Stochastic Geometry Topological and Experimental

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I. **Prior Work**

II. Delaunay Radius

III. Weights

IV. Wrap

I.1 DEFINITIONS

X = Poisson point process in $\mathbb{R}^n$

homogeneous, with density $\varrho > 0$.

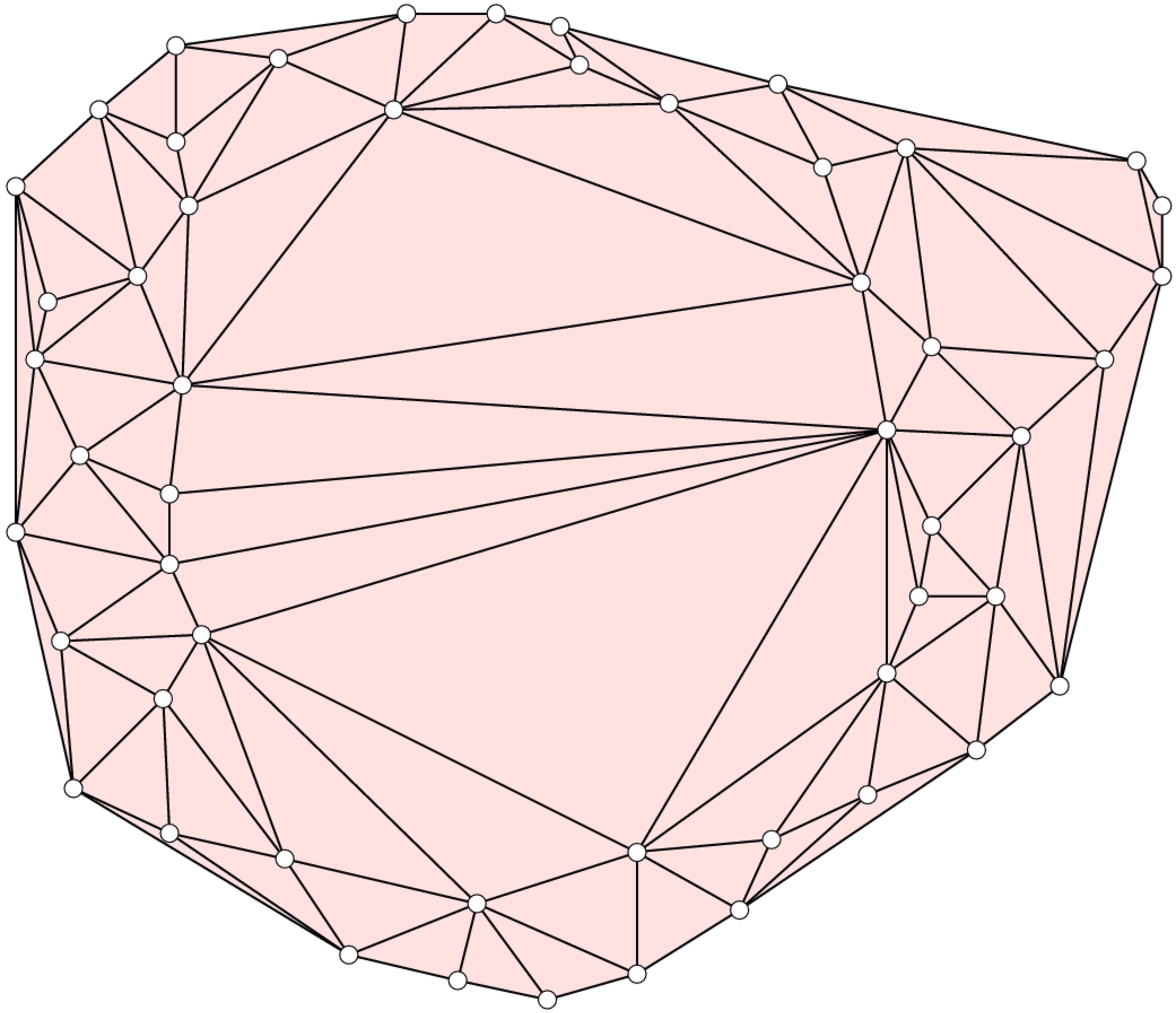
$$\mathbb{P}[\#\text{pts in } \Omega = k] = \frac{(\varrho\|\Omega\|)^k}{k!} e^{-\varrho\|\Omega\|}.$$

$\mathrm{Vor}(X)$ = Voronoi tessellation of X

is collection of Voronoi domains

$\mathrm{Del}(X)$ = Delaunay mosaic of X

is nerve of Voronoi tessellation



I.2 #Simplices

- $\mathbb{E}[d_j^n] = D_j^n \cdot \varrho \|\Omega\|.$ [Miles 1970/71]

D_j^n	$j=0$	1	2	3
$n=1$	1	1		
2	1	3	2	
3	1	7.76	13.53	6.76

[Meijering 1953]

I.3 3D TESSELLATIONS

[Lazar, Mason, MacPherson, Srolovitz 2013]

#faces per Voronoi domain,

#edges per face,

average #faces of neighbors,

volume,

surface area,

sphericity of domains, ...

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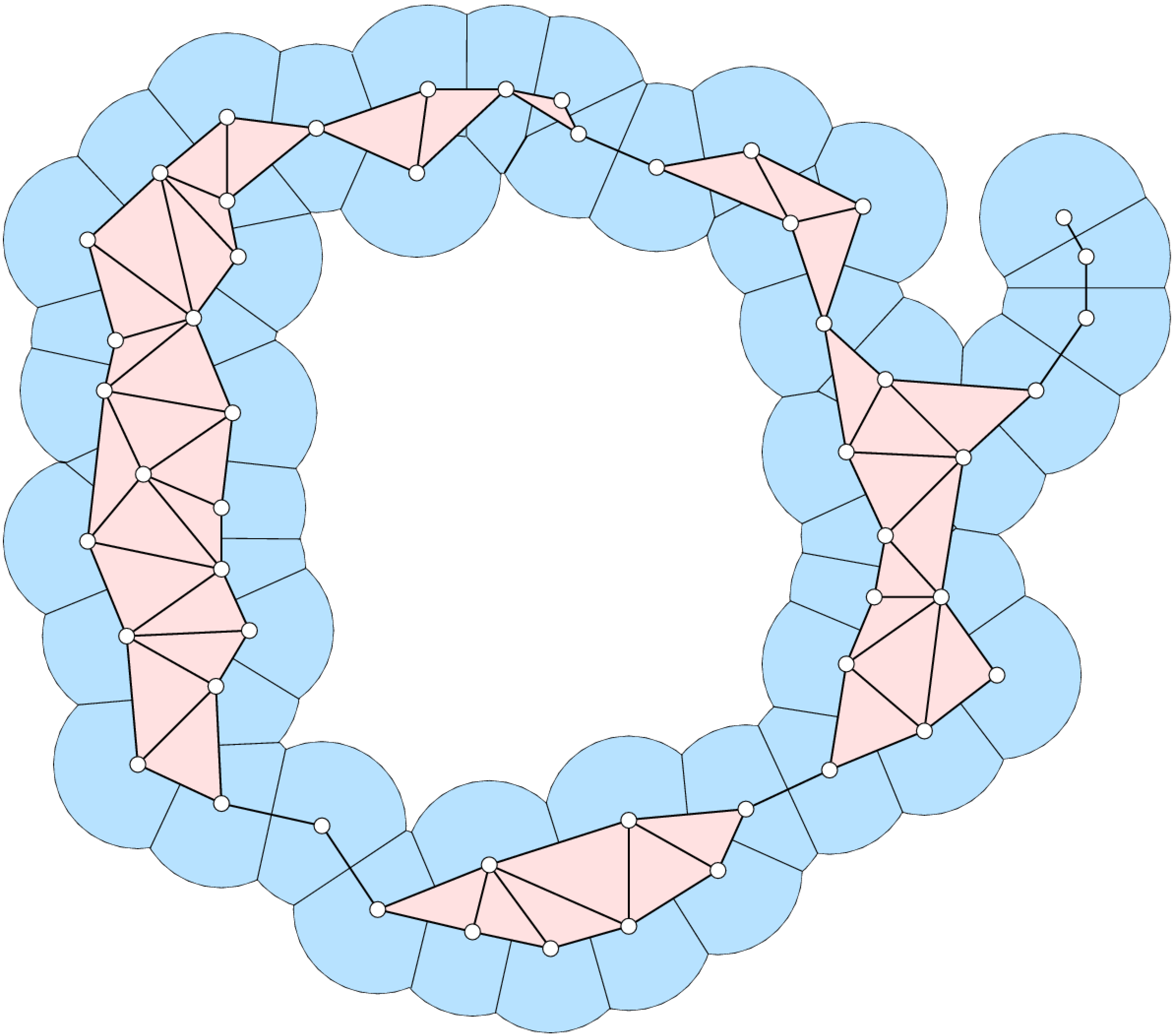
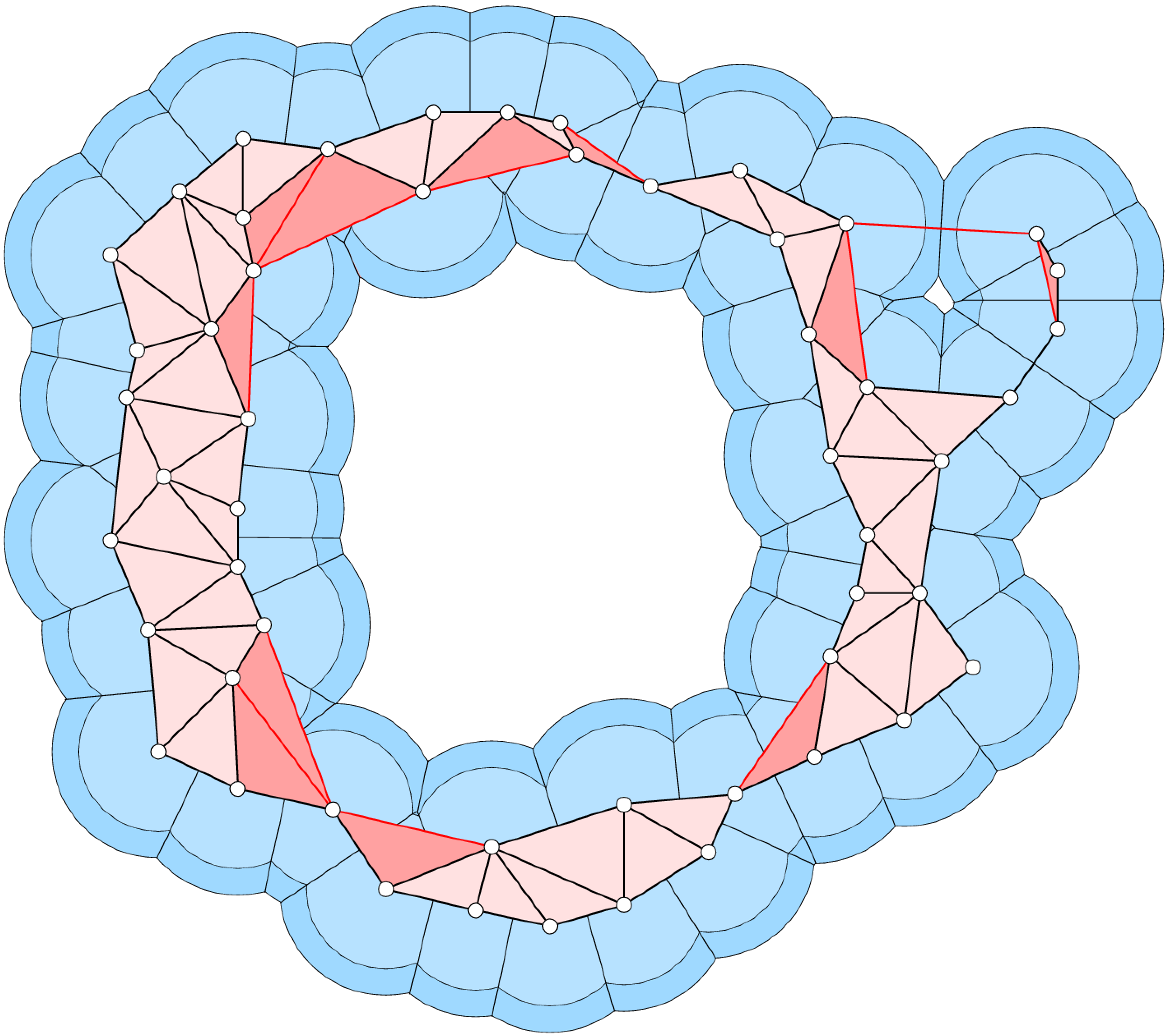
III.1 Radius Function

$\mathcal{R}: \mathrm{Del}(X) \to \mathbb{R}$ dfnd by $\mathcal{R}(Q)$ = min radius

$\mathrm{Alpha}_r(X) = \mathcal{R}^{-1}[0, r]$





II.1 Radius Function

$\mathcal{R} : \mathrm{Del}(X) \to \mathbb{R}$ dfnd by $\mathcal{R}(Q)$ = min radius

$\mathrm{Alpha}_r(X) = \mathcal{R}^{-1}[0,r]$

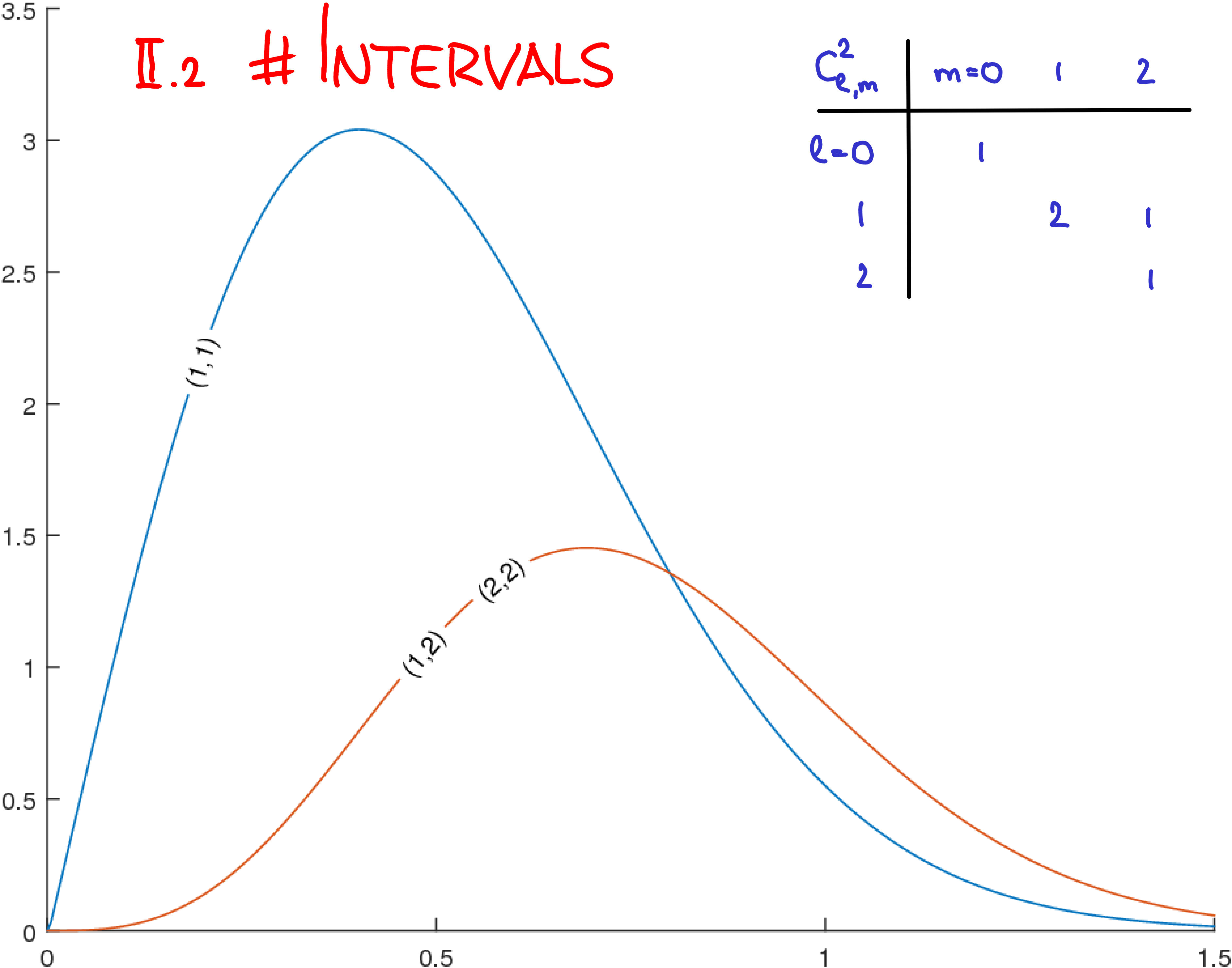
increments are intervals $[L, M] \subseteq \mathrm{Del}(X)$

$\ell = \dim L, \quad m = \dim M$

- $\mathbb{E}[c^n_{\ell,m}(r)] = \dfrac{\gamma(m; \varrho \nu_n r^n)}{\Gamma(m)} \cdot C^n_{\ell,m} \cdot \varrho \, \|\Omega\|$
- $\mathbb{E}[d^n_j(r)] = \sum_{\ell=0}^{j} \sum_{m=j}^{n} \binom{m-\ell}{m-j} \cdot \mathbb{E}[c^n_{\ell,m}(r)]$

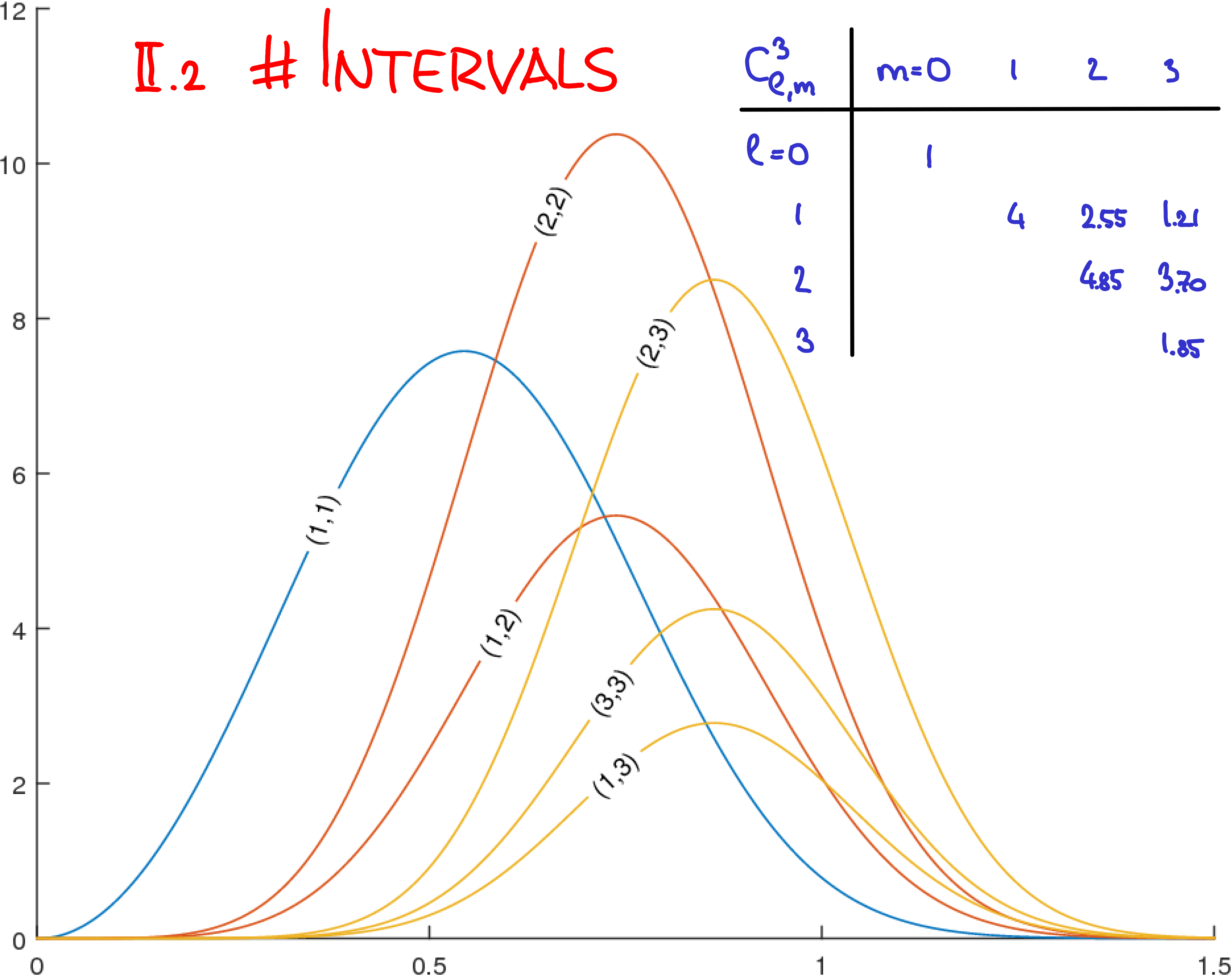
II.2 # INTERVALS

$C^2_{\ell,m}$	$m=0$	1	2
$\ell=0$	1		
1		2	1
2			1

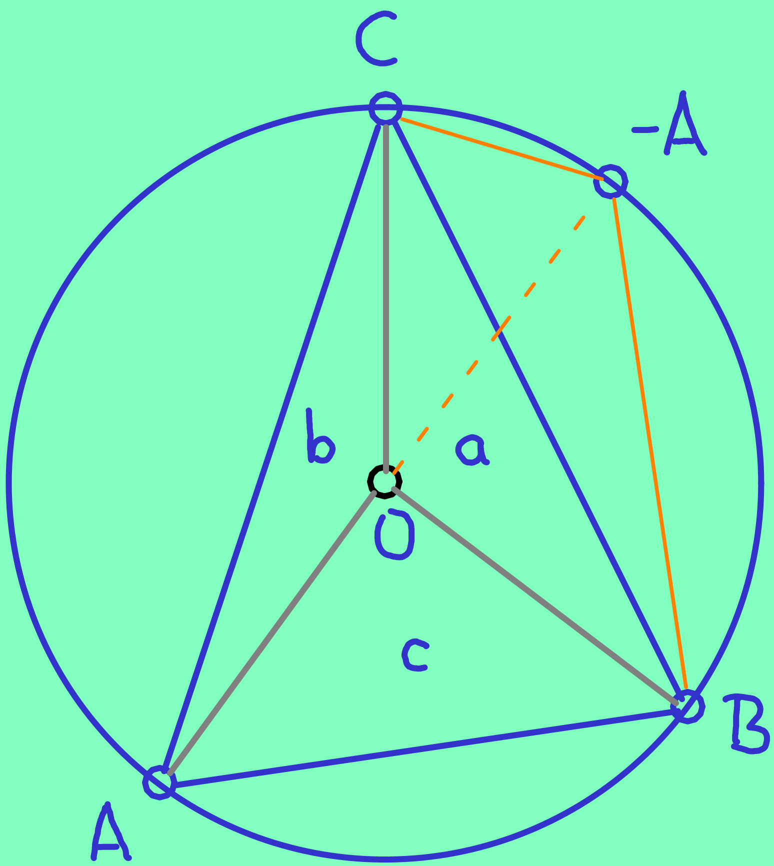


II.2 # INTERVALS

$C^3_{\ell,m}$	$m=0$	1	2	3
$\ell=0$	1			
1		4	2.55	1.21
2			4.85	3.70
3				1.85



II.3 Wendel's Trick in $\mathbb{R}^2$



$$\text{area}(A\,B\,C) = a+b+c$$

$$-A\,B\,C = -a+b+c$$

$$A\,-B\,C = a-b+c$$

$$A\,B\,-C = a+b-c$$

$$-A\,-B\,C = -a-b+c$$

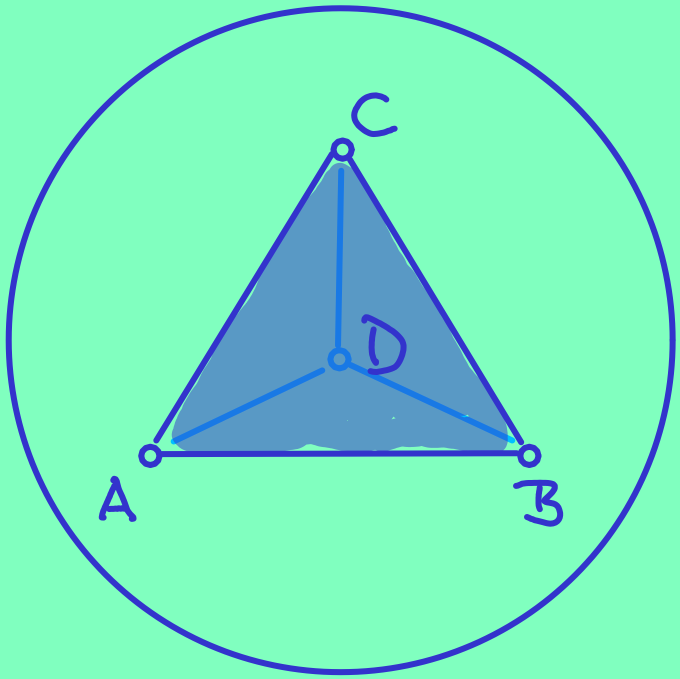
$$-A\,B\,-C = -a+b-c$$

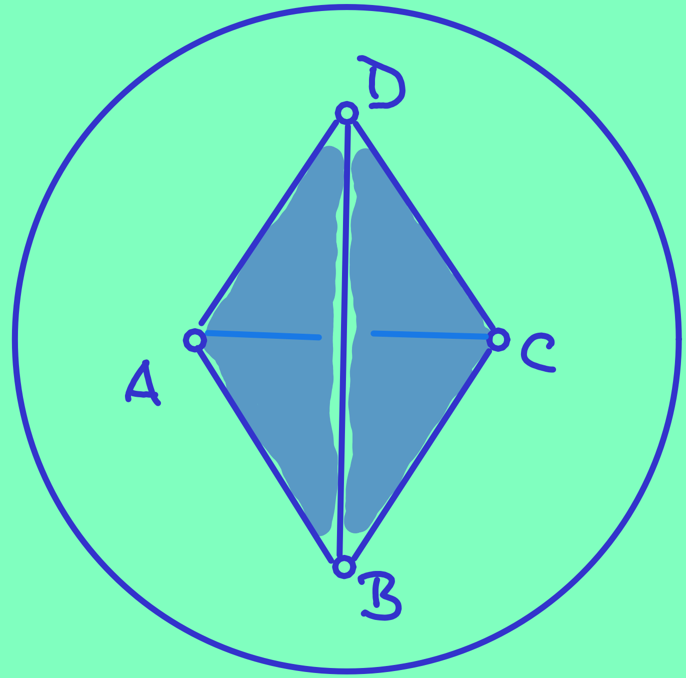
$$A\,-B\,-C = a-b-c$$

$$-A\,-B\,-C = -a-b-c$$

II.3 Wendel's Trick in $\mathbb{R}^3$

$0 \in ABCD$

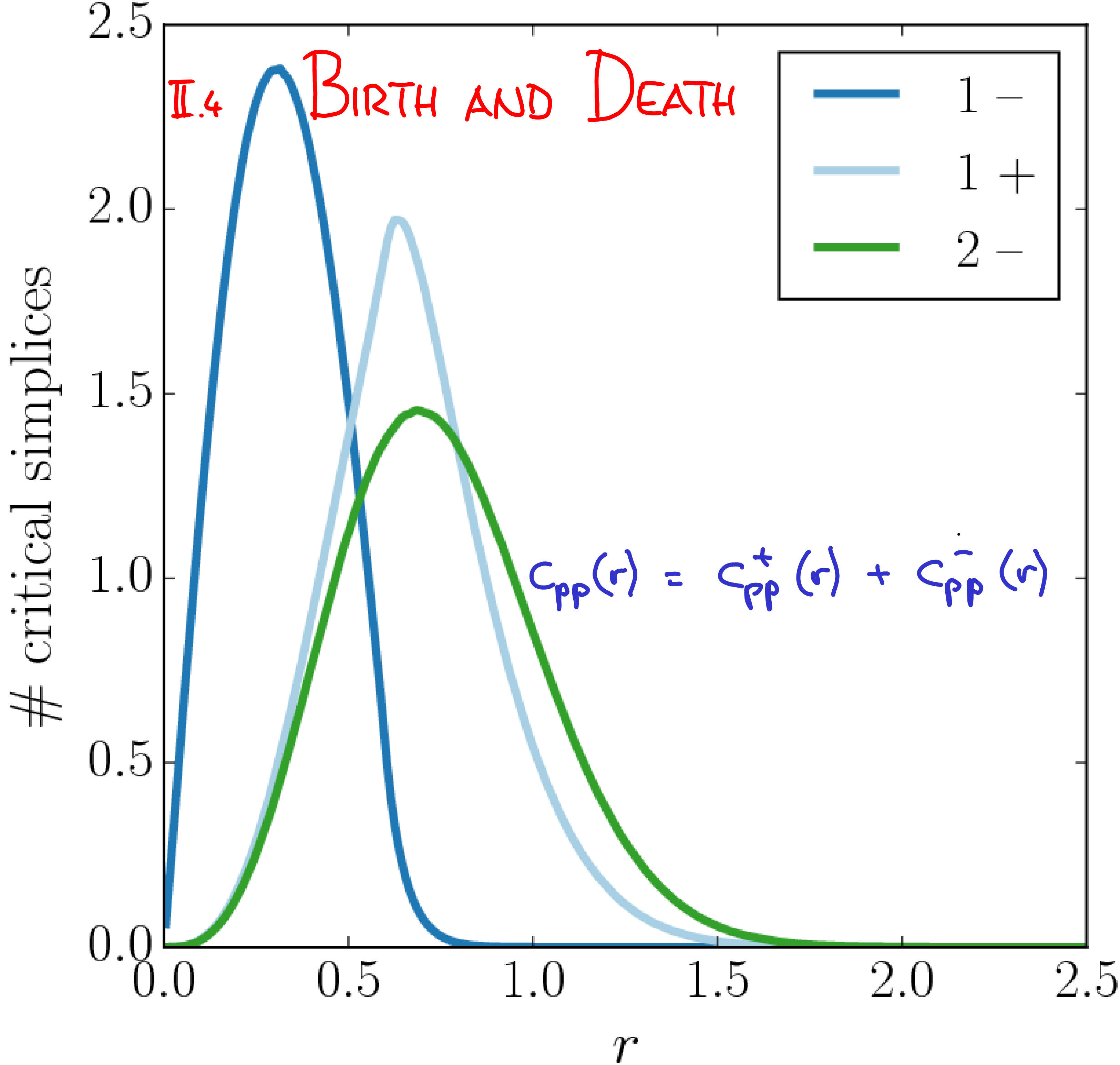




$++++$	$+++-$	$---+$	$++--$	$-++-$
$----$	$++-+$	$--+-$	$+-+-$	$-+-+$
	$+-++$	$-+--$	$+--+$	$--++$
	$-+++$	$+---$		

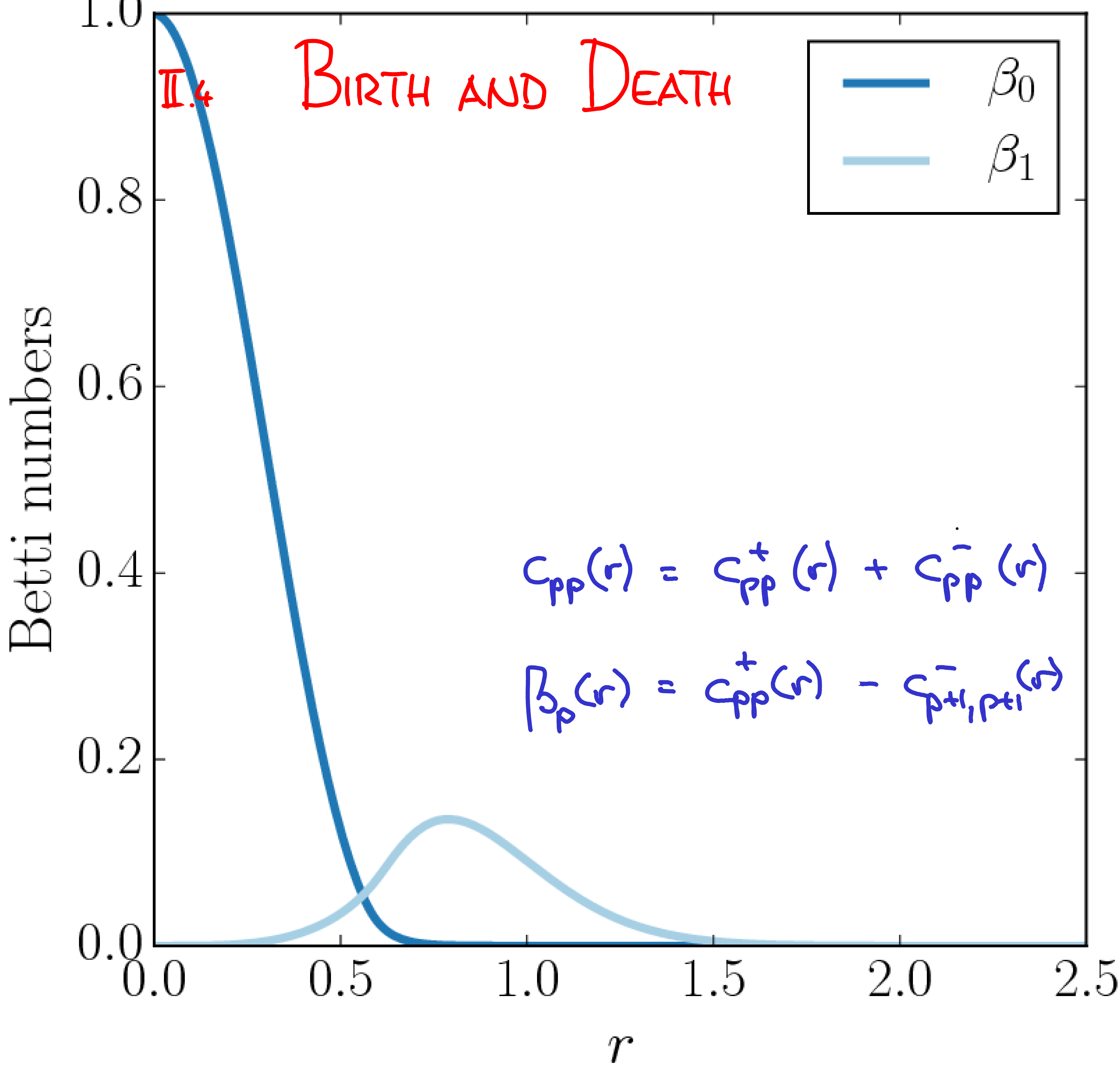
$\text{Prob} = \frac{1}{8}$ $\qquad$ $\text{Prob} = \frac{4}{8}$ $\qquad$ $\text{Prob} = \frac{3}{8}$

II.4 BIRTH AND DEATH



$$c_{pp}(r) = c_{pp}^{+}(r) + c_{pp}^{-}(r)$$

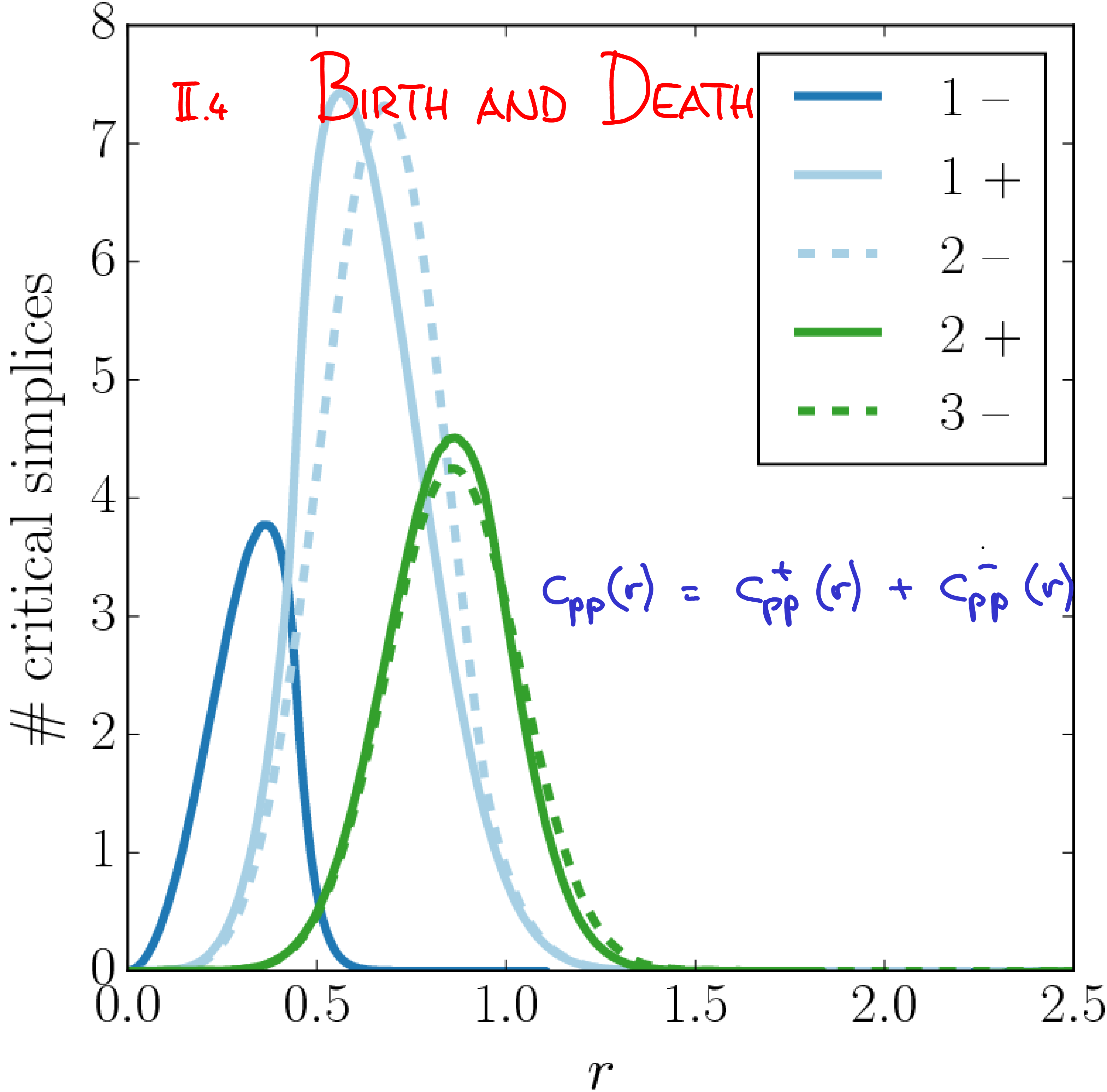
II.4 Birth and Death



$$c_{pp}(r) = c^{+}_{pp}(r) + c^{-}_{pp}(r)$$

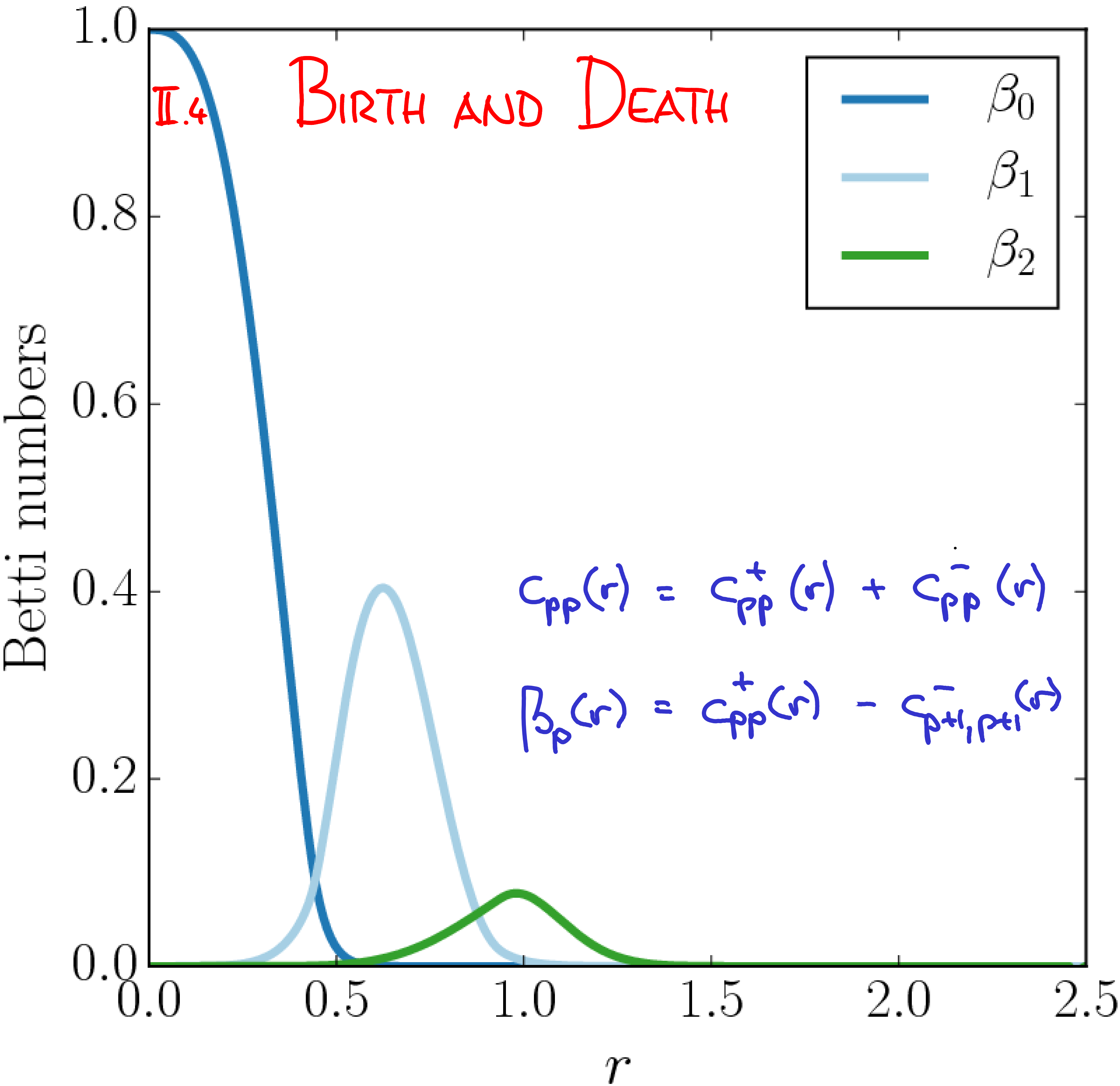
$$\beta_p(r) = c^{+}_{pp}(r) - c^{-}_{p+1,p+1}(r)$$

II.4 Birth and Death



$$c_{pp}(r) = c^{+}_{pp}(r) + c^{-}_{pp}(r)$$

II.4 Birth and Death



$$c_{pp}(r) = c^{+}_{pp}(r) + c^{-}_{pp}(r)$$

$$\beta_p(r) = c^{+}_{pp}(r) - c^{-}_{p+1,p+1}(r)$$

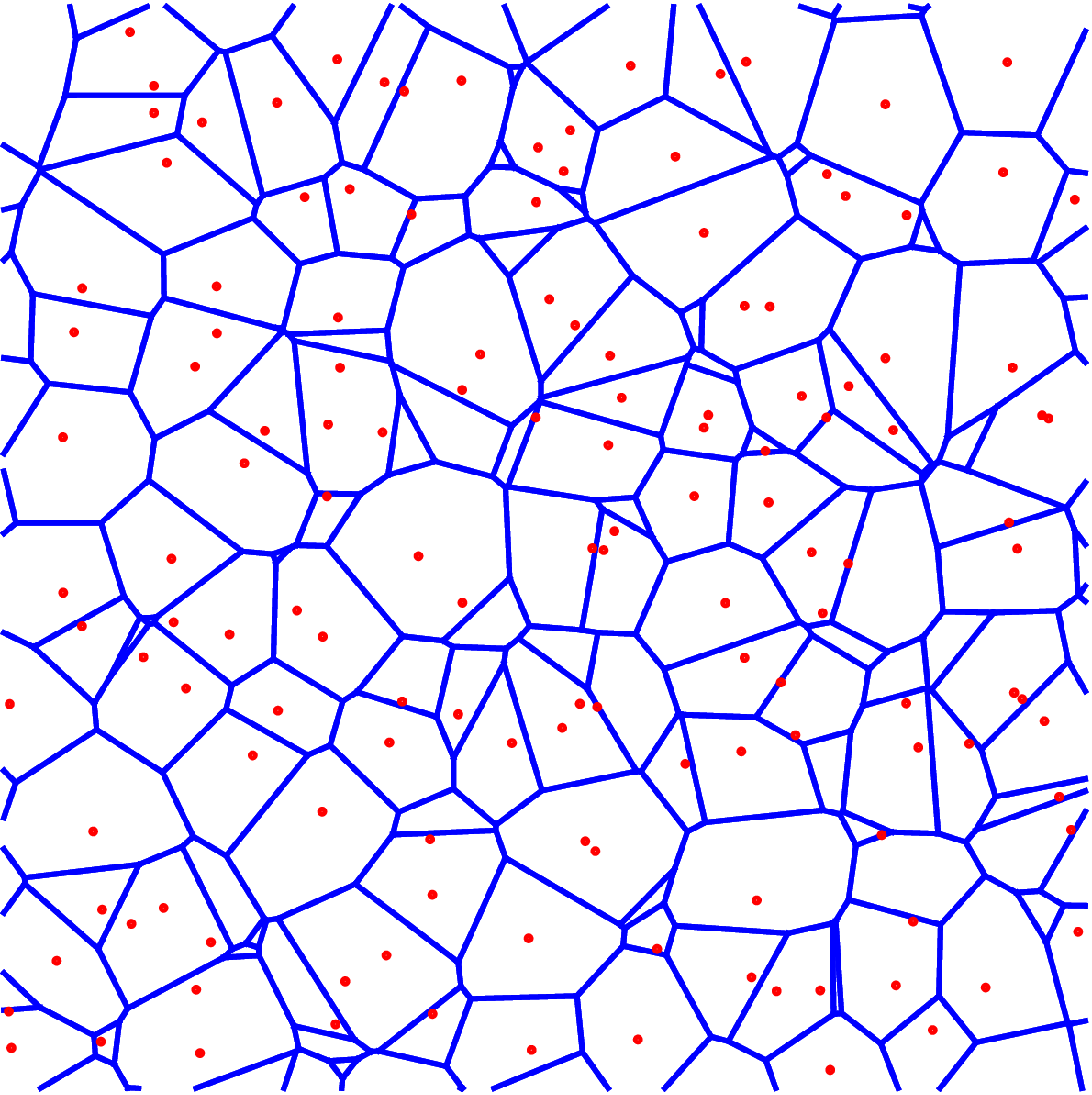
I. PRIOR WORK

II. DELAUNAY RADIUS

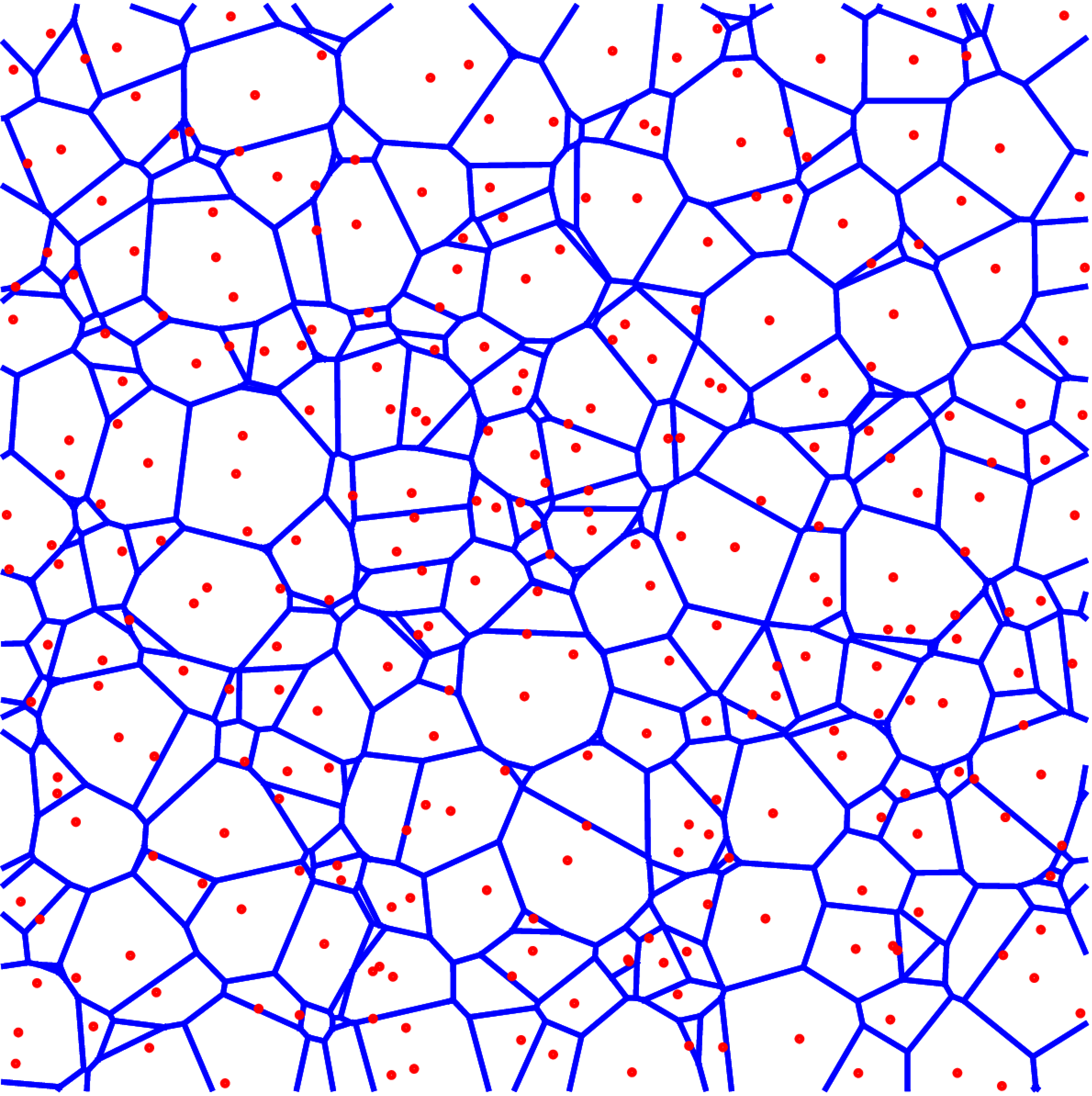
III. WEIGHTS

IV. WRAP

III.1 Slice Construction



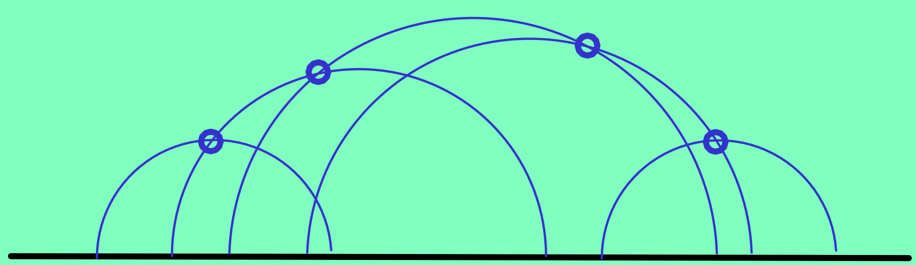
III.1 Slice Construction

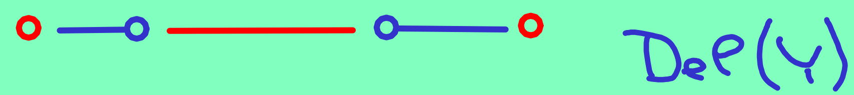


III.2 # INTERVALS

$k = 1, \ n = 2:$

$C^{k,n}_{\ell,m}$	$m=0$	1
$\ell = 0$	1	0.27
1		1





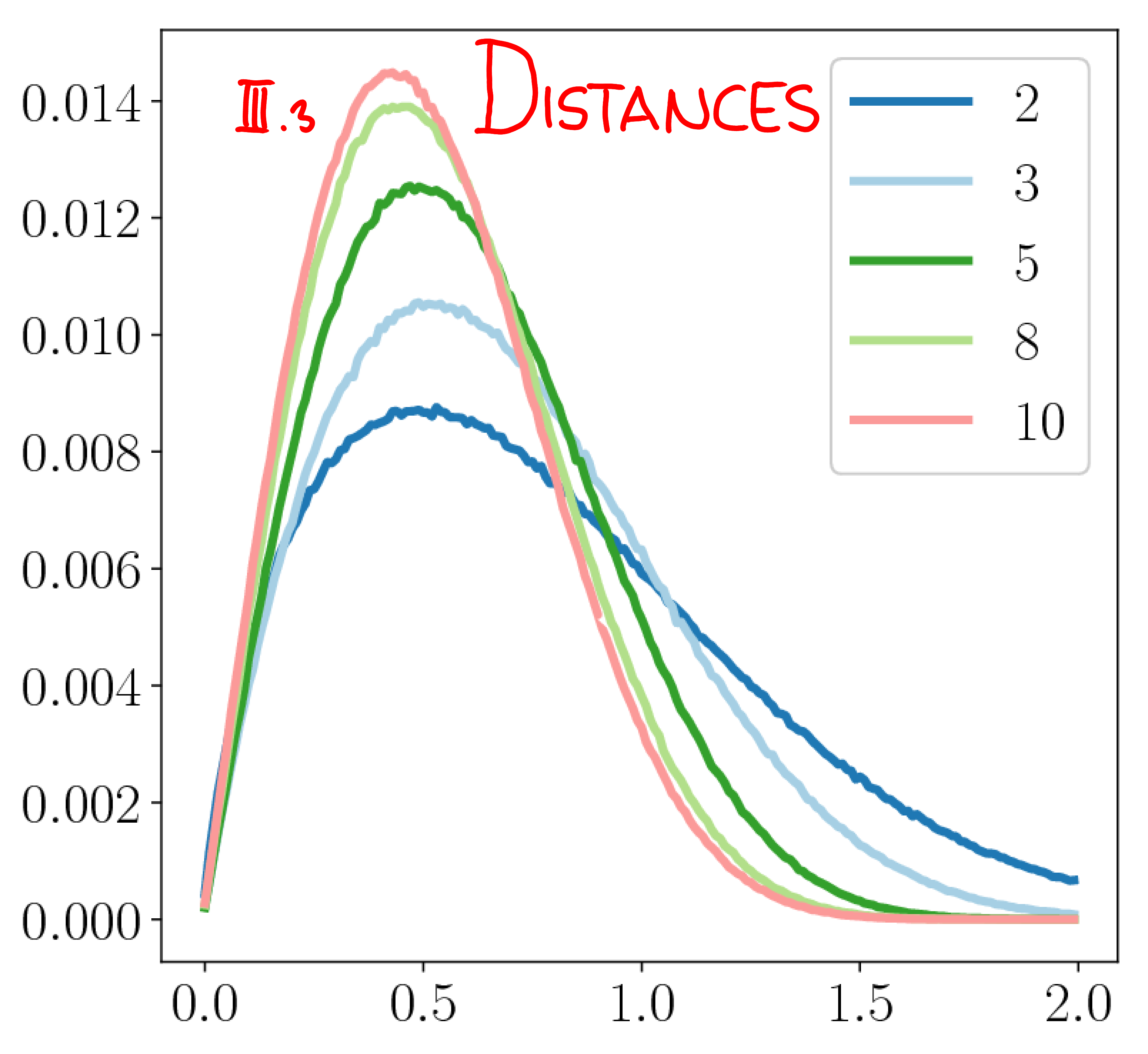
III.2 # Intervals

$k = 2, \; n = 3:$

$C_{\ell,m}^{k,n}$	$m = 0$	1	2
$\ell = 0$	1.11	0.26	0.09
1		2.47	1.46
2			1.37

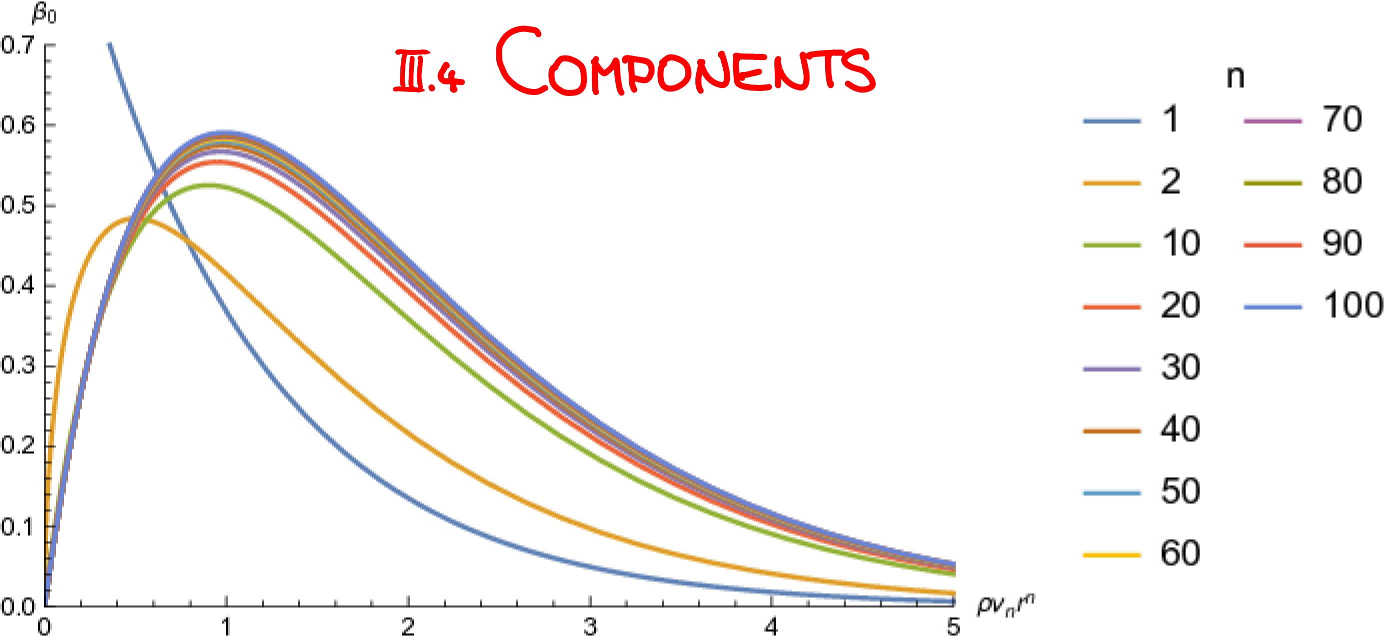
$$C_{22} = C_{00} + C_{01}$$

$$C_{12} = C_{00} + C_{01} + C_{02}$$



III.3
Distances
2
3
5
8
10
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
0.0
0.5
1.0
1.5
2.0

III.4 Components



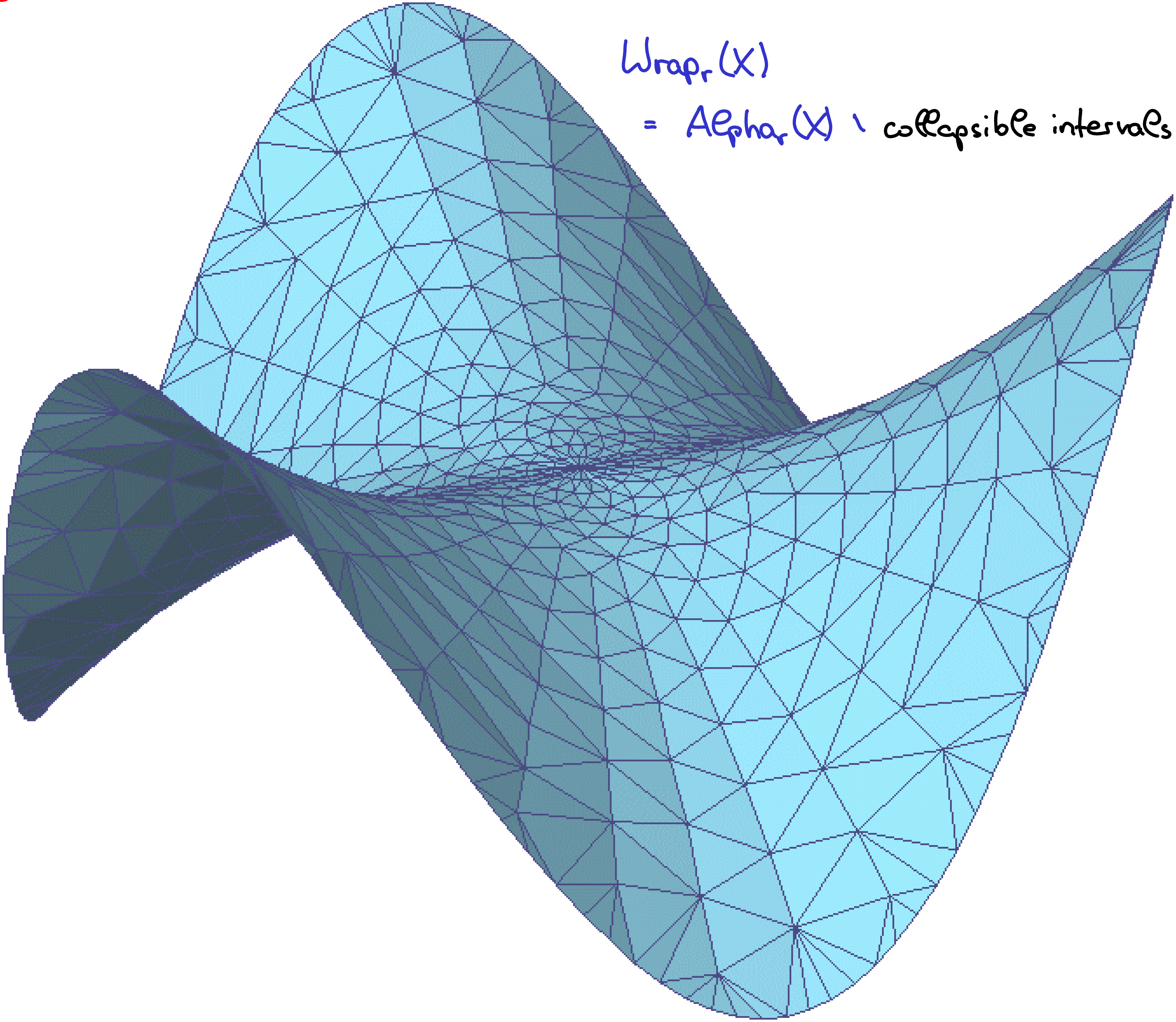
I. Prior Work

II. Delaunay Radius

III. Weights

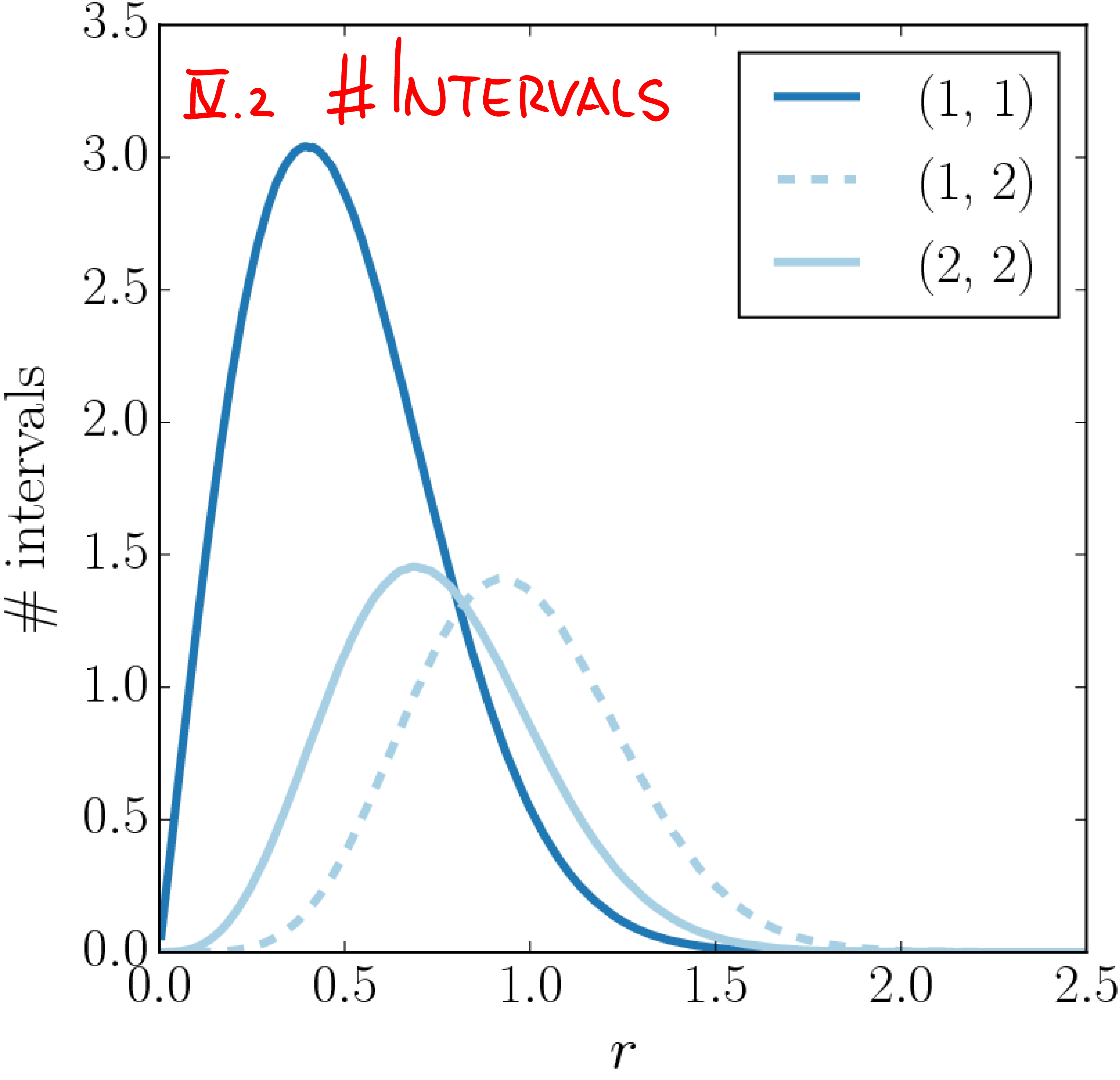
IV. Wrap

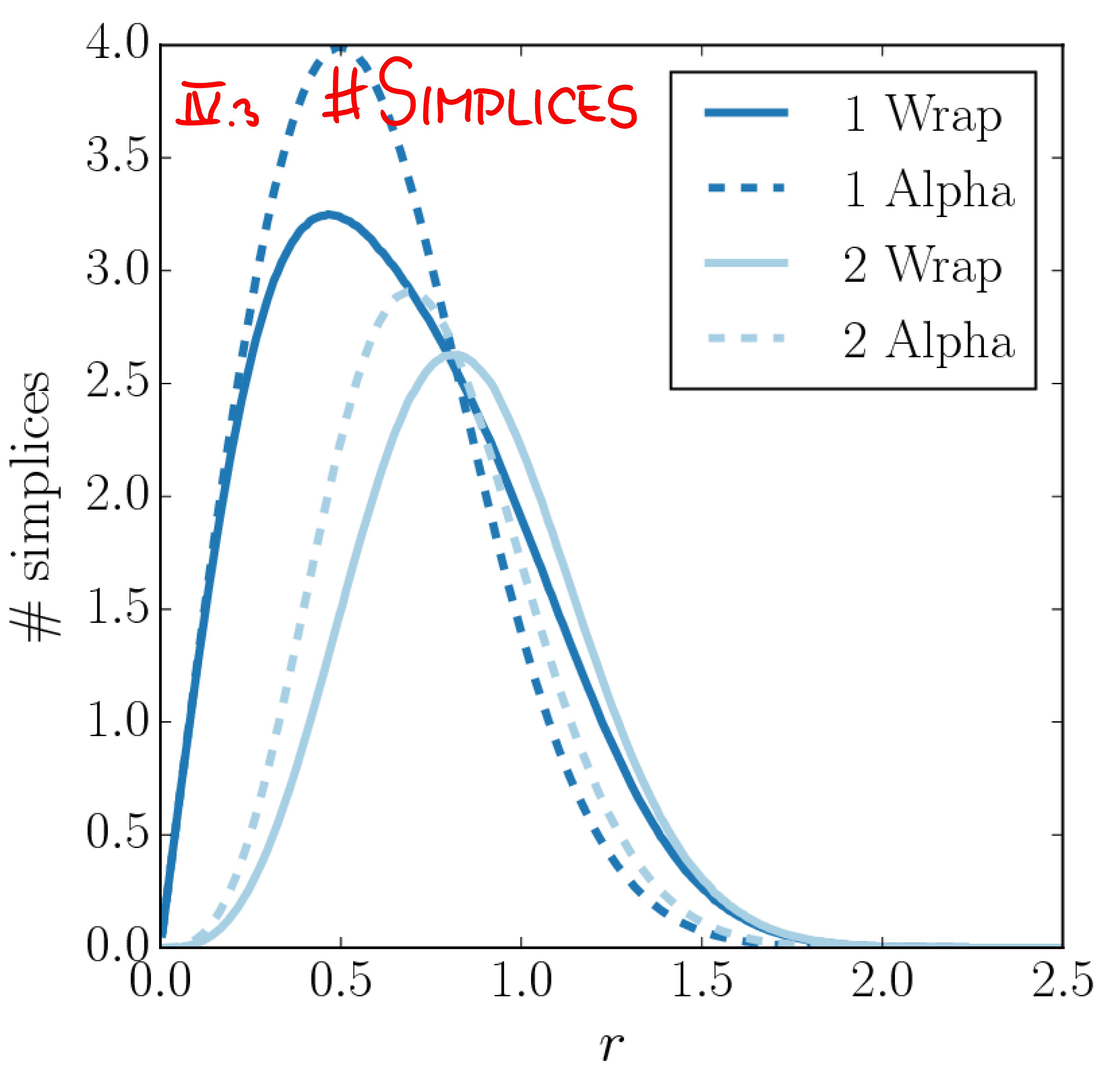
IV.1 Wrap Complex

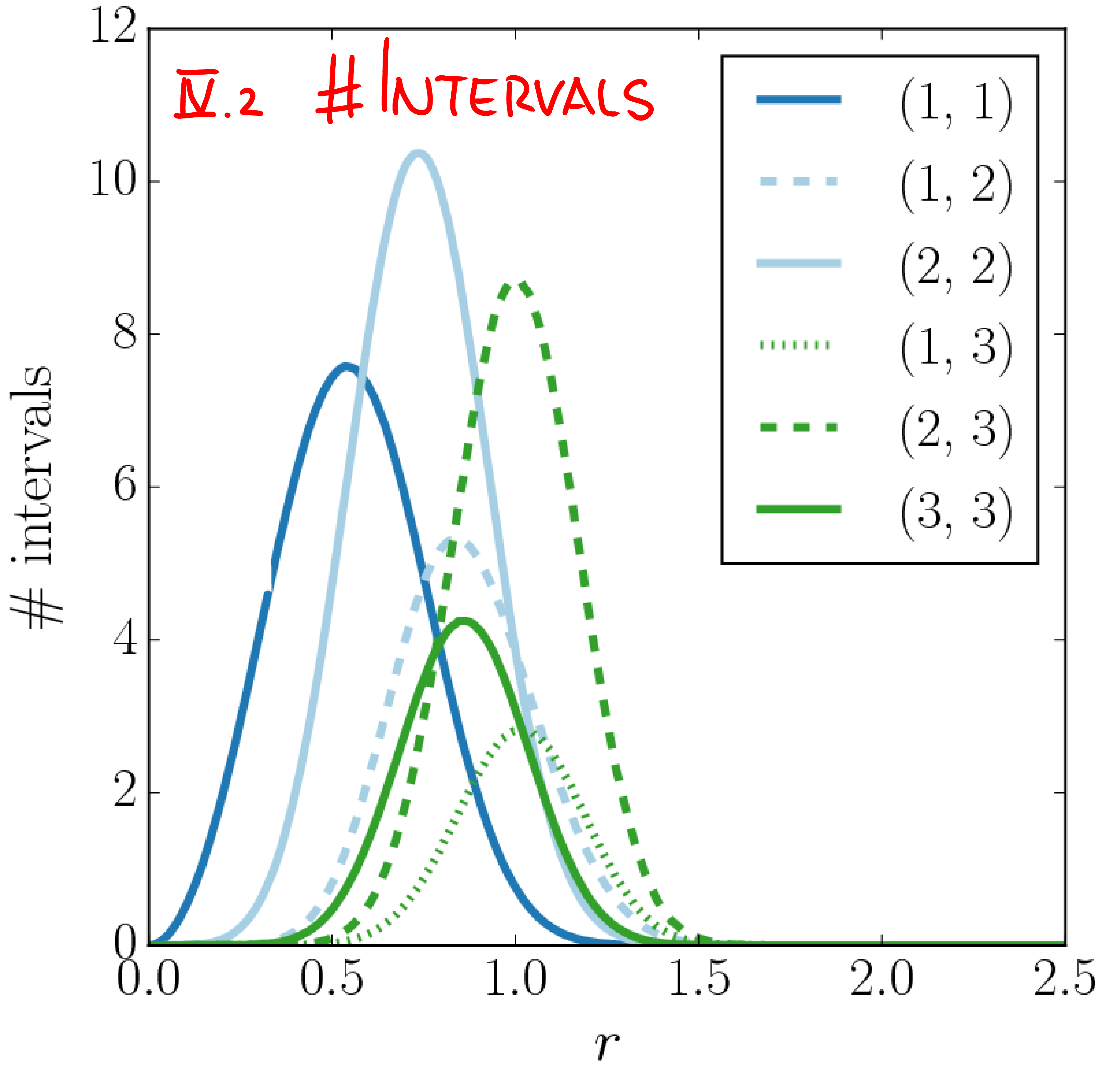


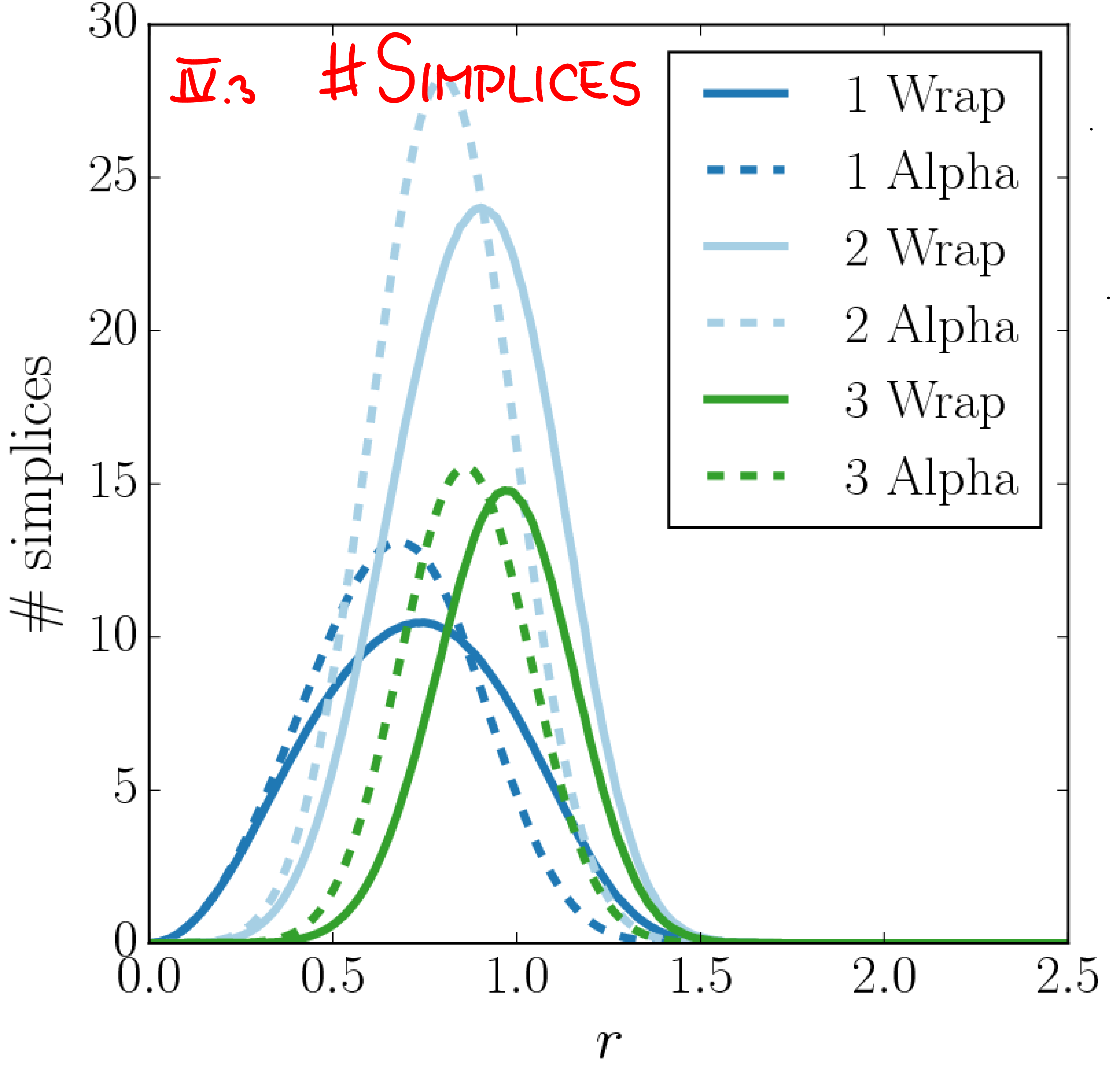
$\mathrm{Wrap}_r(X)$

$= \mathrm{Alpha}_r(X) \setminus$ collapsible intervals


IV.2 #INTERVALS
(1, 1)
(1, 2)
(2, 2)
intervals
r
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5


IV.3 #SIMPLICES
1 Wrap
1 Alpha
2 Wrap
2 Alpha
simplices
r
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0


IV.2 #INTERVALS
intervals
r
(1, 1)
(1, 2)
(2, 2)
(1, 3)
(2, 3)
(3, 3)
0
2
4
6
8
10
12
0.0
0.5
1.0
1.5
2.0
2.5



IV.3 #SIMPLICES
1 Wrap
1 Alpha
2 Wrap
2 Alpha
3 Wrap
3 Alpha
simplices
r
0
5
10
15
20
25
30
0.0
0.5
1.0
1.5
2.0
2.5

THANK YOU