_1^2 m_2}{M^2}\,\boldsymbol{v}^2 \\ &= \frac{1}{2}\,M\boldsymbol{V}^2 + \frac{1}{2}\,m\boldsymbol{v}^2 \ , \end{aligned} \tag{3.14}$$

where we have introduced the *reduced mass*

$$m = \frac{m_1 m_2^2 + m_1^2 m_2}{M^2} = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2} \ . \tag{3.15}$$

The definition may also be written as

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2} \ . \tag{3.16}$$

Note that $2m$ is the *harmonic mean* of the two masses, since

$$\frac{1}{2m} = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \ . \tag{3.17}$$

Note also that $m_1 m_2 = Mm$. Hence, the gravitational force between the two particles is given by the same expression in terms of either pair of masses, either m_1, m_2 or M, m,

$$\boldsymbol{F} = \boldsymbol{F}_{12} = -\boldsymbol{F}_{21} = -\frac{G m_1 m_2}{r^3}\,\boldsymbol{r} = -\frac{GMm}{r^3}\,\boldsymbol{r} \ . \tag{3.18}$$

In summary, what we have done is to separate the two-particle problem into two *independent* one-particle problems. One for the centre of mass $\boldsymbol{R}$, which moves as a free paricle of mass M. And one for the relative position $\boldsymbol{r}$, which moves as a particle of mass m, subject to the force $\boldsymbol{F}$.

3.4 Conservation of energy

Our separation of the two-particle problem into two independent one-particle problems implies that the total mechanical energy

$$E = E_K + V = \frac{1}{2}\,m_1 \boldsymbol{v}_1^2 + \frac{1}{2}\,m_2 \boldsymbol{v}_2^2 - \frac{G m_1 m_2}{r} \tag{3.19}$$

is conserved. We have in fact seen that we may rewrite it as

$$E = \frac{1}{2}\,M\boldsymbol{V}^2 + \frac{1}{2}\,m\boldsymbol{v}^2 - \frac{GMm}{r} = E_{\text{cm}} + E_{\text{rel}} \ , \tag{3.20}$$

where E_{cm} is the energy of the centre of mass motion, and E_{rel} is the energy of the relative motion,

$$E_{\text{cm}} = \frac{1}{2}\,M\boldsymbol{V}^2 \ , \qquad E_{\text{rel}} = \frac{1}{2}\,m\boldsymbol{v}^2 - \frac{GMm}{r} \ . \tag{3.21}$$

We know already that these two energies are separately conserved. However, we will now also prove directly that the energy of the two-particle system is conserved.

The two-particle potential energy

$$V = V(\boldsymbol{r}_1, \boldsymbol{r}_2) = V(r) = -\frac{GMm}{r} \tag{3.22}$$

gives the force on each of the two particles. To see how, first recall that

$$r = |\boldsymbol{r}_1 - \boldsymbol{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \ . \tag{3.23}$$

We now define the gradient operator (nabla operator) with respect to the coordinates of particle 1 as

$$\nabla_1 = \boldsymbol{i}\,\frac{\partial}{\partial x_1} + \boldsymbol{j}\,\frac{\partial}{\partial y_1} + \boldsymbol{k}\,\frac{\partial}{\partial z_1} \ , \tag{3.24}$$

and similarly for particle 2. We have for example that

$$\begin{aligned}\frac{\partial r}{\partial x_1} &= \frac{1}{2r}\,2(x_1-x_2) = \frac{x_1-x_2}{r}\;,\\ \frac{\partial r}{\partial x_2} &= \frac{1}{2r}\,2(x_2-x_1) = -\frac{x_1-x_2}{r}\;,\end{aligned} \tag{3.25}$$

and so on, for all the partial derivatives. It follows that

$$\begin{aligned}\nabla_1\, r &= \frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{r} = \frac{\boldsymbol{r}}{r} = \boldsymbol{e}_r\;,\\ \nabla_2\, r &= \frac{\boldsymbol{r}_2-\boldsymbol{r}_1}{r} = -\frac{\boldsymbol{r}}{r} = -\boldsymbol{e}_r\;.\end{aligned} \tag{3.26}$$

In this way we find that the force on each of the two particles may be written as a negative gradient of the same potential energy,

$$\boldsymbol{F}_{12} = -\frac{GM}{r^3}\,\boldsymbol{r} = -\nabla_1 V\;,\qquad \boldsymbol{F}_{21} = \frac{GM}{r^3}\,\boldsymbol{r} = -\nabla_2 V\;. \tag{3.27}$$

The proof of energy conservation is essentially the same as in the one-particle case, again we have to prove that the time derivative of E vanishes. We have that

$$\begin{aligned}\dot{E} &= \dot{E}_K + \dot{V} = m_1\boldsymbol{a}_1\cdot\boldsymbol{v}_1 + m_2\boldsymbol{a}_2\cdot\boldsymbol{v}_2 + (\nabla_1 V)\cdot\boldsymbol{v}_1 + (\nabla_2 V)\cdot\boldsymbol{v}_2\\ &= (m_1\boldsymbol{a}_1 + \nabla_1 V)\cdot\boldsymbol{v}_1 + (m_2\boldsymbol{a}_2 + \nabla_2 V)\cdot\boldsymbol{v}_2 = 0+0=0\;.\end{aligned} \tag{3.28}$$

Hopefully it should now be reasonably clear how to generalize the law of conservation of mechanical energy to the case of more than two particles.

3.5 Conservation of angular momentum

The total angular momentum is the sum of the angular momenta of the two particles,

$$\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2 = \boldsymbol{r}_1\times\boldsymbol{p}_1 + \boldsymbol{r}_2\times\boldsymbol{p}_2 = \boldsymbol{r}_1\times(m_1\boldsymbol{v}_1) + \boldsymbol{r}_2\times(m_2\boldsymbol{v}_2)\;. \tag{3.29}$$

The proof that it is conserved is straightforward, we show directly that its time derivative vanishes,

$$\begin{aligned}\dot{\boldsymbol{L}} &= \dot{\boldsymbol{r}}_1\times\boldsymbol{p}_1 + \boldsymbol{r}_1\times\dot{\boldsymbol{p}}_1 + \dot{\boldsymbol{r}}_2\times\boldsymbol{p}_2 + \boldsymbol{r}_2\times\dot{\boldsymbol{p}}_2 = 0 + \boldsymbol{r}_1\times\boldsymbol{F}_{12} + 0 + \boldsymbol{r}_2\times\boldsymbol{F}_{21}\\ &= (\boldsymbol{r}_1-\boldsymbol{r}_2)\times\boldsymbol{F}_{12} = 0\;.\end{aligned} \tag{3.30}$$

We use Newton's second law, that $\dot{\boldsymbol{p}}_1 = \boldsymbol{F}_{12}$ and $\dot{\boldsymbol{p}}_2 = \boldsymbol{F}_{21}$, Newton's third law, that $\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$, and the fact that, by Newton's law of gravitation, the force is a vector pointing along the vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$.

It is a rather natural guess that we may also split the total angular momentum as a sum of the angular momentum of the centre of mass, defined as

$$\boldsymbol{L}_{\rm cm} = \boldsymbol{R}\times(M\boldsymbol{V})\;, \tag{3.31}$$

and the relative angular momentum, defined as

$$\boldsymbol{L}_{\rm rel} = \boldsymbol{r}\times(m\boldsymbol{v})\;. \tag{3.32}$$

The proof that $\boldsymbol{L} = \boldsymbol{L}_{\rm cm} + \boldsymbol{L}_{\rm rel}$ is left as an exercise.

This decomposition leads to a second proof that $\boldsymbol{L}$ is conserved. In fact, we know that $\boldsymbol{L}_{\rm cm}$ is conserved, because it is the angular momentum of a free particle. And we also know, from our study of the one-particle problem, that $\boldsymbol{L}_{\rm rel}$ is conserved.

3.6 The virial theorem for two particles

Generalizing from the case of one particle, we introduce the moment of inertia

$$I = \frac{1}{2}\left(m_1 \boldsymbol{r}_1^2 + m_2 \boldsymbol{r}_2^2\right) , \tag{3.33}$$

and its time derivative,

$$U = \dot{I} = \boldsymbol{r}_1 \cdot \boldsymbol{p}_1 + \boldsymbol{r}_2 \cdot \boldsymbol{p}_2 . \tag{3.34}$$

The time derivative of U in turn is

$$\begin{aligned} \dot{U} &= \boldsymbol{v}_1 \cdot \boldsymbol{p}_1 + \boldsymbol{r}_1 \cdot \boldsymbol{F}_{12} + \boldsymbol{v}_2 \cdot \boldsymbol{p}_2 + \boldsymbol{r}_2 \cdot \boldsymbol{F}_{21} \\ &= m_1 \boldsymbol{v}_1^2 - \boldsymbol{r}_1 \cdot (\nabla_1 V) + m_2 \boldsymbol{v}_2^2 - \boldsymbol{r}_2 \cdot (\nabla_2 V) = 2E_K + V . \end{aligned} \tag{3.35}$$

Here E_K is the sum of the kinetic energies of the two particles, $\boldsymbol{r}_1 \cdot \boldsymbol{F}_{12} + \boldsymbol{r}_2 \cdot \boldsymbol{F}_{21}$ is the two particle virial, and $V = -GMm/r$ is the potential energy. We have that

$$\boldsymbol{r}_1 \cdot (\nabla_1 V) + \boldsymbol{r}_2 \cdot (\nabla_2 V) = (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \left(\frac{GMm}{r^3}\,\boldsymbol{r}\right) = \frac{GMm}{r} = -V . \tag{3.36}$$

Again, the relation

$$\boldsymbol{r}_1 \cdot (\nabla_1 V) + \boldsymbol{r}_2 \cdot (\nabla_2 V) = -V \tag{3.37}$$

follows from the general property of the potential energy

$$V(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}} \tag{3.38}$$

that it is a homogeneous function of degree -1. Scaling all six arguments $x_1, y_1, z_1, x_2, y_2, z_2$ by a common factor $\lambda > 0$ gives that

$$V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2) = -\frac{GMm}{\lambda r} = \lambda^{-1}\, V(\boldsymbol{r}) . \tag{3.39}$$

Taking $\lambda = 1 + \epsilon$, where ϵ is infinitesimal, gives that

$$V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2) = V(\boldsymbol{r}_1, \boldsymbol{r}_2) + \epsilon\, \boldsymbol{r}_1 \cdot \nabla_1 V(\boldsymbol{r}_1, \boldsymbol{r}_2) + \epsilon\, \boldsymbol{r}_2 \cdot \nabla_2 V(\boldsymbol{r}_1, \boldsymbol{r}_2) , \tag{3.40}$$

and that

$$V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2) = (1+\epsilon)^{-1}\, V(\boldsymbol{r}_1, \boldsymbol{r}_2) = (1-\epsilon)\, V(\boldsymbol{r}_1, \boldsymbol{r}_2) . \tag{3.41}$$

Comparison of these two expressions gives Equation (3.37), which we wanted to prove.

The remaining part of the derivation of the virial theorem is the same as in the one-particle case. We take the time average of the equation $\dot{U} = 2E_K + V$, and get that

$$\frac{U(t_2) - U(t_1)}{t_2 - t_1} = 2\langle E_K \rangle + \langle V \rangle . \tag{3.42}$$

Again the left hand side of this equation vanishes if it includes one or more complete periods of a periodic orbit, or if the system is bound and we take the limit of an infinitely long time interval. We have to use the centre of mass reference system, in which the centre of mass is at rest. Then we have that

$$2\langle E_K \rangle + \langle V \rangle = 0 . \tag{3.43}$$

And, because the total energy $E = E_K + V$ is constant,

$$E = \langle E_K \rangle + \langle V \rangle = -\langle E_K \rangle = \frac{\langle V \rangle}{2} . \tag{3.44}$$

Chapter 4

The many-particle problem

The gravitational many-particle problem can not be solved explicitly in a similar way as the two-particle problem. But we may formulate the problem, and prove useful general results, such as the conservation of energy and angular momentum, and the virial theorem.

4.1 The equations of motion

We assume now that there are N pointlike masses $m_1, m_2, \ldots, m_N$ acting on each other by gravitational forces. Their time dependent positions are $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N$. Consider one of the particles, say particle number i. The total force acting upon it, $\boldsymbol{F}_i$, is the sum of the gravitational forces from all the other particles,

$$\boldsymbol{F}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \boldsymbol{F}_{ij} = \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(-\frac{G m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} (\boldsymbol{r}_i - \boldsymbol{r}_j) \right) . \tag{4.1}$$

We assume that there are no forces apart from the gravitational forces, then Newton's second law applied to particle i gives that

$$\frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = \frac{\boldsymbol{F}_i}{m_i} = \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(-\frac{G m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} (\boldsymbol{r}_i - \boldsymbol{r}_j) \right) . \tag{4.2}$$

4.2 The centre of mass and conservation of momentum

By definition, the position of the centre of mass of the N particles is

$$\boldsymbol{R} = \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M} \sum_{i=1}^{N} m_i \boldsymbol{r}_i , \tag{4.3}$$

where M is the sum of all the masses. The centre of mass moves with the velocity

$$\boldsymbol{V} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \frac{1}{M} \sum_{i=1}^{N} m_i \boldsymbol{v}_i = \frac{\boldsymbol{P}}{M} , \tag{4.4}$$

where $\boldsymbol{P}$ is the total momentum,

$$\boldsymbol{P} = \sum_{i=1}^{N} \boldsymbol{p}_i = \sum_{i=1}^{N} m_i \boldsymbol{v}_i \ . \tag{4.5}$$

The total momentum is conserved also in the many-particle case, because of Newton's second and third laws,

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \sum_{i=1}^{N} \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \sum_{i=1}^{N} \boldsymbol{F}_i = 0 \ . \tag{4.6}$$

The sum of all forces must vanish, as a consequence of Newton's third law. Here is a more formal proof. We have that

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \boldsymbol{F}_{ij} \ . \tag{4.7}$$

There are various ways to manipulate this double sum. First let us interchange the order of the summations, it gives that

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i\neq j}}^{N} \boldsymbol{F}_{ij} \ . \tag{4.8}$$

Then we rename the summation indices, this is allowed, because the value of a sum does not depend on the name of the summation index. Interchanging the names i and j we get that

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \boldsymbol{F}_{ji} \ . \tag{4.9}$$

It follows that

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} (\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji}) = 0 \ , \tag{4.10}$$

where the last equality follows from Newton's third law.

The conservation of total momentum means that the centre of mass moves with a constant velocity.

4.3 Conservation of energy

The gravitational force $\boldsymbol{F}_i$ on particle i from all the other particles may be obtained as a negative gradient, $\boldsymbol{F}_i = -\nabla_i V$, of a total potential energy V. The correct definition of V should be clear from our study of the two-particle system. In fact, the potential energy of one pair of particles, say the particles number i and j, is the same as before,

$$V_{ij} = V_{ij}(\boldsymbol{r}_i, \boldsymbol{r}_j) = -\frac{G m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \ . \tag{4.11}$$

And the total potential energy is a sum over all the particle pairs,

$$V = V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} V_{ij} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(-\frac{G m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \right) . \tag{4.12}$$

The proof that $\boldsymbol{F}_i = -\nabla_i V$ is left as an exercise, it is essentially a repetition of what we did in the two-particle case.

Obviously, the total kinetic energy is the sum of the kinetic energies of all the particles,

$$E_K = \sum_{i=1}^{N} \frac{1}{2} m_i \boldsymbol{v}_i^{\,2} \,. \tag{4.13}$$

And the total energy is

$$E = E_K + V = \sum_{i=1}^{N} \frac{1}{2} m_i \boldsymbol{v}_i^{\,2} - \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{G m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \,. \tag{4.14}$$

To prove that it is conserved, we have to show that its time derivative vanishes, that $\dot{E} = 0$. In a similar way as in the two-particle case, we have that

$$\dot{E} \;=\; \dot{E}_K + \dot{V} = \sum_{i=1}^{N} (m_i \dot{\boldsymbol{v}}_i \cdot \boldsymbol{v}_i + (\nabla_i V) \cdot \dot{\boldsymbol{r}}_i) = \sum_{i=1}^{N} (m_i \boldsymbol{a}_i + \nabla_i V) \cdot \boldsymbol{v}_i = 0 \,. \tag{4.15}$$

4.4 Conservation of angular momentum

The total angular momentum is the sum of the angular momenta of the N particles,

$$\boldsymbol{L} = \sum_{i=1}^{N} \boldsymbol{L}_i = \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{p}_i \,. \tag{4.16}$$

It is conserved, since its time derivative vanishes. In fact, we have that

$$\dot{\boldsymbol{L}} = \sum_{i=1}^{N} (\dot{\boldsymbol{r}}_i \times \boldsymbol{p}_i + \boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i) = \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_i = \sum_{i=1}^{N} \boldsymbol{r}_i \times \sum_{\substack{j=1 \\ j \neq i}}^{N} \boldsymbol{F}_{ij} = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} \,. \tag{4.17}$$

Newton's third law implies that

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji} = \frac{1}{2} \left(\boldsymbol{F}_{ij} - \boldsymbol{F}_{ji} \right) \,, \tag{4.18}$$

and hence,

$$\dot{\boldsymbol{L}} = \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} (\boldsymbol{r}_i \times \boldsymbol{F}_{ij} - \boldsymbol{r}_i \times \boldsymbol{F}_{ji}) \,. \tag{4.19}$$

By the old trick of changing the order of summations and the names of summation variables we get that

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_{ji} = \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_{ji} = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \boldsymbol{r}_j \times \boldsymbol{F}_{ij} \,, \tag{4.20}$$

and hence,

$$\dot{\boldsymbol{L}} = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}(\boldsymbol{r}_i - \boldsymbol{r}_j)\times \boldsymbol{F}_{ij} = 0\ . \tag{4.21}$$

To get the last equality we use that the two-body force $\boldsymbol{F}_{ij}$ is central, that is, it points along the vector $\boldsymbol{r}_i - \boldsymbol{r}_j$.

Also in the N-particle case we may split the total angular momentum as

$$\boldsymbol{L} = \boldsymbol{L}_{\mathrm{cm}} + \boldsymbol{L}_{\mathrm{rel}}\ , \tag{4.22}$$

defining the angular momentum of the centre of mass as

$$\boldsymbol{L}_{\mathrm{cm}} = \boldsymbol{R}\times\boldsymbol{P} = \boldsymbol{R}\times(M\boldsymbol{V})\ , \tag{4.23}$$

and then defining the relative angular momentum as

$$\boldsymbol{L}_{\mathrm{rel}} = \boldsymbol{L} - \boldsymbol{L}_{\mathrm{cm}} = \sum_{i=1}^{N}(\boldsymbol{r}_i - \boldsymbol{R})\times\boldsymbol{p}_i\ . \tag{4.24}$$

We see from this formula that $\boldsymbol{L}_{\mathrm{rel}}$ is the angular momentum relative to the centre of mass. In the absence of external forces, $\boldsymbol{L}_{\mathrm{cm}}$ is conserved, because it is the angular momentum of a free particle. Since the total angular momentum $\boldsymbol{L}$ is conserved, it follows that the relative angular momentum $\boldsymbol{L}_{\mathrm{rel}}$ is conserved.

4.5 The virial theorem for N particles

The N-particle moment of inertia is defined by the obvious generalization as

$$I = \sum_{i=1}^{N} m_i \boldsymbol{r}_i^2\ . \tag{4.25}$$

We compute its first and second time derivatives,

$$U = \dot{I} = \sum_{i=1}^{N}\boldsymbol{r}_i\cdot\boldsymbol{p}_i\ , \tag{4.26}$$

and

$$\dot{U} = \sum_{i=1}^{N}(\boldsymbol{v}_i\cdot\boldsymbol{p}_i + \boldsymbol{r}_i\cdot\boldsymbol{F}_i) = \sum_{i=1}^{N}\Big(m_i\boldsymbol{v}_i^2 - \boldsymbol{r}_i\cdot(\nabla_i V)\Big) = 2E_K + V\ . \tag{4.27}$$

Here E_K is the sum of the kinetic energies of the N particles, $\sum_i \boldsymbol{r}_i\cdot\boldsymbol{F}_i$ is the N particle virial, and V is the total potential energy, as defined in Equation (4.12).

Like before, in the one- and two-particle cases, the relation

$$\sum_{i=1}^{N}\boldsymbol{r}_i\cdot(\nabla_i V) = -V \tag{4.28}$$

may be proved directly, or we may use that the potential energy V is a homogeneous function of degree -1,

$$V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2, \ldots, \lambda \boldsymbol{r}_N) = \lambda^{-1} V(\boldsymbol{r}, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \; . \tag{4.29}$$

Taking $\lambda = 1 + \epsilon$, where ϵ is infinitesimal, gives that

$$V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2, \ldots, \lambda \boldsymbol{r}_N) = V(\boldsymbol{r}, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) + \epsilon \sum_{i=1}^{N} \boldsymbol{r}_i \cdot \nabla_i V(\boldsymbol{r}, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \; , \tag{4.30}$$

and that

$$V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2, \ldots, \lambda \boldsymbol{r}_N) = (1 - \epsilon)\, V(\boldsymbol{r}, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \; . \tag{4.31}$$

Comparison of these two expressions gives Equation (4.28).

Exactly as before, we conclude that if we take the time average over one or more complete periods of a periodic motion, or if the system is bound and we take the limit of an infinitely long time interval, then

$$2\langle E_K \rangle + \langle V \rangle = 0 \; . \tag{4.32}$$

It is understood that we use the centre of mass reference system, in which the centre of mass is at rest and does not wander off to infinity. Because the total energy $E = E_K + V$ is constant, we have that

$$E = \langle E_K \rangle + \langle V \rangle = -\langle E_K \rangle = \frac{\langle V \rangle}{2} \; . \tag{4.33}$$

If we want to test the virial theorem by observing a cluster of galaxies, for example, we should really observe and average over millions of years, or even hundreds of millions of years. Since that is impossible, we simply assume that the kinetic and potential energies we observe now, at a random moment in the history of the galaxy cluster, are reasonable estimates for the averages that go into the virial theorem.

4.6 The central temperature of the Sun

The virial theorem has, for example, a direct application in understanding the stability of the Sun. The material in the Sun is a gas, consisting mostly of free electrons, protons and helium nuclei. It can be reasonably well described by the ideal gas law,

$$P\mathcal{V} = N k_B T \; , \tag{4.34}$$

where P is the pressure, $\mathcal{V}$ the volume of the gas, N the number of gas particles, T the temperature, and $k_B = 1.38\ 10^{-23}$ J/K is Boltzmann's constant. The temperature and the density vary of course much from the centre of the Sun and out to the surface, thus the surface temperature of $T_s = 5800$ K is essentially zero compared to the central temperature of $T_c = 1.5\ 10^6$ K. It is nevertheless meaningful to talk about an average temperature $\overline{T}$, which will be just a little bit lower than the temperature at the centre, where most of the mass is concentrated.

The Sun is a very stable system, it has existed already some 4.5 10^9 years in essentially the same state. This can happen because it is gravitationally bound, and there exists a stable equilibrium between the gravitational force, directed inwards, and the pressure, directed outwards. The pressure is essentially that of an ideal gas, which is entirely due to the kinetic energy of the gas particles. The average kinetic energy of one particle is

$$E_{\text{av}} = \frac{E_K}{N} = \frac{3}{2}\, k_B \overline{T} \; . \tag{4.35}$$

When energy is radiated away from the surface of the Sun, the internal pressure is maintained by conversion of nuclear energy into kinetic energy. The total gravitational potential energy of the Sun is

$$V = -a\, \frac{GM_\odot^2}{R_\odot} \; , \tag{4.36}$$

where a is a numerical factor close to 1. In particular, $a = 3/5$ in the case of a spherical mass distribution of uniform density. Clearly, $a > 3/5$ in the case of the Sun, since there is a strong concentration of the mass towards the centre.

The stability of the Sun means that the virial theorem $2E_K + V = 0$ applies, that is,

$$2NE_{\text{av}} = \frac{aGM_\odot^2}{R_\odot} \; . \tag{4.37}$$

Here $M_\odot = 2.0\ 10^{30}$ kg is the mass and $R_\odot = 7.0\ 10^8$ m the radius of the Sun. It seems a reasonable guess to take the numerical factor $a = 1$. Thus, we estimate the average temperature of the Sun to be

$$\overline{T} = \frac{aGM_\odot^2}{3Nk_BR_\odot} \approx \frac{GM_\odot^2}{3Nk_BR_\odot} = \frac{GM_\odot \overline{m}}{3k_BR_\odot} \; . \tag{4.38}$$

The ratio $M_\odot/N = \overline{m}$ is the average mass of the gas particles. The gas consists mainly of hydrogen and helium, about 3/4 of the mass is hydrogen, and 1/4 is helium. This means that, on the average, for every 12 free protons, each of mass $m_p = 1.67\ 10^{-27}$ kg, we have one helium nucleus, of mass approximately $4m_p$, and 14 electrons, each of mass $m_e = 9.1\ 10^{-31}$ kg, altogether 27 particles. Since most of the matter in the Sun is completely ionized, most of the electrons are free and not bound in atoms. It follows that the average particle mass is

$$\overline{m} \approx \frac{16m_p}{27} \; . \tag{4.39}$$

Hence,

$$\overline{T} \approx \frac{16GM_\odot m_p}{81k_BR_\odot} = \frac{16 \cdot 6.67\ 10^{-11}\ \text{N}\,\text{m}^2\,\text{kg}^{-2}\ \ 2.0\ 10^{30}\ \text{kg}\ 1.67\ 10^{-27}\ \text{kg}}{81 \cdot 1.38\ 10^{-23}\ \text{J/K}\ 7.0\ 10^8\ \text{m}} = 4.6\ 10^6\ \text{K} \; . \tag{4.40}$$

Remember that this is an estimate of the average temperature of the Sun, and the temperature at the centre must be somewhat higher. It is actually about three times as high, according to the "standard solar model". But we see that the estimate we arrived at gives a rather good idea about the central temperature of the Sun. It must be around ten million kelvin, it must be, for example, higher than one million kelvin, and lower than one hundred million kelvin.

4.7 Gravitational stability of a gas cloud

Imagine a spherical gas cloud of radius R, containing N particles of average mass $\overline{m}$, having a uniform mass density ρ (mass per volume), and a uniform temperature T. The total mass of the cloud is then

$$M = N\overline{m} = \frac{4\pi}{3}\,\rho R^3\ . \tag{4.41}$$

The total kinetic energy is

$$E_K = \frac{3}{2}\,Nk_BT\ , \tag{4.42}$$

and the total gravitational potential energy is

$$V = -\frac{3}{5}\,\frac{GM^2}{R}\ . \tag{4.43}$$

According to the virial theorem, the equation $2E_K + V = 0$, that is,

$$3Nk_BT - \frac{3}{5}\,\frac{GM^2}{R} = 0\ , \tag{4.44}$$

is a necessary condition for gravitational equilibrium of the cloud. We may write this equation in several equivalent ways, for example,

$$k_BT = \frac{GM^2}{5NR} = \frac{GM\overline{m}}{5R} = \frac{4\pi}{15}\,G\overline{m}\rho R^2 = \left(\frac{4\pi\rho M^2}{3}\right)^{\frac{1}{3}}\,\frac{G\overline{m}}{5}\ . \tag{4.45}$$

We should remember that this is only a *global* equilibrium condition. In addition to the global condition, there must hold everywhere a *local* equilibrium condition. In particular, it is also necessary for equilibrium that the pressure increases towards the centre of the cloud in such a way that it balances the gravitational attraction between all parts of the cloud. When the pressure increases towards the centre, the density must also increase.

If equation (4.44) does not hold, then we know immediately that the cloud is not in gravitational equilibrium. For example, if the temperature is too high for a given mass, radius and density, then the kinetic energy is too large for equilibrium, and the cloud must expand. If the temperature is too low, then the cloud must contract as a whole due to internal gravitational forces, because the gas pressure is too low to resist the contraction. Thus, there exists a critical temperature T_J, called the *Jeans temperature*, given by a formula similar to equation (4.38),

$$T_J = \frac{GM\overline{m}}{5k_BR} = \frac{4\pi}{15k_B}\,G\overline{m}\rho R^2 = \left(\frac{4\pi\rho M^2}{3}\right)^{\frac{1}{3}}\,\frac{G\overline{m}}{5k_B}\ . \tag{4.46}$$

The condition for instability against contraction, $T < T_J$, may be written in several equivalent ways. For example, if the temperature T and density ρ are given, the instability condition is that $R > R_J$, where R_J is a critical radius, the *Jeans radius*,

$$R_J = \sqrt{\frac{15k_BT}{4\pi G\rho\,\overline{m}}}\ , \tag{4.47}$$

or equivalently that $M > M_J$, where M_J is the *Jeans mass*,

$$M_J = \frac{4\pi}{3}\,\rho R_J^3 = \sqrt{\frac{3}{4\pi\rho}\left(\frac{5k_BT}{G\,\overline{m}}\right)^3}\ . \tag{4.48}$$

If we take it for granted that the cloud is going to reach a state where it is in gravitational equilibrium, an entirely different question is whether this equilibrium state will be stable or unstable. The virial theorem indicates that it will be stable, by the following reasoning, which is not a strict proof of stability. In order to investigate the stability, we have to ask what happens if the system is perturbed just a little bit away from equilibrium.

We know that the total energy E is a constant of motion of the gas cloud, unless energy is supplied to it or removed from it by interaction with the environment. One important mechanism for removing energy is thermal radiation. However, it takes time to radiate away a substantial amount of energy, perhaps a hundred thousand years for a gas cloud of stellar mass, and on a shorter time scale the energy E is constant, to a good approximation. Hence, any perturbation away from the equilibrium state which is physically realizable on a reasonably short time scale must take the cloud into a state of the same total energy E. What need not hold in the perturbed state is the virial theorem, thus we may have either $2E_K + V > 0$ or $2E_K + V < 0$.

If the perturbed state has

$$E_K + E = 2E_K + V > 0\ , \tag{4.49}$$

then this is a state where the kinetic energy E_K is larger than in an equilibrium state of total energy E. When the kinetic energy is too large for the cloud to be in gravitational equilibrium, it will expand. As a result of the expansion, the potential energy, which is always negative, increases towards zero, that is, it changes from V to $V+\Delta V$, with $\Delta V > 0$. Since $E = E_K+V$ is constant, the kinetic energy changes from E_K to $E_K + \Delta E_K$, where $\Delta E_K = -\Delta V < 0$. Consequently, $2E_K + V$ changes by the amount $2\Delta E_K + \Delta V = -\Delta V < 0$, approaching the equilibrium value of zero.

If, on the other hand, the perturbed state has

$$E_K + E = 2E_K + V < 0\ , \tag{4.50}$$

then it means that the kinetic energy E_K is smaller than in an equilibrium state of energy E, and the cloud will contract. As a result, the potential energy becomes more negative, changing from V to $V + \Delta V$, with $\Delta V < 0$. Since $E = E_K + V$ is still constant, we have that $\Delta E_K = -\Delta V > 0$, and $2E_K + V$ changes by the amount $2\Delta E_K + \Delta V = -\Delta V > 0$, again approaching the equilibrium value of zero. This reasoning is a strong indication, if not a complete proof, that the equilibrium condition $2E_K + V = 0$ is stable.

The equilibrium condition

$$2E_K + V = 0 \tag{4.51}$$

and the energy equation

$$E = E_K + V \tag{4.52}$$

together imply that

$$E_K = -E\ , \qquad V = 2E\ . \tag{4.53}$$

Hence, as long as E is constant, the potential energy V must also be constant, which means that the cloud can neither expand nor contract. The only way that the cloud is able to contract slowly, on a longer time scale, while maintaining its gravitational equilibrium on the shorter time scale, is that it slowly radiates away its energy E. The paradoxical result when the negative energy E is reduced, is that the negative potential energy $V = 2E$ is also reduced, which means that the cloud contracts, and that the kinetic energy $E_K = -E$ increases, which means that the average temperature increases. The only way to halt or to reverse the contraction, at least temporarily, is to convert nuclear energy, or some other nongravitational form of energy, into kinetic energy, as is done inside the Sun.

We see that a gas in gravitational equilibrium has the strange property that when it radiates away energy, its temperature increases. In other words, it has a negative heat capacity.

This process, in which a cloud of gas slowly contracts under its own gravitation, radiating away energy, and simultaneously being heated until the central temperature becomes high enough that thermonuclear reactions can start, is how stars are born.

Chapter 5

Non-interacting particles in quantum mechanics

5.1 One particle in a one dimensional box

The non-relativistic particle in a box is one of the standard problems in quantum mechanics. The energy of the particle is only its kinetic energy

$$E_K = \frac{1}{2}\, m\boldsymbol{v}^2 = \frac{\boldsymbol{p}^2}{2m} \;, \tag{5.1}$$

where m is the mass, $\boldsymbol{v}$ the velocity, and $\boldsymbol{p} = m\boldsymbol{v}$ the momentum of the particle.

The very simplest case is one particle in one dimension, with position x, confined to a box of length $\mathcal{L}$. An *energy eigenstate* of the particle is a solution of the time independent Schrödinger equation ($h = 2\pi\hbar$ is Planck's constant)

$$-\frac{\hbar^2}{2m}\,\psi''(x) = E\,\psi(x) \tag{5.2}$$

for $0 \le x \le \mathcal{L}$, with boundary conditions $\psi(0) = \psi(\mathcal{L}) = 0$. The energy eigenfunctions are

$$\psi(x) = \psi_n(x) = \sqrt{\frac{2}{\mathcal{L}}}\,\sin(kx) \qquad \text{with} \qquad k = \frac{n\pi}{\mathcal{L}} \;, \quad n = 1, 2, 3, \dots \;, \tag{5.3}$$

and the corresponding energy eigenvalues are

$$E = E_n = \frac{\hbar^2 k^2}{2m} = \frac{n^2\pi^2\hbar^2}{2m\mathcal{L}^2} = \frac{n^2 h^2}{8m\mathcal{L}^2} \;. \tag{5.4}$$

These wave functions are normalized such that

$$\int_0^{\mathcal{L}} \mathrm{d}x\; |\psi_n(x)|^2 = 1 \;. \tag{5.5}$$

Note that the energy eigenstates are not momentum eigenstates, since the wave function $\sin(kx)$ is a superposition,

$$\sin(kx) = \frac{1}{2\mathrm{i}}\,(\mathrm{e}^{\mathrm{i}kx} - \mathrm{e}^{-\mathrm{i}kx}) \;, \tag{5.6}$$

of two momentum eigenstates $\mathrm{e}^{\pm\mathrm{i}kx}$ having momentum eigenvalues $p = \pm\hbar k = \pm nh/\mathcal{L}$. This corresponds to the classical picture that the particle is bouncing back and forth between the boundaries $x = 0$ and $x = \mathcal{L}$.

We see that if we pick a momentum interval $-p_1$ to p_1, with $p_1 = n_1 h/\mathcal{L} > 0$, then the number of energy eigenstates with momentum p between $-p_1$ and p_1 is

$$\mathcal{N} = n_1 = \frac{2p_1\mathcal{L}}{h} \,. \tag{5.7}$$

In this formula, $2p_1$ is the length of the momentum interval, and $\mathcal{L}$ is the length of the one dimensional box. Hence, the number of states is

$$\mathcal{N} = \frac{\mathcal{W}}{h} \,, \tag{5.8}$$

where $\mathcal{W} = 2p_1\mathcal{L}$ is the *phase space volume* (or phase space area in this case), equal to the product of the volumes in momentum space and in ordinary space. Of course, a one dimensional volume is a length, and a two dimensional volume is an area.

5.2 A three dimensional box

The generalization from one to three dimensions is straightforward. The three dimensional time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\,\nabla^2\psi(x,y,z) = E\,\psi(x,y,z) \,, \tag{5.9}$$

for $0 \le x \le \mathcal{L}_x$, $0 \le y \le \mathcal{L}_y$, $0 \le z \le \mathcal{L}_z$, with boundary conditions $\psi(x,y,z) = 0$ for $x = 0$, $x = \mathcal{L}_x$, $y = 0$, $y = \mathcal{L}_y$, $z = 0$, $z = \mathcal{L}_z$. The solutions are

$$\psi(x,y,z) = \psi_{n_x,n_y,n_z}(x) = \sqrt{\frac{8}{\mathcal{L}_x\mathcal{L}_y\mathcal{L}_z}}\;\sin(k_x x)\sin(k_y y)\sin(k_z z) \,, \tag{5.10}$$

with

$$k_j = \frac{n_j\pi}{\mathcal{L}_j} \,, \quad n_j = 1,2,3,\ldots \,, \quad j = x,y,z \,, \tag{5.11}$$

and with the energy eigenvalues

$$E = E_{n_x,n_y,n_z} = \frac{\hbar^2(k_x^{\,2} + k_y^{\,2} + k_z^{\,2})}{2m} = \frac{h^2}{8m}\left(\frac{n_x^{\,2}}{\mathcal{L}_x^{\,2}} + \frac{n_y^{\,2}}{\mathcal{L}_y^{\,2}} + \frac{n_z^{\,2}}{\mathcal{L}_z^{\,2}}\right) \,. \tag{5.12}$$

Again, the energy eigenstates are not momentum eigenstates, in this case the wave function is a superposition of 8 momentum eigenstates $\mathrm{e}^{\mathrm{i}(\pm k_x x \pm k_y y \pm k_z z)}$ having momentum components $p_x = \pm\hbar k_x$, $p_y = \pm\hbar k_y$, $p_z = \pm\hbar k_z$. Which is just what we would expect from the classical picture that the particle is bouncing between the walls of the box.

We see that if we pick a rectangular box in momentum space, with

$$-p_1 \le p_x \le p_1 \,, \qquad -p_2 \le p_y \le p_2 \,, \qquad -p_3 \le p_z \le p_3 \,, \tag{5.13}$$

and with

$$p_1 = \frac{n_1 h}{\mathcal{L}_x}\,, \qquad p_2 = \frac{n_2 h}{\mathcal{L}_y}\,, \qquad p_3 = \frac{n_3 h}{\mathcal{L}_z}\,, \tag{5.14}$$

then the number of energy eigenstates with momentum inside the box is again proportional to the phase space volume $\mathcal{W}$,

$$\mathcal{N} = n_1 n_2 n_3 = \frac{(2p_1\mathcal{L}_x)(2p_2\mathcal{L}_y)(2p_3\mathcal{L}_z)}{h^3} = \frac{\mathcal{W}}{h^3}\,. \tag{5.15}$$

We may formulate our result in the following way. Let $\mathcal{V} = \mathcal{L}_x\mathcal{L}_y\mathcal{L}_z$ be the volume of the box containing the particle. Then the number of states in an infinitesimal volume $\mathrm{d}^3\boldsymbol{p}$ in momentum space is

$$\mathrm{d}\mathcal{N} = \frac{\mathcal{V}}{h^3}\,\mathrm{d}^3\boldsymbol{p}\,. \tag{5.16}$$

Spin degeneracy

So far we have not taken into account the spin of the particle. A three dimensional particle may have spin $s = 0, 1/2, 1, 3/2, \ldots$, and for a given momentum there are $g_s = 2s+1$ different spin states. The number of states including the spin degeneracy factor is

$$\mathrm{d}\mathcal{N} = g_s\,\frac{\mathcal{V}}{h^3}\,\mathrm{d}^3\boldsymbol{p}\,. \tag{5.17}$$

Note that photons have spin $s = 1$ but a spin degeneracy factor of $g_s = 2$ instead of $g_s = 2s + 1 = 3$. In fact, a plane polarized photon of momentum $\boldsymbol{p}$ has a polarization vector $\boldsymbol{\epsilon}$ which is orthogonal to $\boldsymbol{p}$, and there are only two such orthogonal directions. The fact that $g_s = 2$ for photons is related to the fact that photons have zero mass.

5.3 Bosons and fermions

Three dimensional particles are either *bosons* or *fermions*. The spin–statistics theorem is a fundamental result in quantum field theory, stating that particles of integer spin are bosons, and particles of half-integer spin are fermions. Thus, photons have spin 1 and are bosons, whereas electrons, protons, neutrons and neutrinos all have spin 1/2 and are fermions.

When two identical particles in a physical system are bosons, then the many-particle wave function must be symmetric under interchange of the arguments of the wave function referring to these two particles (an argument of the wave function is typically a position together with a spin component along some axis). When two identical particles are fermions, then the wave function must be antisymmetric under interchange.

To compute a many-particle wave function is a difficult problem in general, and it does not become easier because the wave function is required to be either symmetric or antisymmetric under interchange of arguments. However, if the particles do not interact with each other, the problem simplifies very much, because the many-particle problem reduces to just a one-particle problem. To construct a many-particle energy eigenfunction we may simply multiply together one-particle energy eigenfunctions, one for each particle, and symmetrize or antisymmetrize afterwards. We get the many-particle energy eigenvalue as a sum of one-particle energy eigenvalues. The assumption of non-interacting particles is often useful, not only because it simplifies a problem, but also because it is a more or less good approximation to reality.

Equilibrium states

The properties of a system of non-interacting particles follow from the very simple rules that one given one-particle state may be occupied by an arbitrary number $n = 0, 1, 2, \ldots$ of identical bosons, but only by $n = 0$ or $n = 1$ fermion of one given species.

The rule that $n = 0$ or $n = 1$ for fermions is the *Pauli exclusion principle.*

An equilibrium state of a system of one single species of identical particles is characterized by two parameters. One parameter is the temperature T. The other parameter is either the number of particles, N, if this is fixed, or else the *chemical potential* μ for this particle species, if N is not fixed. The chemical potential for non-interacting particles may be understood as a zero level for the one-particle energy.

Non-interacting bosons

Given the temperature T and the chemical potential μ, the probablity of having $n = 0, 1, 2, \ldots$ identical non-interacting bosons in a given one-particle quantum state of energy E is

$$P_n = \mathrm{e}^{-\frac{n(E-\mu)}{kT}} (1 - \mathrm{e}^{-\frac{E-\mu}{kT}}) = w^n(1 - w) \;, \tag{5.18}$$

in terms of the Boltzmann factor

$$w = \mathrm{e}^{-\frac{E-\mu}{kT}} \;. \tag{5.19}$$

Here $k = k_B$ is Boltzmann's constant. The sum of these probabilities is

$$\sum_{n=0}^{\infty} P_n = (1 - w) + (w - w^2) + (w^2 - w^3) + \cdots + (w^n - w^{n+1}) + \cdots = 1 \;. \tag{5.20}$$

The average number of particles in the one-particle state, the *occupation number*, is

$$\begin{aligned} f \;&=\; \sum_{n=0}^{\infty} nP_n = (w - w^2) + 2(w^2 - w^3) + \cdots + n(w^n - w^{n+1}) + \cdots \\ &=\; \sum_{n=1}^{\infty} w^n = \frac{w}{1-w} = \frac{1}{\mathrm{e}^{\frac{E-\mu}{kT}} - 1} \;. \end{aligned} \tag{5.21}$$

Non-interacting fermions

The occupation probabilities for fermions are given by the same Boltzmann factor w,

$$P_0 = \frac{1}{1+w} = \frac{1}{1 + \mathrm{e}^{-\frac{E-\mu}{kT}}} \;, \qquad P_1 = \frac{w}{1+w} = \frac{1}{\mathrm{e}^{\frac{E-\mu}{kT}} + 1} \;, \tag{5.22}$$

with $P_0 + P_1 = 1$. We see that $P_0 = P_1 = 1/2$ for $E = \mu$, whereas $P_0 < 1/2 < P_1$ for $E < \mu$ and $P_1 < 1/2 < P_0$ for $E > \mu$.

The average number of particles in the state, the occupation number or occupation fraction, is

$$f \;=\; \sum_{n=0}^{1} nP_n = P_1 = \frac{w}{1+w} = \frac{1}{\mathrm{e}^{\frac{E-\mu}{kT}} + 1} \;. \tag{5.23}$$

The zero temperature limit is particularly simple. Then the occupation fraction is $f = 1$ for $E < \mu$ and $f = 0$ for $E > \mu$. In other words, all one-particle states below the *Fermi energy* $E_F = \mu$ are occupied, and all states above the Fermi energy are empty.

5.4 The photon gas

Photons are bosons that interact very little with each other, hence the above theory applies directly to the photon gas. The electromagnetic field does not interact with itself, at least to a first approximation, because Maxwell's equations are linear in the electromagnetic field.

Another simplification is that $\mu = 0$, the chemical potential of photons is zero. This follows from the fact that photons are their own antiparticles, by an argument which we do not present here.

Photons move with the speed of light, they are the most relativistic particles of all. But this will not prevent us from using the above formula for the number $\mathcal{N}$ of one-particle states in a volume $\mathcal{V}$, which we derived for non-relativistic particles. The formula is indeed valid, by another argument which we do not give here.

The energy of a photon of frequency ν is $E = h\nu = pc$, where $p = |\boldsymbol{p}|$, and $\boldsymbol{p}$ is the photon momentum. From the above results, we obtain the following formula for the number density n of the photon gas, the number of photons per volume,

$$\mathrm{d}n = \frac{f\,\mathrm{d}\mathcal{N}}{\mathcal{V}} = \frac{1}{\mathrm{e}^{\frac{E}{kT}} - 1}\,\frac{g_s}{h^3}\,\mathrm{d}^3\boldsymbol{p} = \frac{2}{h^3}\,\frac{1}{\mathrm{e}^{\frac{E}{kT}} - 1}\,\mathrm{d}^3\boldsymbol{p}\ . \tag{5.24}$$

We introduce polar coordinates $p, \theta\varphi$ for the momentum $\boldsymbol{p}$, and write

$$\mathrm{d}^3\boldsymbol{p} = p^2\,\mathrm{d}p\,\mathrm{d}(\cos\theta)\,\mathrm{d}\varphi = \frac{h^3\nu^2}{c^3}\,\mathrm{d}\nu\,\mathrm{d}\Omega\ . \tag{5.25}$$

In this way we rewrite the number density of photons as

$$\mathrm{d}n = \frac{2\nu^2}{c^3}\,\frac{1}{\mathrm{e}^{\frac{h\nu}{kT}} - 1}\,\mathrm{d}\nu\,\mathrm{d}\Omega\ . \tag{5.26}$$

The energy density of the photon gas is

$$\mathrm{d}\epsilon = E\,\mathrm{d}n = h\nu\,\mathrm{d}n = \frac{2h\nu^3}{c^3}\,\frac{1}{\mathrm{e}^{\frac{h\nu}{kT}} - 1}\,\mathrm{d}\nu\,\mathrm{d}\Omega\ . \tag{5.27}$$

The Stefan–Boltzmann radiation law

In order to derive the Stefan–Boltzmann law, we compute the total energy of the photons crossing an area $\mathcal{A}$ in a time interval $\mathrm{d}t$. Let the area be perpendicular to the z-axis, and consider photons with frequency ν and velocity components $v_x = c\,\sin\theta\cos\varphi$, $v_y = c\,\sin\theta\sin\varphi$, $v_z = c\,\cos\theta$. These photons pass through the given area in the time interval from t to $t + \mathrm{d}t$ if at time t they are inside a certain volume of size

$$\mathrm{d}\mathcal{V} = \mathcal{A}v_z\,\mathrm{d}t = \mathcal{A}c\,\cos\theta\,\mathrm{d}t\ . \tag{5.28}$$

The number of photons is $\mathrm{d}n\,\mathrm{d}\mathcal{V}$, and their energy is $\mathrm{d}\epsilon\,\mathrm{d}\mathcal{V}$. The energy flux $\mathcal{F}$ is the energy per area and per time,

$$\mathrm{d}\mathcal{F} = \frac{\mathrm{d}\epsilon\,\mathrm{d}\mathcal{V}}{\mathcal{A}\,\mathrm{d}t} = \frac{2h\nu^3}{c^3}\,\frac{1}{\mathrm{e}^{\frac{h\nu}{kT}} - 1}\,\mathrm{d}\nu\,\mathrm{d}\Omega\,c\,\cos\theta\ . \tag{5.29}$$

The energy flux integrated over all angles, with the restriction that $v_z = c\cos\theta > 0$, and over all frequences, is

$$\mathcal{F} = \int_0^\infty \mathrm{d}\nu \int_0^1 \mathrm{d}(\cos\theta) \int_0^{2\pi} \mathrm{d}\varphi \, \frac{2h\nu^3}{c^3} \, \frac{1}{\mathrm{e}^{\frac{h\nu}{kT}} - 1} \, c\cos\theta = \int_0^\infty \mathrm{d}\nu \, \frac{2\pi h\nu^3}{c^2} \, \frac{1}{\mathrm{e}^{\frac{h\nu}{kT}} - 1} \; . \tag{5.30}$$

To compute the last integral, we change integration variable to

$$u = \frac{h\nu}{kT} \; , \tag{5.31}$$

and get that

$$\mathcal{F} = \frac{2\pi h}{c^2} \left(\frac{kT}{h}\right)^4 \int_0^\infty \mathrm{d}u \, \frac{u^3}{\mathrm{e}^u - 1} \; . \tag{5.32}$$

We may compute the integral as follows,

$$\int_0^\infty \mathrm{d}u \, \frac{u^3}{\mathrm{e}^u - 1} = \int_0^\infty \mathrm{d}u \, \frac{u^3\,\mathrm{e}^{-u}}{1 - \mathrm{e}^{-u}} = \int_0^\infty \mathrm{d}u \, u^3 \sum_{k=1}^\infty \mathrm{e}^{-ku} = \sum_{k=1}^\infty \frac{6}{k^4} = \frac{\pi^4}{15} \; , \tag{5.33}$$

and the following trick is useful,

$$\int_0^\infty \mathrm{d}u \, u^3\,\mathrm{e}^{-ku} = \left(-\frac{\mathrm{d}}{\mathrm{d}k}\right)^3 \int_0^\infty \mathrm{d}u \, \mathrm{e}^{-ku} = \left(-\frac{\mathrm{d}}{\mathrm{d}k}\right)^3 \frac{1}{k} = \frac{6}{k^4} \; . \tag{5.34}$$

The final result is the Stefan–Boltzmann law for black body radiation. It gives the energy flux radiated from a black surface of temperature T as

$$\mathcal{F} = \sigma T^4 \; , \tag{5.35}$$

where σ is the Stefan–Boltzmann constant,

$$\sigma = \frac{2\pi^5 k^4}{15 h^3 c^2} = 5.67\;10^{-8}\;\mathrm{W/(m^2\;K^4)} \; . \tag{5.36}$$

5.5 The non-relativistic degenerate fermion gas

In order to understand the behaviour of matter in compact stars, the simplified model of a degenerate gas of non-interacting fermions is very useful. We neglect the interactions between the particles, because this enables us to compute the equation of state analytically. As we have seen, we obtain the quantized energy levels of the many-particle system simply by computing the one-particle energies and adding them.

In neutron stars, and in the central regions of white dwarf stars, the electrons have relativistic energies, i.e. velocities close to c, the speed of light. The atomic nuclei, and the neutrons and protons, are non-relativistic even in neutron stars. Here we will treat the non-interacting degenerate electron gas non-relativistically, the relativistic degenerate electron gas is outside our scope.

In a gas of non-interacting fermions, any quantum state of the one-particle system is either empty or filled, the number of particles in the state is either zero or one. The fermion gas is said to become *degenerate* in the limit of zero temperature, $T \to 0$, then all states up to a given energy

$$E_F = \frac{p_F^2}{2m} \tag{5.37}$$

are filled, and all states above this energy are empty. Here $E_F = \mu$ is the Fermi energy, or chemical potential, and p_F is called the Fermi momentum.

The number of energy levels below the Fermi energy E_F is

$$\mathcal{N}_F = g_s \frac{\mathcal{W}}{h^3} = g_s \frac{\mathcal{V}}{h^3} \frac{4\pi p_F^3}{3} , \tag{5.38}$$

where $\mathcal{W}$ is the phase space volume, equal to the product of the volume $\mathcal{V}$ in ordinary space, and the volume $4\pi p_F^3/3$ of a sphere of radius p_F in momentum space.

In this formula there is a spin degeneracy factor $g_s = 2s + 1$. Thus, $g_s = 2$ for electrons and for neutrons, having $s = 1/2$. The factor of 2 appears because the spin component along an arbitrary axis, often chosen to be the z axis, is either $+\hbar/2$, this is called "spin up", or $-\hbar/2$, called "spin down".

Thus, the particle density in the degenerate electron gas, the number of electrons per volume, is

$$n = \frac{\mathcal{N}_F}{\mathcal{V}} = \frac{8\pi p_F^3}{3h^3} = \frac{8\pi (2mE_F)^{\frac{3}{2}}}{3h^3} . \tag{5.39}$$

This expression may also be written as an integral,

$$n = \frac{8\pi}{h^3} \int_0^{p_F} p^2 \, \mathrm{d}p , \tag{5.40}$$

meaning that the contribution to the particle density n from the particles with absolute value of the momentum between p and $p + \mathrm{d}p$ is

$$\mathrm{d}n = \frac{8\pi}{h^3} p^2 \, \mathrm{d}p . \tag{5.41}$$

Since a particle with momentum in this interval has energy $E = p^2/(2m)$, we get the following expression for the energy density,

$$\epsilon = \frac{8\pi}{h^3} \int_0^{p_F} \frac{p^2}{2m} p^2 \, \mathrm{d}p = \frac{4\pi p_F^5}{5h^3 m} = \frac{3}{5} E_F \, n . \tag{5.42}$$

This means that the average energy per particle is 3/5 of the Fermi energy.

The pressure P of the degenerate electron gas is minus the derivative of the energy $E = \epsilon\mathcal{V}$ with respect to the volume $\mathcal{V}$,

$$P = -\frac{\mathrm{d}E}{\mathrm{d}\mathcal{V}} = -\frac{\mathrm{d}\epsilon}{\mathrm{d}\mathcal{V}} \mathcal{V} - \epsilon . \tag{5.43}$$

We let the volume change by $\mathrm{d}\mathcal{V}$ while keeping the number of particles, $N = n\mathcal{V}$, constant, thus we have that

$$0 = \mathrm{d}N = \mathrm{d}n \, \mathcal{V} + n \, \mathrm{d}\mathcal{V} , \tag{5.44}$$

or,

$$\frac{\mathrm{d}\mathcal{V}}{\mathcal{V}} = -\frac{\mathrm{d}n}{n} \; . \tag{5.45}$$

Hence the pressure is

$$P = n\,\frac{\mathrm{d}\epsilon}{\mathrm{d}n} - \epsilon \; . \tag{5.46}$$

The energy density ϵ as a function of the number density n is

$$\epsilon = \frac{3}{5}\,E_F\,n = \frac{3}{5}\,\frac{p_F^2}{2m}\,n = \frac{3}{10m}\left(\frac{3h^3n}{8\pi}\right)^{\frac{2}{3}} n = \frac{3h^2}{40m}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} n^{\frac{5}{3}} \; . \tag{5.47}$$

Taking the logarithm, we get that

$$\ln\epsilon = \text{constant} + \frac{5}{3}\,\ln n \; , \tag{5.48}$$

and hence, by differentiation,

$$\frac{\mathrm{d}\epsilon}{\epsilon} = \frac{5}{3}\,\frac{\mathrm{d}n}{n} \; . \tag{5.49}$$

It follows that

$$n\,\frac{\mathrm{d}\epsilon}{\mathrm{d}n} = \frac{5}{3}\,\epsilon \; , \tag{5.50}$$

and the pressure is

$$P = \frac{2}{3}\,\epsilon = \frac{2}{5}\,E_F\,n = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} n^{\frac{5}{3}} = 2.337\;10^{-38}\;\mathrm{N\;m^3}\;n^{\frac{5}{3}} \; . \tag{5.51}$$

The numerical factor in this formula is obtained when the particle mass m is taken to be the electron mass.

This relation should be compared to the familiar ideal gas law

$$P\mathcal{V} = NkT \; , \tag{5.52}$$

or, when we introduce the number density $n = N/\mathcal{V}$,

$$P = nkT \; . \tag{5.53}$$

This law predicts that the pressure goes to zero when the gas is cooled to zero temperature. In reality the electron gas will become degenerate when the temperature is sufficiently low, and the pressure can never be lower than the degeneration pressure given in equation (5.51). In order to estimate the degeneration temperature T_d, at which the gas becomes degenerate, we set the ideal gas pressure nkT_d equal to the degeneration pressure, and the temperature we then get is

$$T_d = \frac{1}{nk}\,\frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} n^{\frac{5}{3}} = \frac{h^2}{20mk}\left(\frac{3n}{\pi}\right)^{\frac{2}{3}} = 1.693\;10^{-15}\;\mathrm{K\;m^2}\;n^{\frac{2}{3}} \; . \tag{5.54}$$

Again, the numerical factor is obtained by taking m to be the electron mass.

The densities of matter that we encounter in everyday life are of the order of a few thousand kg/m^3. For every electron in such matter there is exactly one proton, to make the total electric charge equal to zero, and approximately one neutron, thus the mass per electron is close to two atomic mass units, $2\text{u} = 3.3\ 10^{-27}$ kg. Thus, the electron density, for a mass density of 10^3 kg/m^3, the density of water, is

$$n = \frac{10^3\ \text{kg/m}^3}{3.3\ 10^{-27}\ \text{kg}} = 3.0\ 10^{29}\ \text{m}^{-3}\ . \tag{5.55}$$

For this electron density the degeneration temperature is 76 000 K. Clearly the solid and liquid matter surrounding us is degenerate.

However, the air we breath is much thinner, its density is about 1000 times smaller, implying that the degeneration temperature should be 100 times smaller, only 760 K. This is still a rather high degeneration temperature, in contradiction to the fact that air is not at all a degenerate gas. On the contrary, its equation of state is well represented by the ideal gas law. We have to admit that it is too crude an approximation to treat air as a gas of free *electrons*. In fact the electrons are not free, they are bound in atoms, and so it is a better approximation to insert in equation (5.54) the atomic mass rather than the electron mass.

At the centre of the Sun, the mass density is 160 times the density of water, and then the degeneration temperature for the electron gas is $2.3\ 10^6$ K. Since the central temperature of the Sun is around $15\ 10^6$ K, the gas there is not quite degenerate.

Chapter 6

Compact stars: white dwarfs and neutron stars

White dwarfs and neutron stars are extreme objects, where the matter densities, gravitational fields, magnetic fields, and everything else are extreme. It may seem strange, therefore, that the structure of a white dwarf is rather easy to understand theoretically, at lest on a basic level. The conditions inside and around a neutron star are much more extreme and complicated, and by no means understood in every detail.

The surfaces of many white dwarfs and most neutron stars are hot in comparison with ordinary stars. Nevertheless, all such compact stars are cold, in a certain sense, and may even be treated as objects of zero temperature. The reason is that the matter density is so high that the average kinetic energy of the particles is much higher than the thermal energy k_BT (as usual, k_B is Boltzmann's constant and T is the temperature). Matter under such conditions is said to be *degenerate*.

6.1 White dwarfs

The pressure keeping a white dwarf in hydrostatic equilibrium is supplied mainly by degenerate electrons. We may derive an approximate relation between the mass M and radius R of a white dwarf by assuming uniform density. We also assume that the electrons are non-relativistic. This model is of course oversimplified, but nevertheless gives some useful insight.

In the approximation that the electrons form a non-interacting degenerate gas with Fermi momentum p_F, the electron density (the number of electrons per volume) is

$$n_e = \frac{8\pi p_F^3}{3h^3} \; . \tag{6.1}$$

The total number of electrons, in the uniform density model, is

$$N_e = n_e \, \frac{4\pi R^3}{3} \; . \tag{6.2}$$

It is believed that the atomic nuclei found in many white dwarfs are mostly ^{12}C and ^{16}O. In that case there are the same number of both protons, neutrons and electrons, hence the mass per electron, disregarding nuclear binding energies, is $m_p + m_n + m_e \approx 2m_p$, so that the total mass of the star is $M = 2m_p N_e$.

The virial theorem,

$$2E_K + V = 0 \ . \tag{6.3}$$

must hold even for a white dwarf. In this case the kinetic energy E_K is mainly the kinetic energy of the degenerate electrons, and since the average kinetic energy of one electron is 3/5 of the Fermi energy E_F, we take

$$E_K = \frac{3}{5} E_F N_e \ . \tag{6.4}$$

The gravitational potential energy of a sphere of uniform density is

$$V = -\frac{3}{5} \frac{GM^2}{R} \ . \tag{6.5}$$

Hence, the virial theorem relates the Fermi energy to the mass and radius of the white dwarf,

$$E_F = \frac{5}{3} \frac{E_K}{N_e} = -\frac{5}{3} \frac{V}{2N_e} = \frac{GM^2}{2RN_e} = \frac{GMm_p}{R} \ . \tag{6.6}$$

But the Fermi energy is also related to the electron density, as follows,

$$E_F = \frac{p_F^2}{2m_e} = \frac{1}{2m_e} \left(\frac{3n_e h^3}{8\pi} \right)^{\frac{2}{3}} = \frac{h^2}{8m_e} \left(\frac{3n_e}{\pi} \right)^{\frac{2}{3}} , \tag{6.7}$$

and using the relation

$$n_e = \frac{3N_e}{4\pi R^3} = \frac{3M}{8\pi m_p R^3} , \tag{6.8}$$

we get that

$$E_F = \frac{h^2}{32 m_e R^2} \left(\frac{9M}{\pi^2 m_p} \right)^{\frac{2}{3}} . \tag{6.9}$$

Equating the two expressions for the Fermi energy, equations (6.6) and (6.9), we get the following expression for the radius as a function of the mass,

$$R = \frac{3h^2}{32\pi G m_e} \left(\frac{3}{\pi m_p^5 M} \right)^{\frac{1}{3}} . \tag{6.10}$$

Since the volume of the white dwarf is $\mathcal{V} = 4\pi R^3/3$ we see that, in our oversimplified model assuming constant density, the mass times volume is the same for all white dwarfs,

$$M\mathcal{V} = \left(\frac{3h^2}{32\pi G m_e} \right)^3 \frac{4}{m_p^5} . \tag{6.11}$$

Remember that we assumed here that the electrons are non-relativistic.

It turns out that in a white dwarf of one solar mass or more, the electrons start becoming relativistic. Because the pressure of a relativistic degenerate electron gas increases more slowly with density than does the pressure of a non-relativistic gas, there is an upper limit to the mass a white dwarf can have without collapsing. This limit is the famous Chandrasekhar mass of 1.4 times the solar mass.

Appendix A

Units, vectors, and other notation

A.1 Units

A general rule is that if a physical unit is the name of a person, then it is either written abbreviated as a capital (or with the first letter in capital if the abbreviation has more than one letter), or written in full with no capital. Examples: N = newton, J = joule, C = coulomb, A = ampere. There is no plural "s" ending in unit names: 10 second, not 10 seconds; 6 newton, not 6 newtons.

The unit of time, the *second*, s, is defined by the frequency of electromagnetic waves that are in resonance with the hyperfine transition in cesium atoms. This resonance frequency is by definition

$$\nu_0 = 9\,192\,631\,770\ \mathrm{s}^{-1} = 9\,192\,631\,770\ \mathrm{Hz}\ . \tag{A.1}$$

This is a good operational definition, because it is possible to build an electric circuit oscillating with this frequenecy, and tune it very accurately to be in resonance with cesium atoms. Then one measures time by counting oscillations.

The speed of light in vacuum,

$$c = 299\,792\,458\ \mathrm{m/s}\ . \tag{A.2}$$

is exact because it defines the *meter*, m, as the unit of length.

A.2 Basic notation

This section is a summary of some standard notation.

Time is usually denoted by the symbol t.

A position in three dimensional space is specified by three coordinates (x, y, z), measured along three orthogonal axes. The coordinates have the dimension of length.

Vectors are written here in boldface, but when writing by hand it is easier to use vector arrows. Thus, $\boldsymbol{A}$ and $\vec{A}$ are two notations for the same vector. A vector in general may have any number of components, but our vectors will usually have three components. The components of the three dimensional vector $\boldsymbol{A}$ are called (A_x, A_y, A_z).

The unit vectors along the three orthogonal (x, y, z) coordinate axes are called $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, or in handwriting $\vec{i}$, $\vec{j}$, $\vec{k}$. In general, a vector $\boldsymbol{A}$ is a linear combination of the basis vectors,

$$\boldsymbol{A} = A_x\,\boldsymbol{i} + A_y\,\boldsymbol{j} + A_z\,\boldsymbol{k}\ . \tag{A.3}$$

The point in space with coordinates (x, y, z) is represented as a position vector

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\ . \tag{A.4}$$

If $\boldsymbol{r}$ is the time dependent position of a pointlike particle, $\boldsymbol{r} = \boldsymbol{r}(t)$, then the first and second time derivatives of the position are respectively the velocity $\boldsymbol{v}$ and the acceleration $\boldsymbol{a}$,

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\ , \qquad \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\ . \tag{A.5}$$

It is often convenient to denote a time derivative by a dot, thus we write

$$\boldsymbol{v} = \dot{\boldsymbol{r}}\ , \qquad \boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}\ . \tag{A.6}$$

All of these equations are vector equations. The equation

$$\boldsymbol{v} = v_x\,\boldsymbol{i} + v_y\,\boldsymbol{j} + v_z\,\boldsymbol{k} = \dot{\boldsymbol{r}} = \dot{x}\,\boldsymbol{i} + \dot{y}\,\boldsymbol{j} + \dot{z}\,\boldsymbol{k}\ , \tag{A.7}$$

for example, consists of the three separate equations

$$v_x = \dot{x}\ , \qquad v_y = \dot{y}\ , \qquad v_z = \dot{z}\ . \tag{A.8}$$

Scalar and vector products of vectors

The *scalar product* (dot product, inner product) of two vectors $\boldsymbol{A} = A_x\,\boldsymbol{i} + A_y\,\boldsymbol{j} + A_z\,\boldsymbol{k}$ and $\boldsymbol{B} = B_x\,\boldsymbol{i} + B_y\,\boldsymbol{j} + B_z\,\boldsymbol{k}$ is defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z\ . \tag{A.9}$$

The scalar product is symmetric, or commutative, $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$. The length of the vector $\boldsymbol{A}$ is written as $|\boldsymbol{A}|$, or A, and the length squared is

$$A^2 = |\boldsymbol{A}|^2 = \boldsymbol{A}^2 = \boldsymbol{A} \cdot \boldsymbol{A} = A_x^{\,2} + A_y^{\,2} + A_z^{\,2}\ . \tag{A.10}$$

Two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are *orthgonal* if $\boldsymbol{A} \cdot \boldsymbol{B} = 0$. The vector $\boldsymbol{A}$ is a *unit* vector, or *normal* vector, if it has unit length, $|\boldsymbol{A}| = 1$. Two or more vectors are *orthonormal* if they are all unit vectors, and if any two of them are orthogonal.

The *vector product* (cross product, outer product) of the same two vectors is defined as

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\,\boldsymbol{i} + (A_z B_x - A_x B_z)\,\boldsymbol{j} + (A_x B_y - A_y B_x)\,\boldsymbol{k}\ . \tag{A.11}$$

The vector product is antisymmetric, or anticommutative, $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$. In particular, the antisymmetry relation $\boldsymbol{A} \times \boldsymbol{A} = -\boldsymbol{A} \times \boldsymbol{A}$ has the unique solution $\boldsymbol{A} \times \boldsymbol{A} = 0$: the vector product of a vector with itself vanishes.

Note that the vector product is not associative, unlike other products we are used to. For example, $\boldsymbol{i} \times (\boldsymbol{i} \times \boldsymbol{j}) = \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j} \neq (\boldsymbol{i} \times \boldsymbol{i}) \times \boldsymbol{j} = 0 \times \boldsymbol{j} = 0$. Thus we have in general that

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \neq (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}\ . \tag{A.12}$$

The scalar product is also not associative, for the simple reason that an expression like $\boldsymbol{A} \cdot (\boldsymbol{B} \cdot \boldsymbol{C})$ is meaningless. The scalar product $\boldsymbol{B} \cdot \boldsymbol{C}$ of the two vectors $\boldsymbol{B}$ and $\boldsymbol{C}$ is a scalar, not a vector, and a scalar product between a vector and a scalar, such as $\boldsymbol{A} \cdot (\boldsymbol{B} \cdot \boldsymbol{C})$, has no meaning.

The *triple product*

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) + A_z(B_x C_y - B_y C_x) \tag{A.13}$$

is completely antisymmetric, for example, $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = -\boldsymbol{A} \cdot (\boldsymbol{C} \times \boldsymbol{B})$, and

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C} = -(\boldsymbol{B} \times \boldsymbol{A}) \cdot \boldsymbol{C} = -\boldsymbol{B} \cdot (\boldsymbol{A} \times \boldsymbol{C}) \ . \tag{A.14}$$

Hence, $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = 0$ whenever two of the three vectors are either equal or proportional. The triple product may also be defined as a determinant,

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ A_z & B_z & C_z \end{vmatrix} \ . \tag{A.15}$$

The geometric interpretation of the scalar product is that $\boldsymbol{A} \cdot \boldsymbol{B} = AB\cos\alpha$, where α is the angle between the two vectors. The length of the vector product, in terms of the same angle, is

$$|\boldsymbol{A} \times \boldsymbol{B}| = AB\sin\alpha \ . \tag{A.16}$$

This is the area of the parallelogram spanned by the two vectors, that is, with the two vectors as two of its sides. The triple product is the (positive or negative) volume of the parallelepiped spanned by the three vectors.

A.3 Scalar and vector fields

A *scalar* is either one number, or a physical quantity such that when it is measured, the result will be one number times a physical unit. The important point is that a scalar has only one component, as opposed to a vector, which has (usually) three components.

A *scalar field* ϕ is a function having a scalar value $\phi(x, y, z, t) = \phi(\boldsymbol{r}, t)$ at any given position $\boldsymbol{r}$ at any given time t. Similarly, a *vector field* $\boldsymbol{A}$ is a function having a vector value $\boldsymbol{A}(x, y, z, t) = \boldsymbol{A}(\boldsymbol{r}, t)$ at the position $\boldsymbol{r}$ at the time t. Very often, when we speak of a scalar or a vector, we actually mean a scalar field or a vector field.

As an example from meteorology, the temperature distribution in the atmosphere is a scalar field, whereas the distribution of wind velocities is a vector field.

A.4 Differentiation

If $f = f(x)$ is a function of one variable x, then the derivative of f with respect to x is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \ . \tag{A.17}$$

We also write

$$f' = \frac{\mathrm{d}f}{\mathrm{d}x} \ . \tag{A.18}$$

A third notation for differentiation is the dot for the time derivative, as introduced above.

We write differentiation with respect to x as $\mathrm{d}/\mathrm{d}x$ when x is the only variable. If $f = f(x, y, z)$ is a function of the three variables x, y, z, then we write the *partial differentiation* with respect to one variable x, for fixed values of y and z, as

$$\frac{\partial f(x,y,z)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} \; . \tag{A.19}$$

The three partial derivatives of f with respect to x, y and z may be regarded as the components of a vector, the *gradient* of f, defined as

$$\nabla f = \frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\,\boldsymbol{j} + \frac{\partial f}{\partial z}\,\boldsymbol{k} \; . \tag{A.20}$$

The gradient vector at one point is orthogonal to the level curve of f (the curve along which f is constant) going through this point. It points in the direction in which f is increasing the fastest, and its length is the rate of increase of f in this direction.

We may think of ∇ (called "nabla", or "del") as an operator,

$$\nabla = \boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z} \; , \tag{A.21}$$

producing a vector field ∇f when it acts on a scalar field f.

The chain rule

Assume, for example, that f is a function of one variable x, which in turn is a function of t, so that f is also a function of t. The chain rule tells us how to differentiate $f = f(x(t))$ with respect to t,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t} \; . \tag{A.22}$$

If g is a function of three variables x, y, x, all of which are functions of t, so that g is in the end a function of the single variable t, then the t derivative of $g = g(x(t), y(t), z(t))$ is

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial g}{\partial z}\,\frac{\mathrm{d}z}{\mathrm{d}t} \; . \tag{A.23}$$

In this formula we write the time derivatives as $\mathrm{d}/\mathrm{d}t$, because we differentiate functions of one single variable t, whereas we write the x derivative as $\partial/\partial x$, because it applies to a function depending not only on x, but also on two other variables y and z. The first kind of derivative, applying to functions of one variable, is called a *total* derivative, and the second kind, applying to functions of several variables, is called a *partial* derivative.

In vector notation we write the same formula as above, i.e. the chain rule for the function $g = g(\boldsymbol{r}(t))$, in the following way,

$$\frac{\mathrm{d}g}{\mathrm{d}t} = (\nabla g) \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \cdot \nabla g \; . \tag{A.24}$$

If g is a function of the four variables x, y, z, t, and if all four of these are functions of a fifth variable u, so that $g = g(x(u), y(u), z(u), t(u))$, then by the same chain rule as above we have that

$$\frac{\mathrm{d}g}{\mathrm{d}u} = \frac{\partial g}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}u} + \frac{\partial g}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}u} + \frac{\partial g}{\partial z}\,\frac{\mathrm{d}z}{\mathrm{d}u} + \frac{\partial g}{\partial t}\,\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \cdot \nabla g + \frac{\partial g}{\partial t}\,\frac{\mathrm{d}t}{\mathrm{d}u} \; . \tag{A.25}$$

In the specaial case $t = u$ we will have that

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial g}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial g}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \cdot \nabla g + \frac{\partial g}{\partial t} \,. \tag{A.26}$$

In this case there is an important difference between the partial time derivative $\partial g/\partial t$, which applies to only the fourth argument of the function $g = g(x, y, z, t)$, and the total time derivative $\mathrm{d}g/\mathrm{d}t$, which applies also to the time dependence of the first three arguments x, y, z.

Appendix B

The Coriolis and centrifugal forces

We will derive here the extra terms that appear in Newton's second law if our coordinate system is not inertial, but is rotating relative to an inertial system. Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be unit vectors in the rotating system, and let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be unit vectors in an inertial system with the same origin.

For simplicity, we consider first the special case of rotation about the z-axis with angular velocity Ω. It is useful to define the angular velocity as a vector along the rotation axis,

$$\boldsymbol{\Omega} = \Omega \boldsymbol{k} = \Omega \boldsymbol{e}_3 \ . \tag{B.1}$$

We take the unit vector $\boldsymbol{e}_3 = \boldsymbol{k}$ to be fixed relative to the inertial system, whereas the two unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ rotate,

$$\begin{aligned} \boldsymbol{e}_1 &= \quad \cos(\Omega t)\, \boldsymbol{i} + \sin(\Omega t)\, \boldsymbol{j} \ , \\ \boldsymbol{e}_2 &= -\sin(\Omega t)\, \boldsymbol{i} + \cos(\Omega t)\, \boldsymbol{j} \ . \end{aligned} \tag{B.2}$$

The position of a particle is given by coordinates x, y, z relative to the rotating coordinate system, so that the position vector is

$$\boldsymbol{r} = x\, \boldsymbol{e}_1 + y\, \boldsymbol{e}_2 + z\, \boldsymbol{e}_3 \ . \tag{B.3}$$

By definition, the velocity and acceleration relative to the rotating coordinate system are

$$\boldsymbol{v} = \dot{x}\, \boldsymbol{e}_1 + \dot{y}\, \boldsymbol{e}_2 + \dot{z}\, \boldsymbol{e}_3 \tag{B.4}$$

and

$$\boldsymbol{a} = \ddot{x}\, \boldsymbol{e}_1 + \ddot{y}\, \boldsymbol{e}_2 + \ddot{z}\, \boldsymbol{e}_3 \ , \tag{B.5}$$

with $\dot{x} = \mathrm{d}x/\mathrm{d}t$, $\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2$, and so on.

In order to apply Newton's second law we need the acceleration relative to the inertial system. The velocity relative to the inertial system is

$$\boldsymbol{v}_{\mathrm{is}} = \dot{\boldsymbol{r}} = \dot{x}\, \boldsymbol{e}_1 + \dot{y}\, \boldsymbol{e}_2 + \dot{z}\, \boldsymbol{e}_3 + x\, \dot{\boldsymbol{e}}_1 + y\, \dot{\boldsymbol{e}}_2 + z\, \dot{\boldsymbol{e}}_3 = \boldsymbol{v} + \boldsymbol{\Omega} \times \boldsymbol{r} \ , \tag{B.6}$$

since

$$\begin{aligned} \dot{\boldsymbol{e}}_1 &= -\Omega \sin(\Omega t)\, \boldsymbol{i} + \Omega \cos(\Omega t)\, \boldsymbol{j} = \quad \Omega \boldsymbol{e}_2 = \boldsymbol{\Omega} \times \boldsymbol{e}_1 \ , \\ \dot{\boldsymbol{e}}_2 &= -\Omega \cos(\Omega t)\, \boldsymbol{i} - \Omega \sin(\Omega t)\, \boldsymbol{j} = -\Omega \boldsymbol{e}_1 = \boldsymbol{\Omega} \times \boldsymbol{e}_2 \ , \\ \dot{\boldsymbol{e}}_3 &= 0 = \boldsymbol{\Omega} \times \boldsymbol{e}_3 \ . \end{aligned} \tag{B.7}$$

In a similar way we find that

$$\dot{\boldsymbol{v}} = \ddot{x}\,\boldsymbol{e}_1 + \ddot{y}\,\boldsymbol{e}_2 + \ddot{z}\,\boldsymbol{e}_3 + \dot{x}\,\dot{\boldsymbol{e}}_1 + \dot{y}\,\dot{\boldsymbol{e}}_2 + \dot{z}\,\dot{\boldsymbol{e}}_3 = \boldsymbol{a} + \boldsymbol{\Omega} \times \boldsymbol{v} \;. \tag{B.8}$$

The acceleration relative to the inertial system is

$$\boldsymbol{a}_{\rm is} = \dot{\boldsymbol{v}}_{\rm is} = \dot{\boldsymbol{v}} + \boldsymbol{\Omega} \times \dot{\boldsymbol{r}} = \boldsymbol{a} + 2\,\boldsymbol{\Omega} \times \boldsymbol{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \;. \tag{B.9}$$

We have to remember the assumption we made that the rotation axis goes through the origin. This implies that $\boldsymbol{\Omega} \times \boldsymbol{r}_0 = 0$ for any $\boldsymbol{r}_0$ on the rotation axis, and hence

$$\boldsymbol{a}_{\rm is} = \boldsymbol{a} + 2\,\boldsymbol{\Omega} \times \boldsymbol{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\boldsymbol{r} - \boldsymbol{r}_0)) \;. \tag{B.10}$$

In this form the formula is generally valid for an arbitrary angular velocity $\boldsymbol{\Omega}$ and a rotation axis going through the point $\boldsymbol{r}_0$, which is also arbitrary.

Newton's second law holds in the inertial system, it tells us that

$$\boldsymbol{F} = m\boldsymbol{a}_{\rm is} \;. \tag{B.11}$$

The force $\boldsymbol{F}$ is called Newtonian, because it determines the motion of the particle relative to the inertial system. In terms of the acceleration $\boldsymbol{a}$ relative to the rotating system, Newton's second law takes the form

$$\boldsymbol{F} = m\,(\boldsymbol{a} + 2\,\boldsymbol{\Omega} \times \boldsymbol{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\boldsymbol{r} - \boldsymbol{r}_0))) \;. \tag{B.12}$$

Moving two terms from the right hand to the left hand side, we obtain the equation

$$\boldsymbol{F} - 2m\,\boldsymbol{\Omega} \times \boldsymbol{v} - m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\boldsymbol{r} - \boldsymbol{r}_0)) = m\,\boldsymbol{a} \;. \tag{B.13}$$

The two extra terms in the equation of motion that appear when we use a rotating coordinate system, are here interpreted as forces, called fictitious forces. They are both proportional to the particle mass m. The velocity dependent term $-2m\,\boldsymbol{\Omega} \times \boldsymbol{v}$ is called the *Coriolis force*. The term $-m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\boldsymbol{r} - \boldsymbol{r}_0))$ is called the *centrifugal force*, it is proportional to the distance from the rotation axis and is directed away from the axis.

Appendix C

Electromagnetism

C.1 Electromagnetic units

Unfortunately, we have to live with at least three different electromagnetic unit systems. We use here the SI system, also called MKSA because it has the basic units meter, kilogram, second and ampere. Many authors use instead the Gaussian or the Heaviside–Lorentz system.

In all three unit systems, Coulomb's law for the force between two point charges q_1 and q_2 at a distance r, in vacuum, is written as

$$F = k\,\frac{q_1 q_2}{r}\ . \tag{C.1}$$

In the MKSA system there is the proportionality constant

$$k = \frac{1}{4\pi\epsilon_0}\ . \tag{C.2}$$

In the Gaussian system we have simply

$$k = 1\ , \tag{C.3}$$

whereas in the Heaviside–Lorentz system we have

$$k = \frac{1}{4\pi}\ . \tag{C.4}$$

Another main difference between the three unit systems is in the expression for the Lorentz force, as described below.

C.2 The electromagnetic field

The *electromagnetic field* consists of the *electric field*

$$\boldsymbol{E} = E_x\,\boldsymbol{i} + E_y\,\boldsymbol{j} + E_z\,\boldsymbol{k} \tag{C.5}$$

and the *magnetic flux density*

$$\boldsymbol{B} = B_x\,\boldsymbol{i} + B_y\,\boldsymbol{j} + B_z\,\boldsymbol{k}\ . \tag{C.6}$$

We write $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{r}, t)$, because the fields are functions of the position $\boldsymbol{r}$ and the time t. $\boldsymbol{E}$ and $\boldsymbol{B}$ are vector fields, each with three components, E_x, E_y, E_z and B_x, B_y, B_z.

We may measure all the six field components, at least in principle, by measuring the force on a pointlike electric charge q which is located at the position $\boldsymbol{r}$ at the time t, and is moving with a velocity $\boldsymbol{v}$. The force $\boldsymbol{F}$ acting on the point charge from the electromagnetic field is called the *Lorentz force*, and is given in the MKSA system by the formula

$$\boldsymbol{F} = q\,(\boldsymbol{E}(\boldsymbol{r}, t) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r}, t))\,. \tag{C.7}$$

Thus, the magnetic field acts on moving charges, but not on charges at rest.

We see from the formula for the Lorentz force that the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ have different dimensions in the MKSA unit system. In both the Gaussian and the Heaviside–Lorentz system these two fields have the same dimension, and we have to compensate for that by writing the Lorentz force as

$$\boldsymbol{F} = q\left(\boldsymbol{E}(\boldsymbol{r}, t) + \frac{1}{c}\,\boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r}, t)\right). \tag{C.8}$$

C.3 Maxwell's equations

The sources of the electromagnetic field are the *electric charge density* and the *electric current density*.

We will denote the electric charge density by $\rho = \rho(\boldsymbol{r}, t)$. It has the dimension of charge per volume, and it depends on position and time. Hopefully, the confusion will not be too large because we use the same symbol ρ elsewhere to denote mass density.

The electric current density, denoted by $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{r}, t)$, has the dimension of charge per area and per time. It also depends on position and time. By definition, $\boldsymbol{n} \cdot \boldsymbol{J}(\boldsymbol{r}, t)$ is the charge passing through a given surface, per area and per time, at the position $\boldsymbol{r}$ and at the time t, when $\boldsymbol{n}$ is the unit normal vector at that point on the surface.

The electromagnetic field is determined from its sources by Maxwell's equations,

$$\begin{array}{lrcl} \text{(I)} & \nabla \cdot \boldsymbol{B} &=& 0\,, \\ \text{(II)} & \nabla \times \boldsymbol{E} + \dfrac{\partial \boldsymbol{B}}{\partial t} &=& 0\,, \\ \text{(III)} & \nabla \cdot \boldsymbol{D} &=& \rho\,, \\ \text{(IV)} & \nabla \times \boldsymbol{H} - \dfrac{\partial \boldsymbol{D}}{\partial t} &=& \boldsymbol{J}\,. \end{array} \tag{C.9}$$

The field quantities going into Maxwell's equations are the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$, and the electric and magnetic flux densities $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$ and $\boldsymbol{B} = \mu_0 \boldsymbol{H}$.

Both ϵ_0 and μ_0 are constants of nature that have defined values in the MKSA unit system. ϵ_0 is the *permittivity of vacuum*, and μ_0 is the *permeability of vacuum*. The definition of μ_0 defines the ampère, A, as the unit of electric current. The two constants are related by the speed of light, defined to be $c = 299\,792\,458$ m/s, and their values are

$$\begin{aligned} \mu_0 &= 4\pi\ 10^{-7}\ \text{N/A}^2\,, \\ \epsilon_0 &= \frac{1}{\mu_0 c^2} = 8.854\,187\,817\ \ldots\ 10^{-12}\ \text{F/m}\,. \end{aligned} \tag{C.10}$$

The four Maxwell's equations (I), (II), (III), and (IV) are actually eight equations, because the equations (II) and (IV) are vector equations. A vector equation has three components, hence it actually represents three equations, one equation for each vector component.

In the equations (III) and (IV) the electric charge density and electric current density appear on the right hand sides as sources of the field. There are no such sources on the right hand sides of the equations (I) and (II), they are source free equations, and the reason is that magnetic charges have never been observed.

The electromagnetic scalar and vector potentials

We obtain the general solution of Maxwell's equations (I) and (II) by introducing the scalar potential $\Phi = \Phi(\boldsymbol{r}, t)$ and the vector potential $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r}, t)$, and then writing

$$\begin{aligned} \boldsymbol{B} &= \nabla \times \boldsymbol{A} \,, \\ \boldsymbol{E} &= -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t} \,. \end{aligned} \tag{C.11}$$

C.4 Electromagnetic waves

One particular type of solutions of Maxwell's equations, in a region where there are no free electric charges acting as sources of the field, so that $\rho = 0$ and $\boldsymbol{J} = 0$, are the plane wave solutions, of the form

$$\Phi(\boldsymbol{r}, t) = 0 \,, \qquad \boldsymbol{A}(\boldsymbol{r}, t) = \boldsymbol{\epsilon} \cos(\omega t - \boldsymbol{\kappa} \cdot \boldsymbol{r} + \varphi_0) \,. \tag{C.12}$$

A plane wave is characterized by an angular frequency ω, a wave vector, or wave number vector, $\boldsymbol{\kappa}$, and a polarization vector $\boldsymbol{\epsilon}$. In addition, we may introduce an arbitrary constant phase angle φ_0. The length of the wave vector, $\kappa = |\boldsymbol{\kappa}|$, is called the wave number.

The period of oscillation of the plane wave is $\Delta t = 2\pi/\omega$, this is the time in which the phase $\omega t - \boldsymbol{\kappa} \cdot \boldsymbol{r} + \varphi_0$ increases by 2π, at a fixed position $\boldsymbol{r}$. The frequency is

$$\nu = \frac{1}{\Delta t} = \frac{\omega}{2\pi} \,. \tag{C.13}$$

The equation

$$\omega t - \boldsymbol{\kappa} \cdot \boldsymbol{r} + \varphi_0 = \text{constant} \tag{C.14}$$

defines a surface of constant phase, which at a given time t is a plane perpendicular to the wave vector $\boldsymbol{\kappa}$. The plane is moving in the direction of its unit normal vector

$$\boldsymbol{n} = \frac{\boldsymbol{\kappa}}{\kappa} \,, \tag{C.15}$$

with a velocity ω/κ. The fact that the surfaces of constant phase are planes, is of course the reason for the name plane wave. By definition, the phase changes by -2π if we go one wave length λ along the normal vector $\boldsymbol{n}$, that is, $\boldsymbol{\kappa} \cdot (\lambda \boldsymbol{n}) = \lambda\kappa = 2\pi$, so that the wave length is

$$\lambda = \frac{2\pi}{\kappa} \,. \tag{C.16}$$

Maxwell's equations impose restrictions on the quantities ω, $\boldsymbol{\kappa}$ and $\boldsymbol{\epsilon}$. Define

$$\begin{aligned} f &= \cos(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \varphi_0)\ , \\ f' &= -\sin(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \varphi_0)\ , \\ f'' &= -\cos(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \varphi_0) = -f\ , \end{aligned} \tag{C.17}$$

so that $\boldsymbol{A} = \boldsymbol{\epsilon} f$. Since the gradient of f is $\nabla f = -\boldsymbol{\kappa} f'$, the magnetic flux density is

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = (\nabla f)\times\boldsymbol{\epsilon} = (-\boldsymbol{\kappa} f')\times\boldsymbol{\epsilon} = -\boldsymbol{\kappa}\times\boldsymbol{\epsilon}\, f' = \boldsymbol{\kappa}\times\boldsymbol{\epsilon}\,\sin(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \varphi_0)\ . \tag{C.18}$$

The electric field is

$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial\boldsymbol{A}}{\partial t} = -\omega\boldsymbol{\epsilon}\, f' = \omega\boldsymbol{\epsilon}\,\sin(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \varphi_0)\ . \tag{C.19}$$

Maxwell's equation (III) with charge density $\rho = 0$ says that

$$0 = \nabla\cdot\boldsymbol{E} = \nabla\cdot(-\omega\boldsymbol{\epsilon}\, f') = -\omega(\nabla f')\cdot\boldsymbol{\epsilon} = -\omega(-\boldsymbol{\kappa} f'')\cdot\boldsymbol{\epsilon} = \omega(\boldsymbol{\kappa}\cdot\boldsymbol{\epsilon}) f''\ . \tag{C.20}$$

It imposes the condition

$$\boldsymbol{\kappa}\cdot\boldsymbol{\epsilon} = 0\ . \tag{C.21}$$

That is, the polarization vector $\boldsymbol{\epsilon}$ has to be orthogonal to the wave vector $\boldsymbol{\kappa}$. We say that plane electromagnetic waves are *transversely polarized*.

The other condition comes from Maxwell's equation (IV) with current density $\boldsymbol{J} = 0$,

$$\nabla\times\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial t} = 0\ . \tag{C.22}$$

Multiplying with μ_0 and using the relation $\mu_0\epsilon_0 = 1/c^2$, we get the equation

$$\nabla\times\boldsymbol{B} - \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} = 0\ . \tag{C.23}$$

Since $\boldsymbol{B} = -\boldsymbol{\kappa}\times\boldsymbol{\epsilon}\, f'$ and $\boldsymbol{E} = -\omega\boldsymbol{\epsilon}\, f'$, we get that

$$\nabla\times\boldsymbol{B} = -(\nabla f')\times(\boldsymbol{\kappa}\times\boldsymbol{\epsilon}) = \boldsymbol{\kappa}\times(\boldsymbol{\kappa}\times\boldsymbol{\epsilon})\, f'' = -\kappa^2\boldsymbol{\epsilon}\, f''\ , \tag{C.24}$$

and

$$\frac{\partial\boldsymbol{E}}{\partial t} = -\omega^2\boldsymbol{\epsilon}\, f''\ . \tag{C.25}$$

We see that Maxwell's equation (IV) holds if

$$\omega = c\kappa\ . \tag{C.26}$$

Dispersion

The relation between the wave length and frequency, or equivalently between the wave number κ and angular frequency ω, is called the *dispersion relation.* The phase velocity of the electromagnetic wave is equal to the speed of light,

$$\frac{\omega}{\kappa} = c \,, \tag{C.27}$$

independent of the wave number. In other words, there is no dispersion.

The absence of dispersion is a property of electromagnetic waves in vacuum. However, perfect vacuum exists nowhere, even in interstellar space there are some atoms floating around, and the dispersion of radio waves travelling hundred of light years may be noticable. For example, in the case of a pulsar with a period of around one millisecond, the dispersion may smear out the pulses so much that when the signal arrives at the Earth, no pulses are observed directly. To see the pulses, one has to Fourier transform the signal and apply corrections for the different time delays of different frequencies.

Appendix D

English and Norwegian terminology

The Norwegian terminology is not always a straightforward translation from English (or vice versa). Some of the less obvious translations are collected here.

English to Norwegian

angular momentum — dreieimpuls, impulsmoment.

angular resolution — vinkeloppløsning.

angular velocity — vinkelhastighet.

brightness or **apparent brightness** (of a star, unit W/m^2) — lysstyrke; tilsynelatende lysstyrke.

Circle, Arctic Circle, Antarctic Circle (66.5° north or south latitude) — polarsirkel, nordlige polarsirkel, sørlige polarsirkel.

constellation — stjernebilde.

eclipse (solar e., lunar e.) — formørkelse (solf., månef.).

equinox (vernal equinox, autumnal equinox) — jevndøgn (vårjevndøgn, høstjevndøgn).

fictitious force (non-Newtonian force, present in an accelerated reference frame, not subject to Newton's 3rd law) — fiktivkraft.

focal length, focal point (of a lens or a mirror) — brennvidde, brennpunkt (for en linse eller et speil).

latitude and longitude — breddegrad og lengdegrad.

limb darkening (we see the Sun darkening towards the edge) — randfordunkling.

luminosity (of a star, unit W) — luminositet; absolutt lysstyrke.

magnitude (dimensionless measure of the brightness of a star) — størrelsesklasse.

absolute magnitude (dimensionless measure of the luminosity) — absolutt størrelsesklasse.

the main sequence (in a Hertzsprung–Russell (HR) diagram) — hovedserien.

mean free path — midlere fri veilengde.

momentum — impuls (bevegelsesmengde, massefart).

objective, eyepiece (of a telescope) — objektiv, okular (til et teleskop).

power (physical quantity, energy per time, unit W) — effekt.

right ascension and declination (celestial coordinates) — rektasensjon og deklinasjon.

Tropic; Tropic of Cancer; Tropic of Capricorn (23.5° north or south latitude) — vendesirkel; nordlige vendesirkel, Krepsens vendesirkel; sørlige vendesirkel, Steinbukkens vendesirkel.

Norwegian to English

breddegrad og lengdegrad — latitude and longitude.

brennvidde, brennpunkt (for en linse eller et speil) — focal length, focal point (of a lens or a mirror).

dreieimpuls, impulsmoment — angular momentum.

effekt (fysisk størrelse, energi pr. tid, enhet W) — power.

fiktivkraft (ikke-Newtonsk kraft, opptrer i et akselerert referansesystem, oppfyller ikke Newtons 3. lov) — fictitious force.

formørkelse (solf., månef.) — eclipse (solar e., lunar e.).

hovedserien (i et Hertzsprung–Russell (HR) diagram) — the main sequence.

impuls (bevegelsesmengde, massefart) — momentum.

jevndøgn (vårjevndøgn, høstjevndøgn) — equinox (vernal equinox, autumnal equinox).

luminositet eller **absolutt lysstyrke** (til en stjerne, enhet W) — luminosity.

lysstyrke eller **tilsynelatende lysstyrke** (til en stjerne, enhet W/m^2) — brightness.

midlere fri veilengde — mean free path.

objektiv, okular (til et teleskop) — objective, eyepiece (of a telescope).

polarsirkel, nordlige polarsirkel, sørlige polarsirkel (66,5° nordlige eller sørlige breddegrad) — Circle, Arctic Circle, Antarctic Circle.

randfordunkling (at solskiven ser mørkere ut nærmere kanten) — limb darkening.

rektasensjon og deklinasjon (himmelkoordinater) — right ascension and declination.

stjernebilde — constellation.

størrelsesklasse (dimensjonsløst mål for lysstyrken til en stjerne) — magnitude.

absolutt størrelsesklasse (dimensjonsløst mål for luminositeten) — absolute magnitude.

vendesirkel; nordlige vendesirkel, Krepsens vendesirkel; sørlige vendesirkel, Steinbukkens vendesirkel (23.5° nordlige eller sørlige breddegrad) — Tropic; Tropic of Cancer; Tropic of Capricorn.

vinkelhastighet — angular velocity.

vinkeloppløsning — angular resolution.