Option Pricing in Continuous Time
the famous Black and Scholes option pricing formula

Nico van der Wijst
1. Modelling stock returns in continuous time
   - Logarithmic returns
   - Properties of log returns
   - Transforming probabilities: loading a die

2. Pricing options
   - Brownian motion
   - Transforming probabilities: changing measure
   - The Black & Scholes option pricing formula

3. Working with Black and Scholes
   - Interpretation and determinants
   - An example
   - Dividends
   - A closer look at volatility
Logarithmic stock returns

Recall from second chapter: discretely compounded returns:

\[ r = \frac{(S_t - S_{t-1})}{S_{t-1}} \]

\( r \) is return, \( S_{t,t-1} \) stock prices end - begin period.
Discretely compounded stock returns

- are easily aggregated across investments:
  - attractive in portfolio analysis
- but non-additive over time:
  - 5% p. year over 10 years = 62.9% return \((1.05^{10})\) not 50%

Option pricing uses individual returns over time

- makes continuously compounded returns convenient
Continuously compounded returns calculated as:

\[ \frac{S_T}{S_0} = e^{rT} \quad \text{or} \quad S_T = S_0e^{rT} \]

Taking natural log’s gives the log returns:

\[ \ln \frac{S_T}{S_0} = \ln e^{rT} = rT \]

Log returns additive over time:

\[ \ln \left( \frac{S_1}{S_0} \times \frac{S_2}{S_1} \right) \Rightarrow \ln \frac{S_1}{S_0} + \ln \frac{S_2}{S_1} = \ln e^{r_1} + \ln e^{r_2} = r_1 + r_2 \]

- convenient to use in continuous time models

But: non-additive across investments:

- log is non-linear → ln of sum ≠ sum of ln’s
Properties of log returns

Have to describe return behaviour over time
Done by making one critical assumption:

$log \text{ returns are independently and identically distributed (iid)}$

- Looks innocent assumption for convenience
- Has far reaching consequences:
  - iid assumption means we can invoke Central Limit Theorem: 
    $\text{sum of } n \text{ iid variables is } \pm \text{ normally distributed}$
Consequences of normally distributed returns:

- returns = ln stock prices
  - if returns \( \sim N \) \( \Rightarrow \) stock prices \( \sim \log N \).

- sum 2 indep. normal variables is also normal with
  - mean = sum 2 means
  - variance = sum 2 variances

- extend to many (T) time periods \( \Rightarrow \) mean & variance grow linearly with time:
  - so \( R_T \sim N(\mu T, \sigma^2 T) \)
  - \( R_T = \) continuously compounded return time \([0, T]\)
  - expectation \( E[R_T] = \mu T \)
  - variance \( \text{var}[R_T] = \sigma^2 T \)
  - instantaneous return = \( \mu \)
Some more consequences:

- iid returns follow a random walk
- random walks have *Markov property* of memorylessness
  - past returns & patterns useless to predict future returns
  - means market is weak form efficient.

Assumptions & consequences fit the real world well but real life stock returns have:

- fatter tails
- more skewness,
- more kurtosis

than normal distribution

Fatter tails give underpricing of financial risks
Transforming probabilities: loading a die

In discrete time, risk neutral probabilities followed 'naturally' from analysis (discounted state prices)
In continuous time specific action is needed:

- change of probability measure

Idea of changing probabilities is counter-intuitive, illustrate with example of loading a die
Simple die game: sixes bet

- aka de Méré’s problem
- even money bet that player will roll a ‘six’ at least once in four rolls of a single die

What is the probability the player will win?

- NOT: 4/6; often used wrong calculation:
  - prob. of rolling ‘six’ is 1/6
  - 4 rolls, so total prob. is 4/6

- Different formulation of same mistake (all over internet):
  - game should be played with 3 rolls
  - to make it fair, even money bet
Correct calculation:

- prob. player wins in first roll is $1/6 = 0.16667$
- prob. in the second roll is $5/6 \times 1/6 = 0.13889$, etc.
- total probability of winning is:

$$\frac{1}{6} + \frac{5}{6} \times \frac{1}{6} + \left(\frac{5}{6}\right)^2 \times \frac{1}{6} + \left(\frac{5}{6}\right)^3 \times \frac{1}{6} = 0.51775$$

Simpler calculation:

- $1 - \text{the probability of losing:}$

$$1 - (5/6)^4 = 0.51775$$

Player has 'edge' of 0.01775, can expected to earn money in the long run
We now turn the problem around:

- what must the probability of rolling a 'six' be
- to make three-roll sixes bet an even money game?

Mathematically reformulated:

- how must we transform the die’s probability measure
- to make the probability of winning 50%?

A *probability measure* is a rule (or function) that assigns probabilities to set of events.

Probability measure for fair die:

- each side equal probability of coming up
- each outcome has probability of 1/6
Problem assumes probabilities that specific die faces (1,2,...6) come up can be manipulated.
Crooked gamblers found several ways to do that:

- Dice that are not perfect cubes ('shapes')

Notice: This is for educational purposes only.
Dice with sticky substance on the side with one spot
- glue activated by blowing on it ('for luck’) is harder to detect

Dice hollowed out on one side, weight is removed
- called 'floats’ or 'floaters’

'Tappers’, dice with small mercury reservoir at the center
- connected to other reservoir at side by thin tube
- tapping makes mercury flow to side reservoir: die becomes loaded

Loaded dice, with a weight on one side
- placing the weight towards side with one spot
- increases probability die will land on side with one spot, so that 'six' comes up
A loaded die, the concentric circles represent the weight. This loading increases the probability that the die will land on a side with one spot ('six' comes up) and decreases the probability that 'one' comes up.
Assume we can load a die very accurately
- can move probability mass from 'one' to 'six' in any degree

We can then calculate what probability of rolling a 'six' will make three-roll sixes bet an even money game
- Calculate it from the probability of losing, $p$, which is also 0.5
- so in a three roll game:

\[ p^3 = 0.5 \Rightarrow p = \sqrt[3]{0.5} = 0.7937 \]

- probability of rolling a 'six' is then:

\[ 1 - 0.7937 = 0.2063 \]
What must prob. of rolling ‘six’ be to give player same ’edge’ of 0.01775 as in four-roll game with fair die?

- Probability of winning then is 0.51775
- Probability of losing becomes $1 - 0.51775 = 0.48225$
- Reusing symbol $p$ we get:

$$p^3 = 0.48225 \Rightarrow p = \sqrt[3]{0.48225} = 0.7842$$

- So probability of rolling a ‘six’ is

$$1 - 0.7842 = 0.2158$$
A more complex gambling game with a die:

- you have to pay to get in
- payoff = number of spots turning up: 1, 2,.., 6

What is a fair price to enter the game?

- With a fair die, all outcomes equal probability 1/6
- expected payoff $\sum p_i R_i = 3.5$, (payoffs = $R$, prob. = $p$)
- variance = $\sum p_i (R_i - (\sum p_i R_i))^2 = 2.917$

With 3.5 entry price:

- both players have equal expected gain - loss (zero)
- game is fair
But organizers want to make money, not to have fair games
Can be done in several ways:

- raise the entrance price:
  - 4.5 gives exp. payoff 1 for organizer, same loss for player
  - looks silly, but is basis of all lotteries

- adjust spots:
  - blot out 6 (replacing with 0) reduces exp. payoff to 2.5,
    variance same 2.917
  - also looks silly, but is done in roulette

- change the probabilities:
  - by tampering with the die, e.g. loading it

How should the die be loaded to give the organizer an exp. payoff of 1?
Reformulated as scientific problem:

*can probability measure for a die be transformed by a formula that affects all probabilities in such a way that expected payoff = 2.5 and variance left unchanged?*

Restrictions:

- measures must be equivalent  
  (means they assign positive prob. to same events)
- $0 < \text{probabilities} < 1$, and sum to 1
- for convenience, additional ‘smoothness’ restriction: 
  probability of 1 spot $\geq$ prob. 2 spots $\geq$ prob. 3 spots, etc.
Probabilities for fair die are: $p_{fair} = 1/6 = .1667$

We want to load die so that:

- sides with few spots get higher probability
- sides with many spots get lower probability

Probabilities $= f($no.sots $X$)

function $\pm$ hyperbola, increase curvature with a power:

$$p_{loaded} = \left( \frac{\alpha}{X} \right)^\beta + \gamma$$

coefficients $\alpha$, $\beta$ and $\gamma$ easily found by solver spreadsheet:

$\alpha = 0.6$, $\beta = 2$ and $\gamma = 0.077$  
Gives:
Probabilities of a fair and a loaded die
Or in table form:

<table>
<thead>
<tr>
<th>spots</th>
<th>prob.</th>
<th>expectation</th>
<th>variance (contr.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.437</td>
<td>.437</td>
<td>.9833</td>
</tr>
<tr>
<td>2</td>
<td>.167</td>
<td>.334</td>
<td>.0418</td>
</tr>
<tr>
<td>3</td>
<td>.117</td>
<td>.351</td>
<td>.0293</td>
</tr>
<tr>
<td>4</td>
<td>.100</td>
<td>.400</td>
<td>.2250</td>
</tr>
<tr>
<td>5</td>
<td>.091</td>
<td>.455</td>
<td>.5688</td>
</tr>
<tr>
<td>6</td>
<td>.087</td>
<td>.522</td>
<td>1.0658</td>
</tr>
<tr>
<td>sum:</td>
<td>.999</td>
<td>2.499</td>
<td>2.914</td>
</tr>
</tbody>
</table>

Transformed probabilities for a die
We can express one measure as a function of the other: (the likelihood ratio of two probability measures is called their *Radon-Nikodym* derivative)

\[
\frac{p_{\text{loaded}}}{p_{\text{fair}}} = \frac{(\frac{.6}{X})^2 - .0897}{.1667} = \frac{2.16}{X^2} + .462
\]

then write them as 'measure transformation functions':

\[
p_{\text{loaded}} = \left(\frac{2.16}{X^2} + .462\right) p_{\text{fair}}
\]

\[
p_{\text{fair}} = \frac{p_{\text{loaded}}}{\left(\frac{2.16}{X^2} + .462\right)}
\]

ensures equivalence: zero \( p_{\text{fair}} \) cannot be transformed in positive \( p_{\text{loaded}} \) and vice versa
What have we accomplished?

- changed probability measure (loaded the die)
- with a formula (also works in reverse)
- left ‘probability process’ in tact (we still roll the die)
- process now produces different expectation (2.5 instead of 3.5)
- variance remains 2.9

Apply same idea to model of stock prices by *changing probability measure*
Modelling stock returns: Brownian motion

Have to model properties of stock return in a forward looking way

- In discrete time - variables:
  - we list all possibilities as:
    - states of the world or
    - values in binomial tree

- In continuous time - variables:
  - infinite number of possibilities, cannot be listed
  - have to express in probabilistic way.

Standard equipment: stochastic process
Most used process is *Brownian motion*, or Wiener process

- Discovered ±1825 by botanist Robert Brown
- looked through microscope at pollen floating on water
- observed pollen moving around

Physics described by Albert Einstein in 1905
Mathematical process described by Norbert Wiener in 1923

We use the

- term *Brownian motion*
- and the symbol $\mathcal{W}$ or $\tilde{\mathcal{W}}$ (for Wiener)
Standard Brownian motion = continuous time analogue of random walk

- can be thought of as series of very small steps
- each drawn randomly from standard normal distribution

Definition

Process \( \tilde{W} \) is standard Brownian motion if:

- \( \tilde{W}_t \) is continuous and \( \tilde{W}_0 = 0 \),
- has independent increments
- increments \( \tilde{W}_{s+t} - \tilde{W}_s \sim N(0, \sqrt{t}) \), which implies:
  - increments are stationary: only function of length of time interval \( t \), not of location \( s \).
From definition follows:

- Brownian motion has Markov property

Discrete representation over short period $\delta t$:

- $\epsilon \sqrt{\delta t}$, $\epsilon$ = random drawing from standard normal distribution

Brownian motion has remarkable properties:

- wild: no upper - lower bounds, will eventually hit any barrier
- continuous everywhere, differentiable nowhere:
  - never 'smooths out' if scale is compressed or stretched
  - that why special, stochastic calculus is required
- is a fractal
Standard Brownian motion poor model of stock price behavior:

- Catches only random element
- Misses individual parameter for stock’s volatility
- Misses expected positive return (positive drift)
- Misses proportionality: changes should be in % not in amounts

Missing elements expressed by adding:

- deterministic drift term for expected return
- parameter for stock’s volatility
- proportionality: return and random movements (or volatility) in proportion to stock’s value
Standard model is *geometric Brownian motion* in a stochastic differential equation:

\[
dS_t = \mu S_t dt + \sigma S_t d\tilde{W}_t
\]

\[S_0 > 0\]

- \(d\) = next instant's incremental change
- \(S_t\) = stock price at time \(t\)
- \(\mu\) = drift coefficient, exp. instantaneous stock return
- \(\sigma\) = diffusion coefficient, stock’s volatility (stand. dev. returns), 'scales' random term
- \(\tilde{W}\) = standard Brownian motion, stochastic disturbance term
- \(S_0\) = initial condition (a process has to start somewhere)
- \(\mu, \sigma\) are assumed to be constants
Geometric Brownian motion has all the properties we set out to model
But is also restricted:

- constant volatility
- no jumps or 'catastrophes'

Formula (1) is stochastic differential equation (SDE)

- is a differential equation with a stochastic process in it
- Need a special, stochastic calculus to manipulate SDEs
Financial market also contains risk free debt, $D$

- defined in similar, but simpler, manner:

$$dD_t = rD_t\,dt \quad (2)$$

- $r$ is short for $r_f$, risk free rate (also called money market account or bond)
- risk free $\rightarrow$ no stochastic disturbance term
- natural interpretation for $r$ is short interest rate
- $r$ is assumed to be constant
Sample paths of geometric Brownian motion with $\mu = 0.15$, $\sigma = 0.3$ and $T=250$
Technique of changing measure

Want to change probabilities such that they embed market price of risk

- so that all assets can be discounted at risk free rate

Mathematical instrument for that is *Girsanov’s theorem*:

- Transforms stochastic process, that is a Brownian motion under one probability measure
- into another stochastic process that is a Brownian motion under another probability measure;
- transformation done with third process, Girsanov kernel
The expression for Girsanov kernel is:

\[ d\tilde{W}_t = \theta_t dt + dW_t \]  \hspace{1cm} (3)

- \( \tilde{W} = \) original process, Brownian motion under original, real probability measure called \( Q \)
- \( W = \) transformed process, Brownian motion under new probability measure called \( P \)
- \( \theta = \) Girsanov kernel

Inserting (3) into (1) gives stock price dynamics under \( P \) measure:

\[ dS_t = \mu S_t dt + \sigma S_t (\theta_t dt + dW_t) \]
Collecting terms:

\[ dS_t = (\mu + \sigma \theta_t) S_t dt + \sigma S_t dW_t \] (4)

original process \( \tilde{W} \) replaced with new process \( W \)
- we have changed measure!

Looks futile:
- switched from \( Q \)-Brownian motion with drift \( \mu \)
- to \( P \)-Brownian motion with drift \( (\mu + \sigma \theta_t) \)

But latter contains process \( \theta \), is not yet defined
We know desired result from definition:

- process should contain pricing information
- similar to state prices in binomial model
- so that proper discount rate = drift = risk free rate $r$

Solution: define $\theta$ as minus the market price of risk:

$$\theta = -\frac{\mu - r}{\sigma}$$

We have seen $\theta$ before:

- price of risk in CML and SML
- also used in Sharpe ratio
The Girsanov kernel $-\frac{\mu - r}{\sigma}$ is very simple:

- it is deterministic (no stochastic term)
- it is constant ($\mu$, $\sigma$ and $r$ are constants)

Substituting for $\theta$ in the drift term we get:

$$\mu + \sigma \theta_t = \mu + \sigma \left( -\frac{\mu - r}{\sigma} \right) = r$$

(5)

So we have a dynamic process with drift of risk free rate and, under measure $P$, BM disturbance term:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

(6)
Solving the sde

- SDEs are notoriously difficult to solve
- Deterministic equivalent of (6) simplified by taking logarithms
- Try same transformation here
  - that is how it is done, trial & error

Have to use stochastic calculus (Ito’s lemma), result:

\[
d(\ln S_t) = \left( r - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t
\]  
(7)

changes \( \ln(\text{stock price}) \) follow BM, drift \( (r - \frac{1}{2} \sigma^2) \), diffusion \( \sigma \)
Term $-\frac{1}{2} \sigma^2$ in drift follows from stochastic nature of returns
Illustrate intuition with example:

- security has return $(1+r)$ over 2 periods
- plus random term of $\varepsilon$ in one period, $-\varepsilon$ in other
- Compound return:

\[
((1+r) + \varepsilon) \times ((1+r) - \varepsilon) = (1+r)^2 - \varepsilon^2
\]

- cross terms $+$ and $-(1+r)\varepsilon$ cancel out, $-\varepsilon \times +\varepsilon = -\varepsilon^2$ not
- volatility reduces compound return
- that is why geometric average $< \text{arithmetic average}$
Recall: increments Brownian motion normally distributed and notice: drift and diffusion of

\[ d(\ln S_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t \]

are constants \(\Rightarrow d(\ln S_t)\) also normally distributed:

\[
\ln S_T - \ln S_0 \sim N((r - \frac{1}{2}\sigma^2)T, \sigma^2T) \\
\text{or} \quad \ln S_T \sim N(\ln S_0 + (r - \frac{1}{2}\sigma^2)T, \sigma^2T)
\]

We use this property later on

Constant drift and diffusion make process for \(d(\ln S_t)\) very simple SDE
can be integrated directly over time interval $[0, T]$, result:

$$S_T = S_0e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T}$$ (8)

- since $\ln S_T$ is normally distributed
- $S_T$ must be lognormally distributed

$E[S_t]$ follows from properties lognormal distribution:
- expectation of lognormally distributed variable is

$$e^{m+\frac{1}{2}s^2}$$

- $m$ and $s$ are mean and variance of corresponding normal distribution
We have

\[ \ln S_T \sim N(\ln S_0 + \left( r - \frac{1}{2} \sigma^2 \right) T, \sigma^2 T) \]

So expectation of \( S_T \) is:

\[ E[S_T] = e^{\ln S_0 + (r - \frac{1}{2} \sigma^2) T + \frac{1}{2} \sigma^2 T} = S_0 e^{rT} \]

\[ E[S_T] = S_0 e^{rT} \quad \text{means} \quad e^{-rT} E[S_T] = S_0 \]

discounted future exp. stock price = current stock price under prob. measure \( P \)

- risky assets can be discounted with risk free rate
- as long as expectations are under measure \( P \)

The exact equivalent of Binomial model
The Black & Scholes formula

Formula can be obtained in several ways:

1. Black & Scholes original work uses partial differential equations (outline in appendix)
2. Cox, Ross Rubinstein show that binomial approach converges to B&S formula
3. Martingale method (used here)
   - prices by directly calculating expectation under probability measure $Q$
   - discount result with risk free rate
Problem:

- price now (t=0) of European call option $O^E_{c,0}$,
  - exercise price $X$,
  - matures at time $T$,
  - written on non-dividend paying stock

Using martingale method:

$$O_{c,0} = e^{-rT} E [O_{c,T}]$$  \hspace{1cm} (9)$$

$r$ is the risk free rate
Option’s payoff at maturity:

\[ O_{c,T} = \begin{cases} 
S_T - X & \text{if } S_T > X \\
0 & \text{if } S_T \leq X 
\end{cases} \]

can be written as:

\[ O_{c,T} = (S_T - X)1_{S_T > X} \quad (10) \]

\(1_{S_T > X}\) is step function:

\[ 1_{S_T > X} = \begin{cases} 
1 & \text{if } S_T > X \\
0 & \text{if } S_T \leq X 
\end{cases} \]
Substituting step function (10) into option value (9):

\[ O_{c,0} = e^{-rT}E\left[ (S_T - X)1_{S_T>X} \right] \]  \hspace{1cm} (11)

To prepare for rest of derivation, we write option value (11) as:

\[ O_{c,0} = e^{-rT}E\left[ (e^{\ln S_T} - e^{\ln X})1_{\ln S_T>\ln X} \right] \]  \hspace{1cm} (12)

We use two key elements:

1. \( \ln S_T \) is normally distributed, mean = \( \ln S_0 + (r - \frac{1}{2}\sigma^2)T \), var. = \( \sigma^2 T \)

2. We can regard step function as truncation of distribution of \( S_T \) on left: values \(< X \) replaced by zero (truncated distributions are well researched, formula for truncated normal distribution available)
Lognormally distributed stock price ($\ln(S) \sim N(10, 2)$, dashed), and its left truncation at $\ln(S) = 11$ (solid)
We use following step function for normally distributed variable $Y$ with mean $M$ and variance $\nu^2$ truncated at $A$:

\[
E \left[ \left( e^Y - e^A \right) 1_{Y>A} \right] = e^{M+\frac{1}{2} \nu^2} N \left( \frac{M + \nu^2 - A}{\nu} \right) - e^A N \left( \frac{M - A}{\nu} \right) \tag{13}
\]

$N(.)$ is cum. standard normal distr.
Has same form as (12), apply to option pricing problem:
$M = \ln S_0 + (r - \frac{1}{2} \sigma^2) T$
$\nu^2 = \sigma^2 T \rightarrow \nu = \sigma \sqrt{T}$
$Y = \ln S_T$
$A = \ln X$
Substituting:

- Details of our problem \((M, \nu^2, \gamma, A)\) into formula (13) for the expectation of truncated distribution
- that expectation formula in our option pricing formula

and collecting terms we get the famous Black and Scholes formula:

\[
O_{c,0} = S_0 N \left( \frac{\ln(S_0/X) + (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \right) - Xe^{-rT}N \left( \frac{\ln(S_0/X) + (r - \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \right) \tag{14}
\]
Defining, as is commonly done:

\[ d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \]  

(15)

and

\[ d_2 = \frac{\ln(S_0/X) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \]  

(16)

we get the usual form of the Black & Scholes option pricing formula:

\[ O_{c,0} = S_0N(d_1) - Xe^{-rT}N(d_2) \]  

(17)

with the corresponding value of a European put:

\[ O_{p,0} = Xe^{-rT}N(-d_2) - S_0N(-d_1) \]  

(18)
Interpretation:

\[ O_{c,0} = \left( S_0 \right)_{\text{stock price}} N\left( d_1 \right)_{\text{option delta}} - \left( X e^{-rT} \right)_{\text{PV (exerc.p.)}} N\left( d_2 \right)_{\text{prob. of exercise}} \]

\[ N(d_1) = \text{option delta}, \text{ has different interpretations:} \]

- *hedge ratio*: \# shares needed to replicate option
- *sensitivity*: of option price for changes in stock price
- technical: partial derivative w.r.t. stock price:
  \[ \frac{\partial O_{c,0}}{\partial S_0} = N(d_1) \]
- not just prob. of exercise, also encompasses in-the-moneyness
What is *not* in the Black and Scholes formula:

- real drift parameter $\mu$
- investors’ attitudes toward risk
- other securities or portfolios

Greediness, in $\text{max}[]$ expressions, implicit in analysis.

Reflects conditional nature of B&S:
As the binomial model, B&S only translates existing security prices on a market into prices for additional securities.
Determinants of option prices

In B&S, stock price + four other variables
Option price sensitivity for these 4 derived in same way as $\Delta$ (partial derivatives), called 'the Greeks'

<table>
<thead>
<tr>
<th>Determinant</th>
<th>Greek</th>
<th>Effect on call option</th>
<th>Effect on put option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exercise price</td>
<td></td>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>Stock price</td>
<td>Delta</td>
<td>$0 &lt; \Delta_c &lt; 1$</td>
<td>$-1 &lt; \Delta_p &lt; 0$</td>
</tr>
<tr>
<td>Volatility</td>
<td>Vega</td>
<td>$\nu_c &gt; 0$</td>
<td>$\nu_p &gt; 0$</td>
</tr>
<tr>
<td>Time to maturity</td>
<td>Theta</td>
<td>$-\Theta_c &lt; 0$</td>
<td>$-\Theta_p &lt;&lt; 0$</td>
</tr>
<tr>
<td>Interest rate</td>
<td>Rho</td>
<td>$\rho_c &gt; 0$</td>
<td>$\rho_p &lt; 0$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\Gamma_c &gt; 0$</td>
<td>$\Gamma_p &gt; 0$</td>
<td></td>
</tr>
</tbody>
</table>
'The Greeks' is a bit of a misnomer

- X is determinant without Greek
- Vega is not a Greek letter
- Gamma is Greek without determinant, gamma is:
  - effect of increase in stock price on delta
  - second derivative option price w.r.t. stock price

Generally, option value increases with time to maturity

- American options always do
- European call on dividend paying stock may decrease with time to maturity if dividends are paid in 'extra' time.
- Value of deep in the money European puts can decrease with time to maturity: means longer waiting time before exercise money is received
An example:

Calculate value of at the money European call

- matures in one year
- strike price of 100
- underlying stock pays no dividends
- has annual volatility of 20%
- risk free interest rate is 10% per year.
We have our five determinants:
$S_0 = 100, \ X = 100, \ r = .1, \ \sigma = .2$ and $T = 1$.

\[
d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}
\]
\[
= \frac{\ln(100/100) + (.1 + \frac{1}{2}(.2)^2)1}{.2\sqrt{1}} = .6
\]

\[
d_2 = d_1 - \sigma\sqrt{T} = .6 - .2\sqrt{1} = .4
\]

Areas under normal curve for values of $d_1$ and $d_2$ can be found:

- table in compendium (good enough for this course),
- calculator, spreadsheet, etc.:
### d = 0 0.01 0.02 0.03 0.04 0.05 0.06 0.09

<table>
<thead>
<tr>
<th>d</th>
<th>0</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
<th>0.06</th>
<th>0.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.500</td>
<td>0.504</td>
<td>0.508</td>
<td>0.512</td>
<td>0.516</td>
<td>0.520</td>
<td>0.524</td>
<td>0.536</td>
</tr>
<tr>
<td>0.1</td>
<td>0.540</td>
<td>0.544</td>
<td>0.548</td>
<td>0.552</td>
<td>0.556</td>
<td>0.560</td>
<td>0.564</td>
<td>0.575</td>
</tr>
<tr>
<td>0.2</td>
<td>0.579</td>
<td>0.583</td>
<td>0.587</td>
<td>0.591</td>
<td>0.595</td>
<td>0.599</td>
<td>0.603</td>
<td>0.614</td>
</tr>
<tr>
<td>0.3</td>
<td>0.618</td>
<td>0.622</td>
<td>0.626</td>
<td>0.629</td>
<td>0.633</td>
<td>0.637</td>
<td>0.641</td>
<td>0.652</td>
</tr>
<tr>
<td>0.4</td>
<td>0.655</td>
<td>0.659</td>
<td>0.663</td>
<td>0.666</td>
<td>0.670</td>
<td>0.674</td>
<td>0.677</td>
<td>0.688</td>
</tr>
<tr>
<td>0.5</td>
<td>0.691</td>
<td>0.695</td>
<td>0.698</td>
<td>0.702</td>
<td>0.705</td>
<td>0.709</td>
<td>0.712</td>
<td>0.722</td>
</tr>
<tr>
<td>0.6</td>
<td>0.726</td>
<td>0.729</td>
<td>0.732</td>
<td>0.736</td>
<td>0.739</td>
<td>0.742</td>
<td>0.745</td>
<td>0.755</td>
</tr>
<tr>
<td>0.7</td>
<td>0.758</td>
<td>0.761</td>
<td>0.764</td>
<td>0.767</td>
<td>0.770</td>
<td>0.773</td>
<td>0.776</td>
<td>0.785</td>
</tr>
<tr>
<td>0.8</td>
<td>0.788</td>
<td>0.791</td>
<td>0.794</td>
<td>0.797</td>
<td>0.800</td>
<td>0.802</td>
<td>0.805</td>
<td>0.813</td>
</tr>
<tr>
<td>0.9</td>
<td>0.816</td>
<td>0.819</td>
<td>0.821</td>
<td>0.824</td>
<td>0.826</td>
<td>0.829</td>
<td>0.831</td>
<td>0.839</td>
</tr>
<tr>
<td>1</td>
<td>0.841</td>
<td>0.844</td>
<td>0.846</td>
<td>0.848</td>
<td>0.851</td>
<td>0.853</td>
<td>0.855</td>
<td>0.862</td>
</tr>
<tr>
<td>2.5</td>
<td>0.994</td>
<td>0.994</td>
<td>0.994</td>
<td>0.994</td>
<td>0.994</td>
<td>0.995</td>
<td>0.995</td>
<td>0.995</td>
</tr>
</tbody>
</table>
NormalDist(.6) = 0.72575, NormalDist(.4) = 0.65542,
Option price becomes:

\[ O_{c,0} = 100 \times (0.72575) - 100e^{-0.1} (0.65542) = 13.27 \]

Value put option calculated with equation or the put call parity:

\[ O_{p,0} = O_{c,0} + Xe^{-rT} - S_0 \]
\[ = 13.27 + 100e^{-0.1} - 100 = 3.75 \]
Call option prices for $\sigma = 0.5$ (top), 0.4 and 0.2 (bottom)
Call option prices for $T = 3$ (top), 2 and 1 (bottom)
Dividends

Black & Scholes assumes

- European options
- on non dividend paying stocks

Can be adapted to allow for deterministic (non-stochastic) dividends (can be predicted with certainty)

Dividends:

- stream of value out of the stock
- stream accrues to stockholders
- not option holders
Stock price for stockholders has:

- stochastic part (stock without dividends)
- deterministic part (PV dividends)

Stock price for option holders:

- only stochastic part relevant

Adaptation Black & Scholes formula:

- subtract PV(dividends) from stock price ($S_0$)
- dividends certain $\rightarrow$ discount with risk free rate
- (implicitly redefines volatility parameter $\sigma$ for stochastic part only)

Other determinants ($X, T$ and $r$) unaffected by dividends
Example:

- same stock used before
- pays semi-annual dividends of 2.625
  - first after 3 months
  - then after 9 months

Stock price = 100, volatility 20%, risk free interest rate 10% per year.

*What is value European call, maturity 1 year, strike price = 100?*
$S_0 = 100, X = 100, r = .1, \sigma = .2$ and $T = 1$.

Start by calculating PV dividends:

- $2.625e^{-0.25 \times 0.1} + 2.625e^{-0.75 \times 0.1} = 5$.
- makes adjusted stock price $S_0 = 100 - 5 = 95$

Then we can proceed as before:

\[
d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}
\]

\[
= \frac{\ln(95/100) + (.1 + \frac{1}{2}.2^2)1}{.2\sqrt{1}}
\]

\[
= 0.34353
\]

\[
d_2 = d_1 - \sigma\sqrt{T} = 0.34353 - .2\sqrt{1} = 0.14353
\]
Areas under normal curve for values $d_1$ and $d_2$ are:

- $\text{NormalDist}(0.34353) = 0.6344$ and
- $\text{NormalDist}(0.14353) = 0.5571$.

So the option price becomes:

$$O_{c,0} = 95 \times (0.6344) - 100e^{-0.1} (0.5571) = 9.86$$

- value call lowered by dividends
- from 13.27 to 9.86
Value of a put (same specifications) calculated with equation

$$O_{p,0} = X e^{-rT} N(-d_2) - S_0 N(-d_1)$$

Just calculated that \(d_1 = 0.34353\) and \(d_2 = 0.14353\)

NormalDist\((-0.34353)\) = 0.3656 and

NormalDist\((-0.14353)\) = 0.44294

\(\bullet\) In table use symmetric property \(N(-d) = 1 - N(d)\)

Value of the put is:

$$O_{p,0} = 100 \times e^{-0.1} (0.44294) - 95 \times (0.3656) = 5.35$$

\(\bullet\) value of put increased by dividends

\(\bullet\) from 3.75 to 5.35
Matching discrete and continuous time volatility

We have expressed volatility in 2 ways:

- In binomial model:
  - difference between up and down movement

- In Black and Scholes model:
  - volatility parameter $\sigma$ used to scale $\tilde{W}$

If we want to switch models

- we have match the parameters
- recalculate $\mu$ and $\sigma$ into $u$, $d$ and $p$
Looking at small time interval $\delta t$

- we can equate the return expressions:

$$e^{r\delta t} = pu + (1 - p)d$$

$r = \text{risk free rate}$ and $p = \text{risk neutral probability}$

- we can also equate variance expressions:

$$\sigma^2 \delta t = pu^2 + (1 - p)d^2 - [pu + (1 - p)d]^2$$

notice:

- continuous variance increases with time ($\delta t$)
- discrete variance uses definition:
  variance of a variable $A$ is $E(A^2) - [E(A)]^2$
This gives us 2 expressions:

- 1 for return, 1 for variance
- For 3 unknowns: \( p, u \) and \( d \)
- Need additional assumption for third equation

Most common assumption is:

\[
    u = \frac{1}{d}
\]

Three equations give (after much algebra):

\[
    u = e^{\sigma \sqrt{\delta t}}, \quad d = e^{-\sigma \sqrt{\delta t}} \quad \text{and} \quad p = \frac{e^{r \delta t} - d}{u - d}
\]

Same definition of \( p \) we found in binomial model
Implied volatility

Black & Scholes formula has 5 determinants of option prices:

- $X, T, S, r, \sigma$ are model inputs
- 6 if dividends are included

4 of then are easy to obtain:

- $X, T, S, r$ are, at least in principle, observable:
  - $X$ and $T$ are determined in option contract
  - $S$ and $r$ are market determined
- $\sigma$ is not observable
There are 2 ways of obtaining numerical value for $\sigma$:

1. Estimate from historical values and extrapolate into future;
   - assumes, like Black & Scholes, that volatility is constant
   - known not to be the case
     (volatility peaks around events as quarterly reports)

2. Estimate from prices of other options;
   - given $X, T, S, r$ each value for $\sigma$ corresponds to 1 B&S price and vice-versa
   - for given price, run B&S in reverse (numerically) and find $\sigma$
   - called *implied volatility*
Implied volatility is commonly used:

- option traders quote option prices in volatilities
- not $ or € amounts.

Can also be used to test validity of B&S model

How do you use implied volatility to test B&S?
Black & Scholes assumes constant volatility:
Options with different $X$ and $T$ should give same implied volatility.
Implied volatility typically not constant:

- far in- and out-of the money options give higher implied volatilities than at the money options
  - called *volatility smile* after its graphical representation
  - implies more kurtosis (peakedness) of stock prices than lognormal distribution
  - also fatter tails, but intermediate values less likely

- Stock options may also imply volatility skewness:
  - far out of the money calls priced lower than far out of the money puts (or far in the money calls)
  - implies skewed distribution of stock prices
  - left tail fatter than right tail

- Implied volatility may also increase with time to maturity