Value of information for spatial decision situations

Jo Eidsvik,
IMF, NTNU

Bayesian model

- All the currently available information is contained in the prior model for the variables:

$$p(\boldsymbol{x})$$

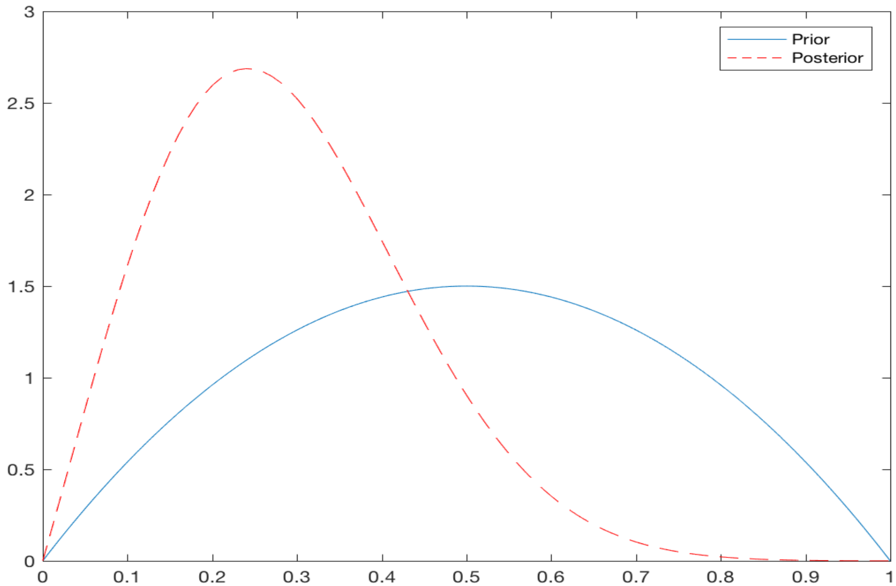
- New data (and the data gathering scheme) is represented by a likelihood model:

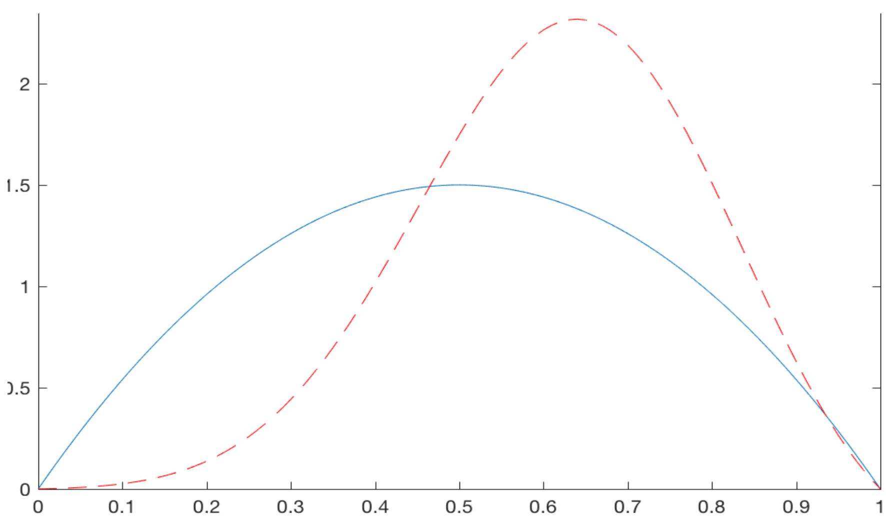
$$p(\boldsymbol{y} \mid \boldsymbol{x})$$

- If we collect data, the model is updated to the posterior, conditional on the new observations:

$$p(\boldsymbol{x} \mid \boldsymbol{y}) = \frac{p(\boldsymbol{y} \mid \boldsymbol{x})\, p(\boldsymbol{x})}{p(\boldsymbol{y})}, \qquad p(\boldsymbol{y}) = \sum_{\boldsymbol{x}} p(\boldsymbol{y} \mid \boldsymbol{x})\, p(\boldsymbol{x})$$

Bayesian updating





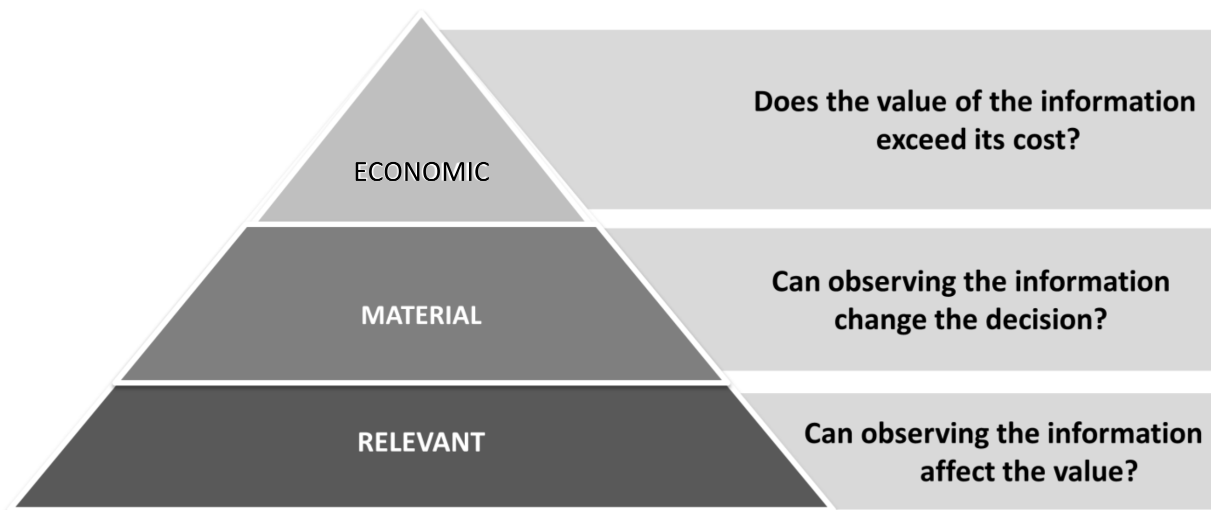
$$p(\mathbf{x})$$

$$p(\mathbf{x} \mid \mathbf{y})$$

- What data is valuable?
- Study the **expected effect of data,** before it is collected.
- We gather data not only to reduce uncertainty, but to make better **decisions**. We have a goal, a clear question we want to answer.

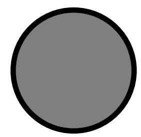
Value of information (VOI)

In many Earth science applications we consider purchasing more data before making difficult decisions under uncertainty. The value of information (VOI) is useful for quantifying the value of the data, before it is acquired and processed.



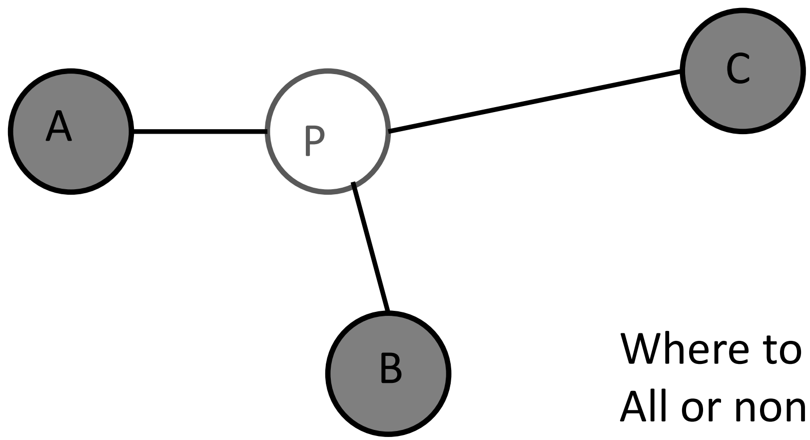
This pyramid of conditions - VOI is different from other information criteria (entropy, variance, prediction error, etc.)

What if several projects or treasures?



Relatively easy for univariate
situations.

What if several projects or treasures?



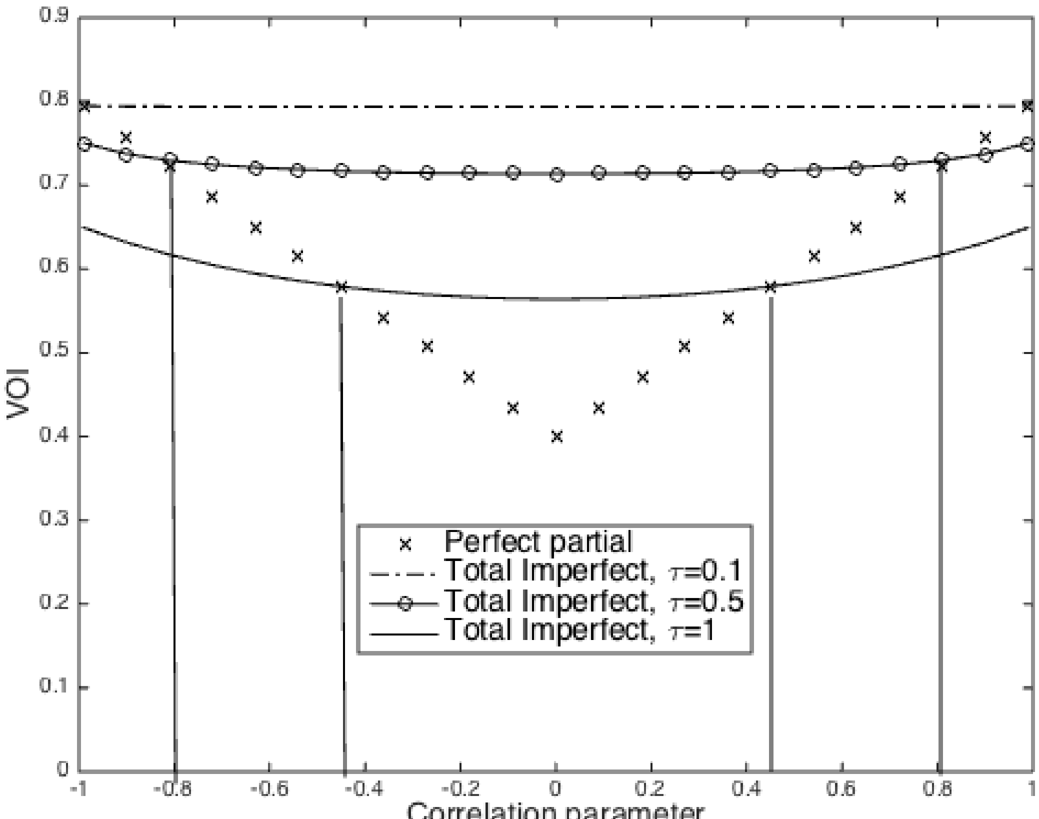
Where to dig?
All or none? Free to choose as many as profitable? One at a time, then choose again?

Where should one collect data? All or none? One only? Or two? One first, then maybe another?

Gaussian projects results

$$PoV(\boldsymbol{y}) = \left(\sqrt{R_{1,1}} + \sqrt{R_{2,2}}\right) \Big/ \sqrt{2\pi}, \qquad \boldsymbol{R} = \boldsymbol{\Sigma C}^{-1}\boldsymbol{\Sigma}$$

$$PoV(x_1) = \left(1 + |\rho|\right) \Big/ \sqrt{2\pi}$$



VOI and spatial models

- Uncertainties are multivariate (spatial). Dependence. $\boldsymbol{x} = (x_1, \ldots, x_n)$
- Alternatives are multivariate (spatial): select sites for drilling, development, conservation, harvesting, etc. $\boldsymbol{a} = (a_1, \ldots, a_N)$
- Value function (spatial). Flow simulation. «physics» and economic attributes. $v(\boldsymbol{x}, \boldsymbol{a})$
- Data are multivariate (spatial). And variety of spatial tests (seismic, electromagnetic, etc.) $\boldsymbol{y} = (y_1, \ldots, y_m)$

Decision analysis – Prior value

- Uncertain variables: $\boldsymbol{x} = (x_1, \ldots, x_n)$
- Alternatives (Where? How? When?) $\boldsymbol{a} = (a_1, \ldots, a_N)$
- Value function is revenues, subtracted costs. $v(\boldsymbol{x}, \boldsymbol{a})$
- Risk neutral decision maker will **maximize expected value**:

$$PV = \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \right) \right\}, \qquad E\left(v(\boldsymbol{x}, \boldsymbol{a}) \right) = \sum_{\boldsymbol{x}} v(\boldsymbol{x}, \boldsymbol{a}) \, p(\boldsymbol{x})$$

VOI

Prior value:

$$PV = \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \right) \right\}$$

Posterior value:

$$PoV\left(\boldsymbol{y}\right) = \int \max_{\boldsymbol{a} \in \boldsymbol{A}} \left\{ E\left(v\left(\boldsymbol{x}, \boldsymbol{a}\right) \mid \boldsymbol{y} \right) \right\} p\left(\boldsymbol{y}\right) d\boldsymbol{y}$$

VOI = Expected posterior value – Prior value

$$VOI\left(\boldsymbol{y}\right) = PoV\left(\boldsymbol{y}\right) - PV$$

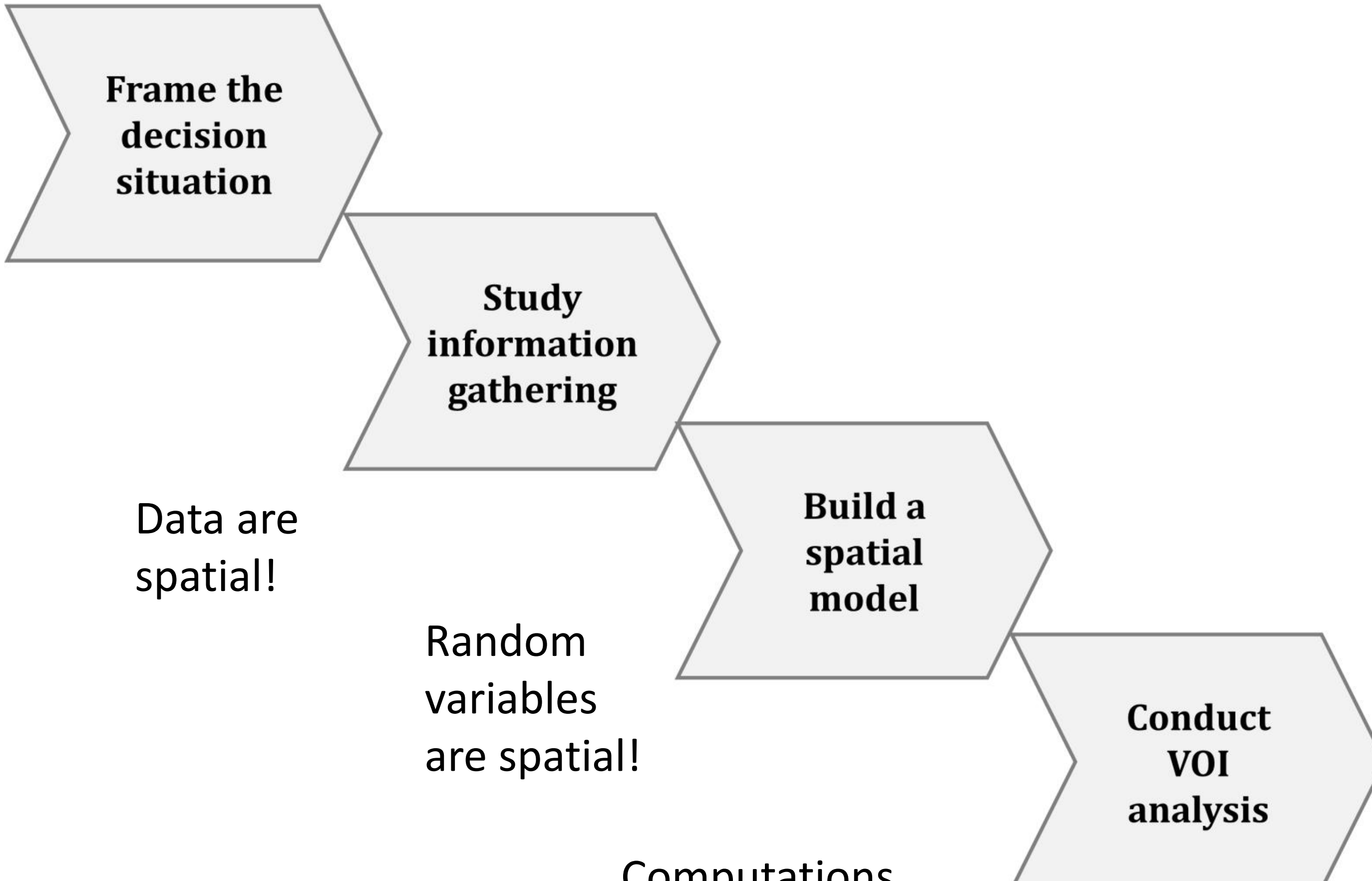
$\boldsymbol{x}$ - Uncertainties

$\boldsymbol{a}$ - Alternatives

$v\left(\boldsymbol{x}, \boldsymbol{a}\right)$ - Value function

$\boldsymbol{y}$ - Data

VOI workflow for spatial applications



Alternatives are spatial!

Data are spatial!

Random variables are spatial!

Computations are demanding!

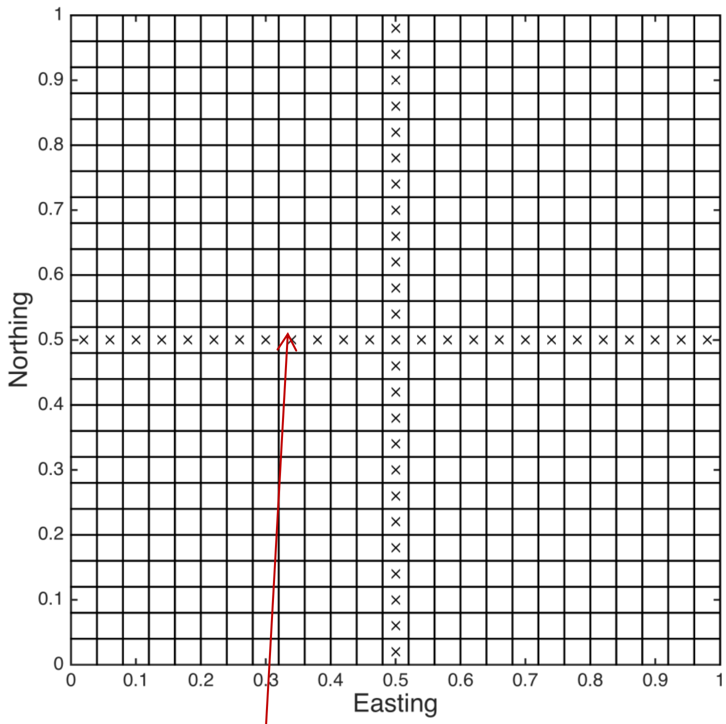
Information gathering

	Perfect	Imperfect
Total	Exact observations are gathered for all locations. This is rare, occurring when there is extensive coverage and highly accurate data gathering. $$\boldsymbol{y} = \boldsymbol{x}$$	Noisy observations are gathered for all locations. This is common in situations with remote sensors with extensive coverage, e.g. seismic, radar, satellite data. $$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\varepsilon}$$
Partial	Exact observations are gathered at some locations. This might occur, for instance, when there is careful analysis of rock samples along boreholes in a reservoir or a mine. $$\boldsymbol{y}_{\mathbb{K}} = \boldsymbol{x}_{\mathbb{K}}, \qquad \mathbb{K} \text{ subset}$$	Noisy observations are gathered at some locations. Examples include hand-held (noisy) meters to observe grades in mine boreholes, electromagnetic testing along a line, biological surveys of species, etc. $$\boldsymbol{y}_{\mathbb{K}} = \boldsymbol{x}_{\mathbb{K}} + \boldsymbol{\varepsilon}_{\mathbb{K}}, \qquad \mathbb{K} \text{ subset}$$

Illustration – forestry example

Farmer must decide whether to harvest forest units, or not.

Another decision is whether to collect data before making these decisions. If so, how and where should data be gathered.



Where to put survey lines for timber volumes information? Typically partial, imperfect information.

Decision situations and values

	Assumption: Decision Flexibility	Assumption: Value Function
Low decision flexibility; Decoupled value	Alternatives are easily enumerated $a \in A$	Total value is a sum of value at every unit $v(\mathbf{x}, a) = \sum_j v(x_j, a)$
High decision flexibility; Decoupled value	None $\mathbf{a} \in \mathbf{A}$	Total value is a sum of value at every unit $v(\mathbf{x}, \mathbf{a}) = \sum_j v(x_j, a_j)$
Low decision flexibility; Coupled value	Alternatives are easily enumerated $a \in A$	None $v(\mathbf{x}, a)$
High decision flexibility; Coupled value	None $\mathbf{a} \in \mathbf{A}$	None $v(\mathbf{x}, \mathbf{a})$

Decoupling – values are sums

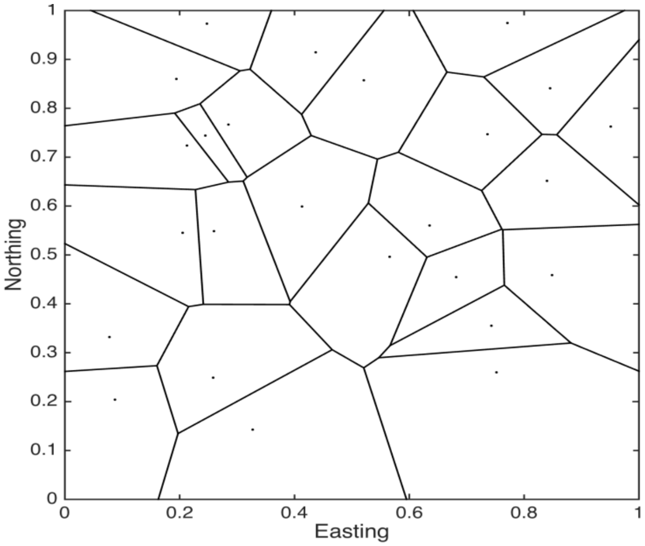
	Assumption: Decision Flexibility	Assumption: Value Function
Low decision flexibility; Decoupled value	Alternatives are easily enumerated $a \in A$	Total value is a sum of value at every unit $v(\mathbf{x},a)=\sum_j v(x_j,a)$
High decision flexibility; Decoupled value	None $\mathbf{a} \in \mathbf{A}$	Total value is a sum of value at every unit $v(\mathbf{x},\mathbf{a})=\sum_j v(x_j,a_j)$
Low decision flexibility; Coupled value	Alternatives are easily enumerated $a \in A$	None $v(\mathbf{x},a)$
High decision flexibility; Coupled value	None $\mathbf{a} \in \mathbf{A}$	None $v(\mathbf{x},\mathbf{a})$

Profit is sum of timber volumes from units.

Decoupled versus coupled value

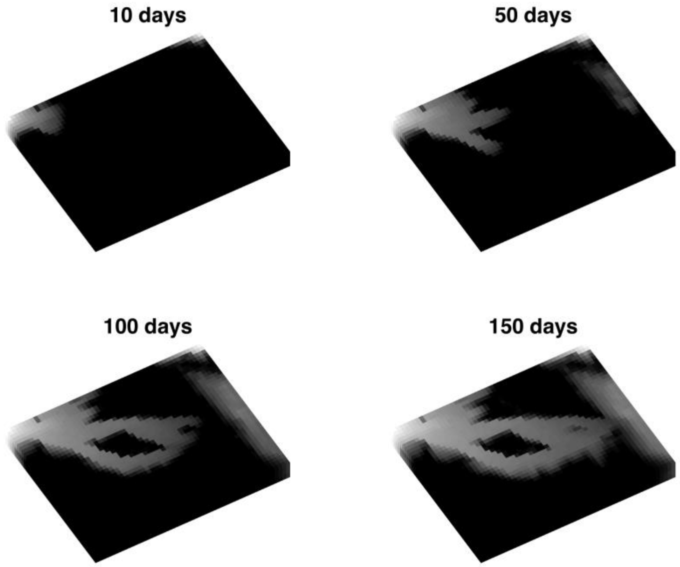
Farmer must decide whether to harvest at forest units, or not.

Value decouples to sum over units.



Petroleum company must decide how to produce a reservoir.

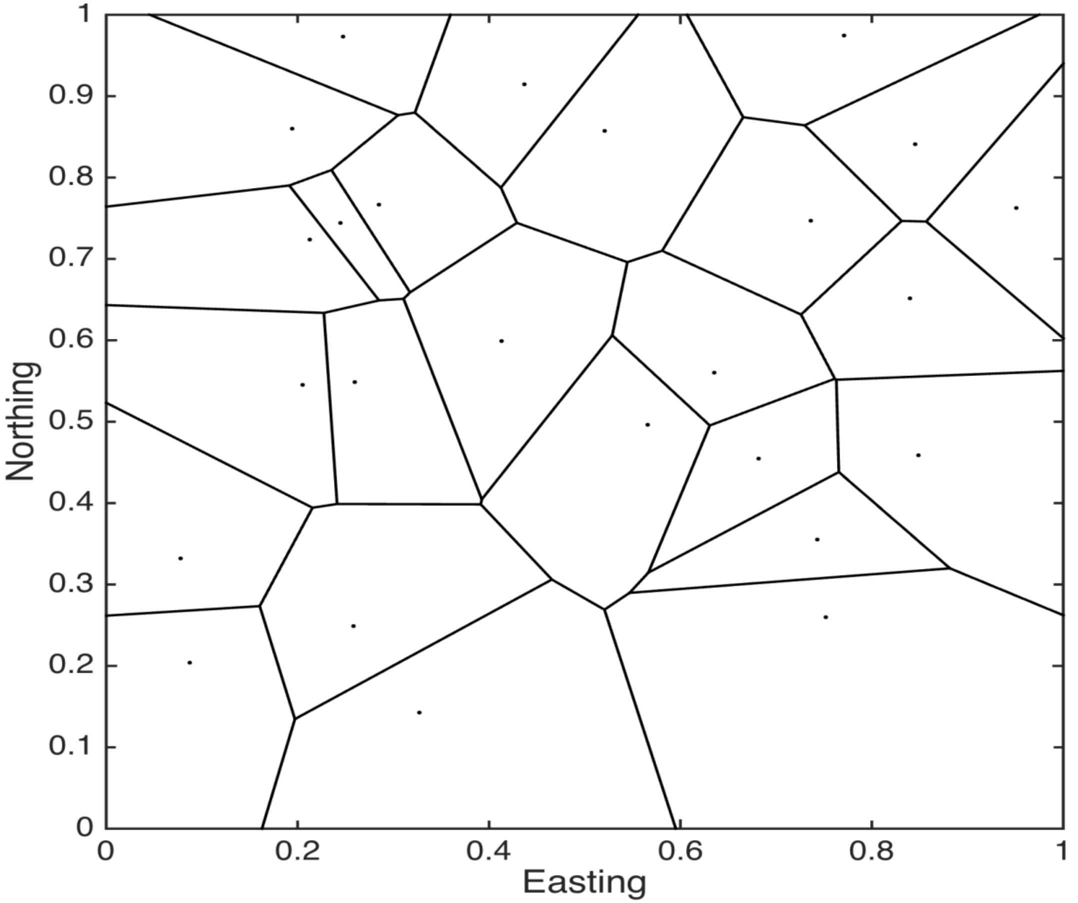
Value involves complex coupling of drilling strategies, and reservoir properties.



Low versus high flexibility

High flexibility:
Can select individual units.

Low flexibility:
Must select all units, or none.



Computation - Formula for VOI

$$PV = \max_{\boldsymbol{a}\in \boldsymbol{A}} \left\{ E\left(v(\boldsymbol{x},\boldsymbol{a}) \right) \right\} = \max_{\boldsymbol{a}\in \boldsymbol{A}} \left\{ \int_{\boldsymbol{x}} v(\boldsymbol{x},\boldsymbol{a}) p\left(\boldsymbol{x}\right) d\boldsymbol{x} \right\}$$

$$PoV\left(\boldsymbol{y}\right) = \int \max_{\boldsymbol{a}\in \boldsymbol{A}} \left\{ E\left(v\left(\boldsymbol{x},\boldsymbol{a}\right) \mid \boldsymbol{y} \right) \right\} p\left(\boldsymbol{y}\right) d\boldsymbol{y}$$

Computations :

- Easier with low decision flexibility (less alternatives).
- Easier if value decouples (sums or integrals split).
- Easier for perfect, total, information (upper bound on VOI).
- Sometimes analytical solutions. Otherwise approximations and Monte Carlo.

Formula for total perfect information

$$PV = \max_{\boldsymbol{a} \in A} \left\{ E\left(v(\boldsymbol{x}, \boldsymbol{a}) \right) \right\} = \max_{\boldsymbol{a} \in A} \left\{ \int_{\boldsymbol{x}} v(\boldsymbol{x}, \boldsymbol{a}) \, p\left(\boldsymbol{x} \right) d\boldsymbol{x} \right\}$$

$$PoV\left(\boldsymbol{x} \right) = \int \max_{\boldsymbol{a} \in A} \left\{ v\left(\boldsymbol{x}, \boldsymbol{a} \right) \right\} p\left(\boldsymbol{x} \right) d\boldsymbol{x}$$

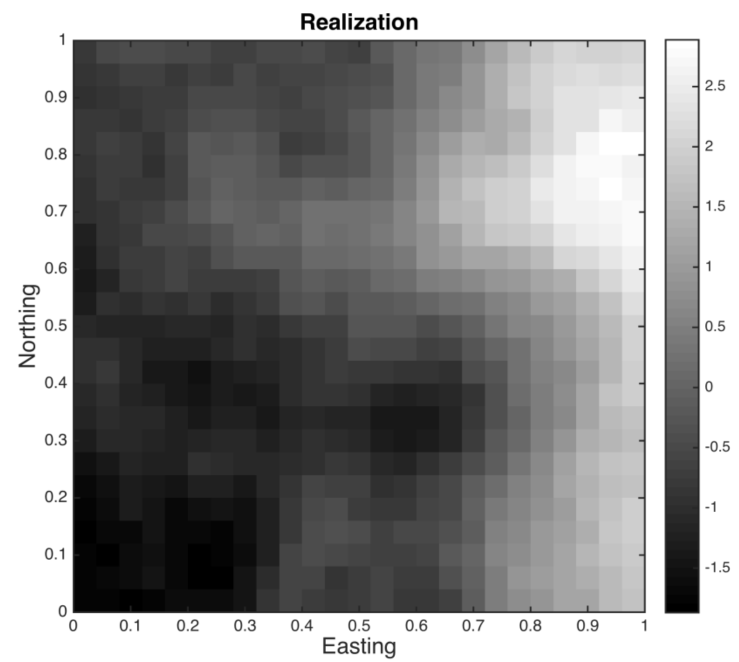
$$VOI\left(\boldsymbol{x} \right) = PoV\left(\boldsymbol{x} \right) - PV.$$

Upper bound on any information gathering scheme.

Gaussian process example

- Analytical solution.
- Decoupled value.

- Compare high-low decision flexibility.
- Compare different data gathering opportunities.
- Study sensitivity of VOI for different parameter settings.



Gaussian process for value

$$v = \sum_i v(x_i, a = 1) = \sum_i x_i, \qquad v(x_i, a = 0) = 0$$

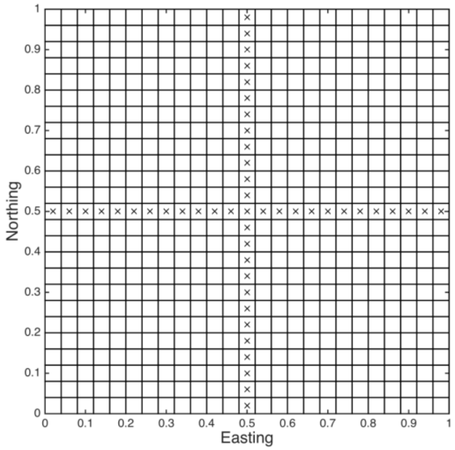
Global alternatives

$$v(x_i, a_i = 1) = x_i, \qquad v(x_i, a_i = 0) = 0$$

Local alternatives

Motivation, uncertainties on a grid - model profits directly.

Forest units.
Uncertainty is value in each cell.



Gaussian process

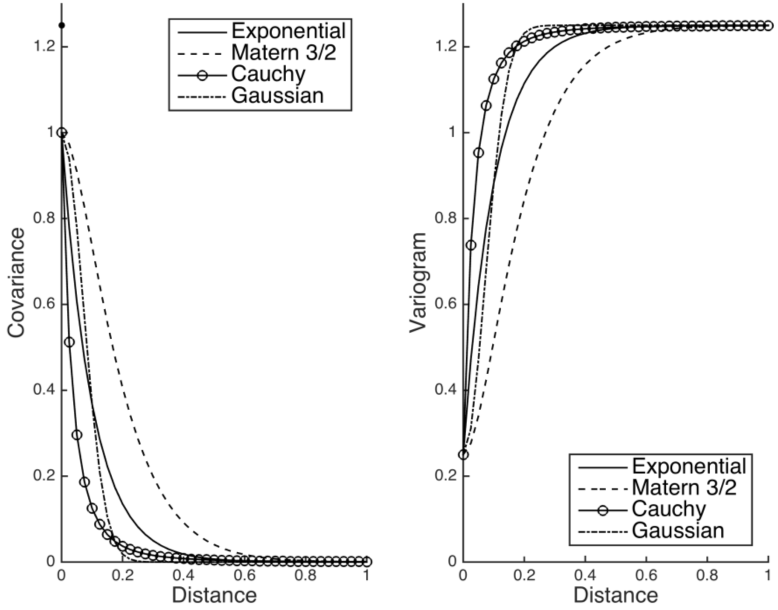
$$\boldsymbol{x} = (x_1, \ldots, x_n)$$

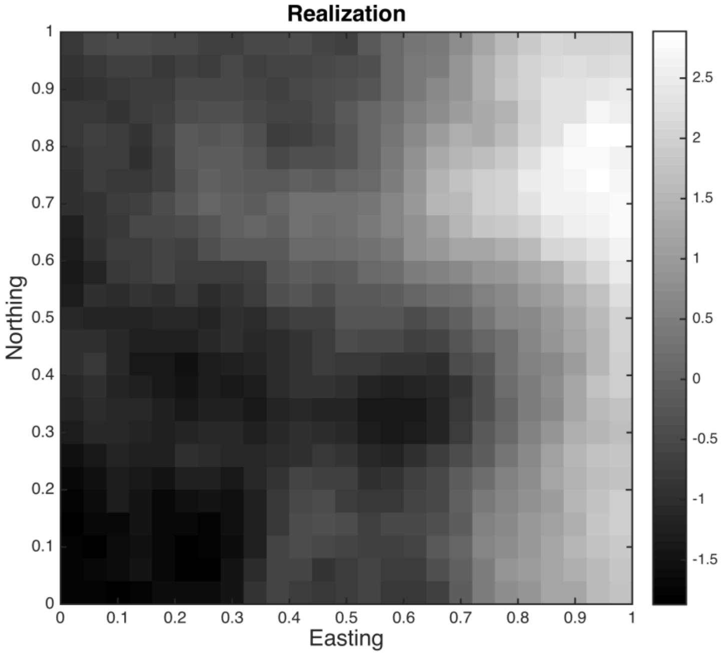
$$p(\boldsymbol{x}) = N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$\boldsymbol{\mu} = \mathbf{1}\mu$$

$$\Sigma_{ij} = \Sigma(\mathbf{s}_i, \mathbf{s}_j; \theta)$$

Spatial dependence, Matern covariance.





Formulas for Gaussian models

$$p(\boldsymbol{x}) = N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

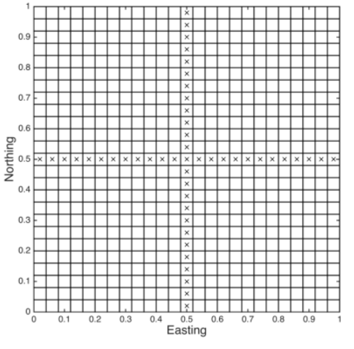
Prior for values.
Uncertainties are timber values.

$$\boldsymbol{y} = \boldsymbol{F}\boldsymbol{x} + N(\boldsymbol{0}, \tau^2 \boldsymbol{I})$$

$$p(\boldsymbol{y} \mid \boldsymbol{x}) = N(\boldsymbol{F}\boldsymbol{x}, \tau^2 \boldsymbol{I})$$

Likelihood,
design matrix, picks
data locations.

Motivation, uncertainties on a grid – forest units.



Conditioning – Gaussian models

$$E(\boldsymbol{x} \mid \boldsymbol{y}) = \boldsymbol{\mu} + \boldsymbol{\Sigma F}^t \left(\tau^2 \boldsymbol{I} + \boldsymbol{F \Sigma F}^t\right)^{-1} \left(\boldsymbol{y} - \boldsymbol{F} \boldsymbol{\mu}\right)$$

$$Var(\boldsymbol{x} \mid \boldsymbol{y}) = \boldsymbol{\Sigma} - \boldsymbol{R}, \qquad \boldsymbol{R} = \boldsymbol{\Sigma F}^t \left(\tau^2 \boldsymbol{I} + \boldsymbol{F \Sigma F}^t\right)^{-1} \boldsymbol{F \Sigma}$$

$$r_i = \sqrt{R_{ii}}$$

Prediction, Kriging.

VOI – Gaussian models

$$PV = \max\left\{0, E\left(\sum_{i=1}^{n} x_i\right)\right\} = \max\left\{0, \sum_{i=1}^{n} \mu_i\right\}$$

Low flexibility:
Must select all units, or none.

Value decouples to sum.

$$\boldsymbol{R} = \boldsymbol{\Sigma F}^t\left(\tau^2 \boldsymbol{I} + \boldsymbol{F\Sigma F}^t\right)^{-1}\boldsymbol{F\Sigma}$$

$$\mu_w = \sum_{i=1}^{n} \mu_i \qquad r_w^2 = \sum_{k=1}^{n}\sum_{l=1}^{n} R_{kl}$$

$$PoV(\boldsymbol{y}) = \int \max\left\{0, E\left(\sum_{i=1}^{n} x_i \mid \boldsymbol{y}\right)\right\} p(\boldsymbol{y})\,d\boldsymbol{y} = \mu_w \Phi\left(\mu_w / r_w\right) + r_w \phi\left(\mu_w / r_w\right)$$

$\phi(z), \Phi(z)$ standard Gaussian density and cumulative function

VOI – Gaussian models

High flexibility:
Can select individual units.

$$PV = \sum_{i=1}^{n} \max\left\{0, E\left(x_i\right)\right\} = \sum_{i=1}^{n} \max\left\{0, \mu_i\right\}$$

Value decouples to sum.

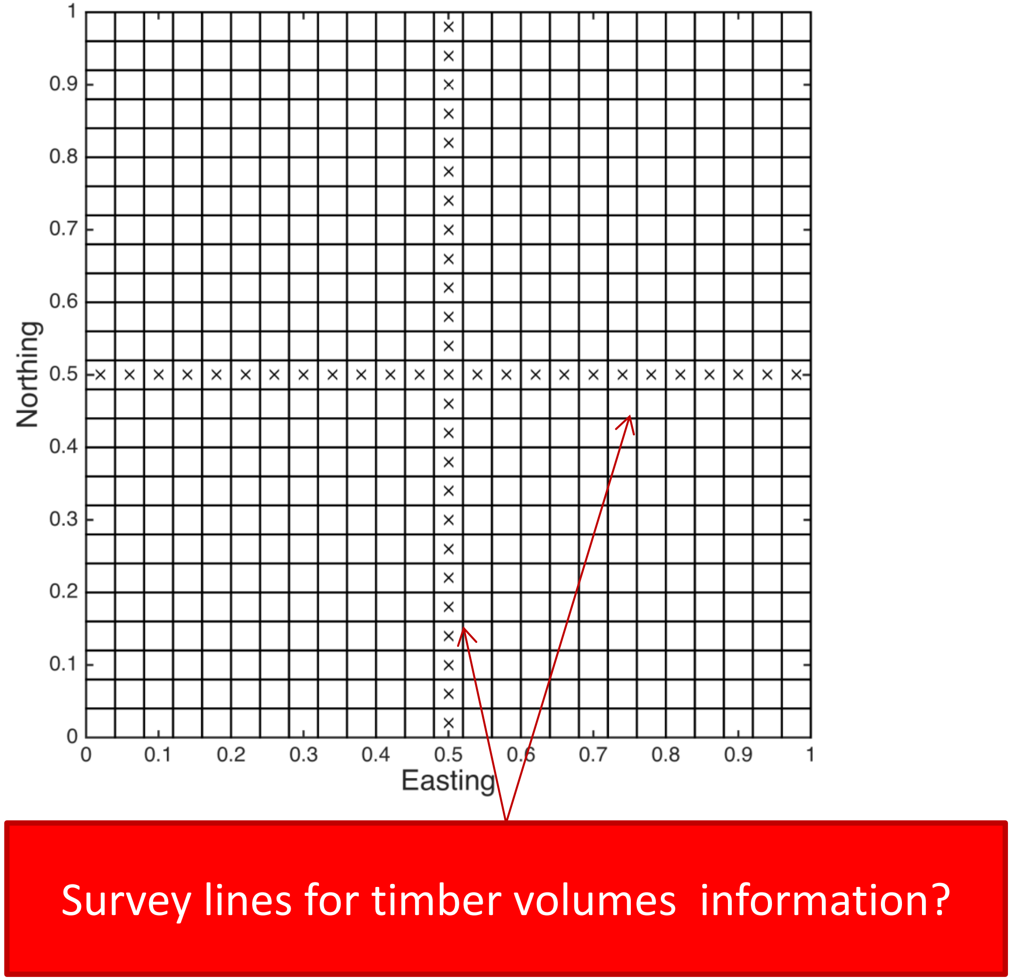
$$\boldsymbol{R} = \boldsymbol{\Sigma F}^t\left(\tau^2 \boldsymbol{I} + \boldsymbol{F\Sigma F}^t\right)^{-1} \boldsymbol{F\Sigma} \qquad r_i = \sqrt{R_{ii}}$$

$$PoV\left(\boldsymbol{y}\right) = \sum_{i=1}^{n} \int \max\left\{0, E\left(x_i \mid \boldsymbol{y}\right)\right\} p\left(\boldsymbol{y}\right) d\boldsymbol{y} = \sum_{i=1}^{n}\left(\mu_i \Phi\left(\mu_i / r_i\right) + r_i \phi\left(\mu_i / r_i\right)\right)$$

$\phi\left(z\right), \Phi\left(z\right)$ standard Gaussian density and cumulative function

Forestry example - information

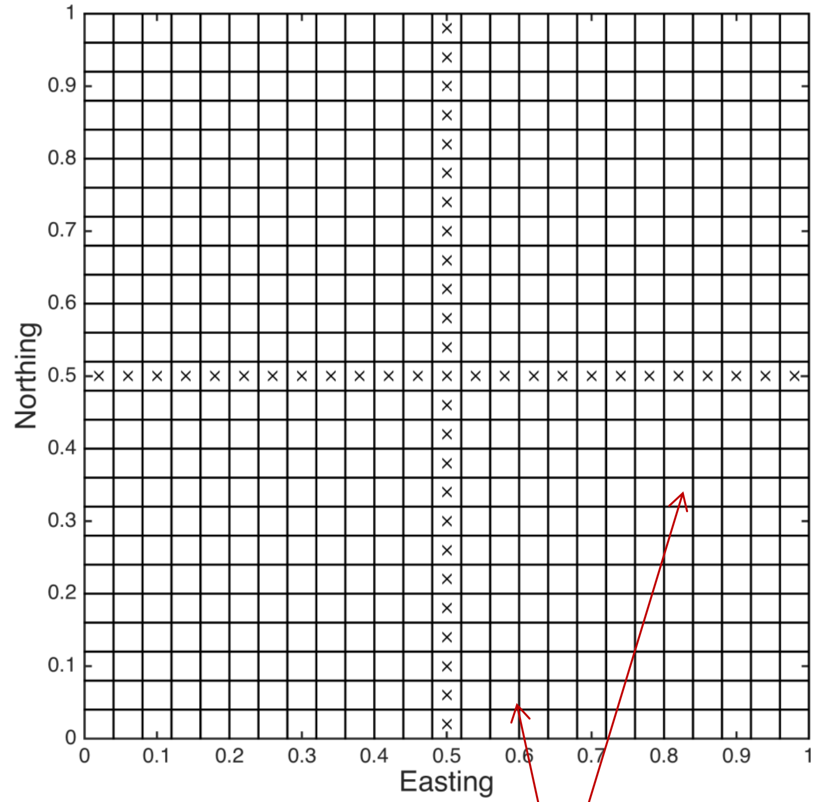
Farmer must decide whether to harvest forest units, or not.



Forestry example - information

Three data designs:

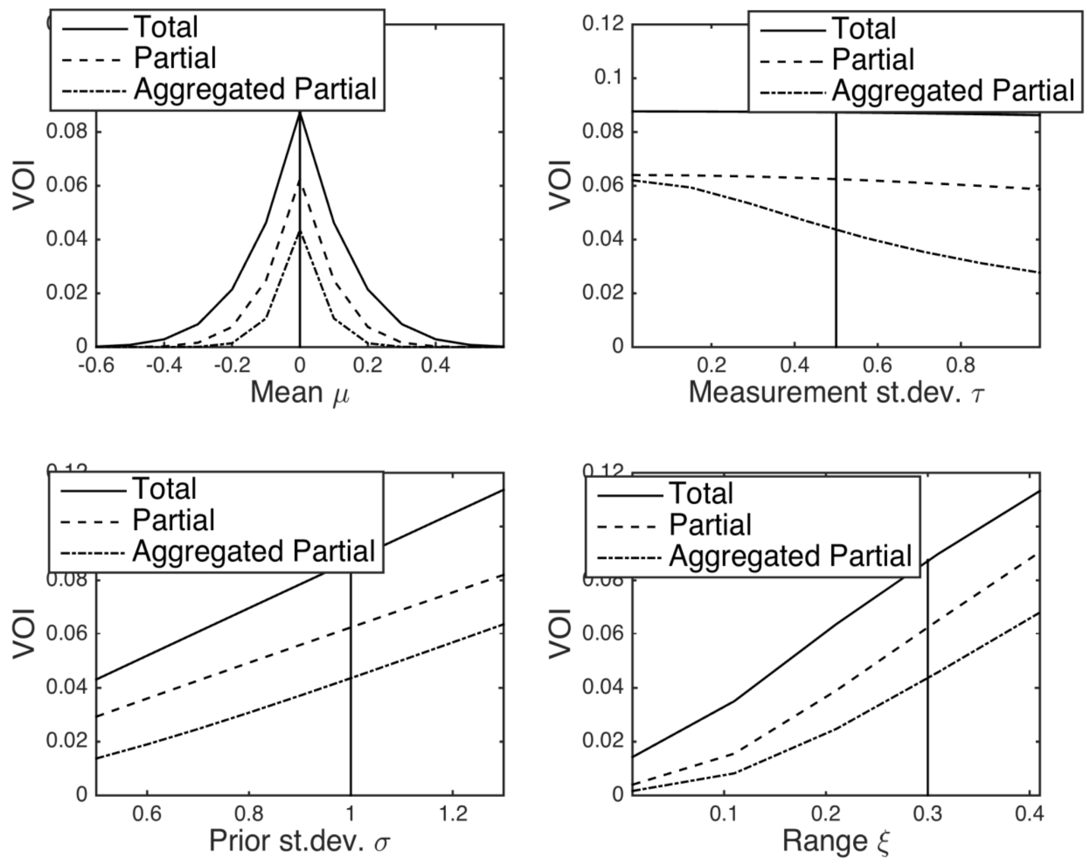
- Total (all cells)
- Partial (all cells along center lines)
- Aggregate partial (sums along the two center lines).



Survey lines for timber volumes information?

VOI - Forestry example

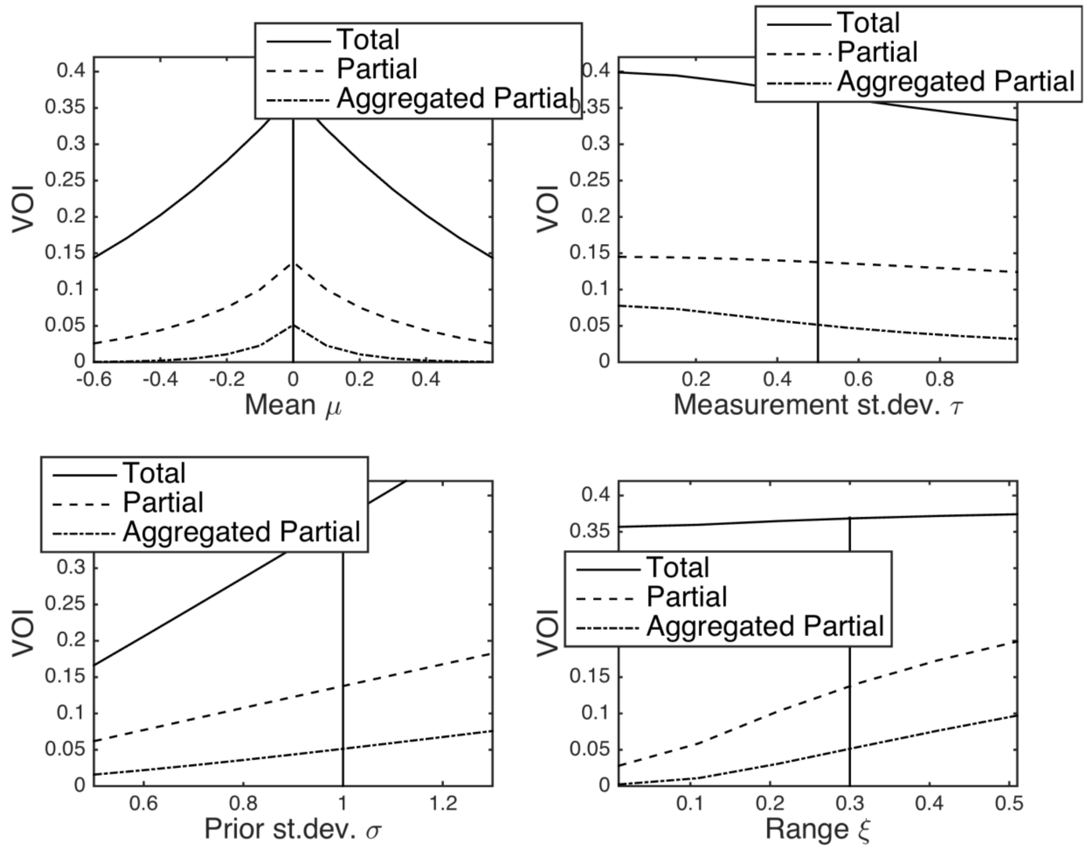
Low flexibility:
Must select all units, or none.



Total: all cells. Partial: Every cell along center lines. Aggregated partial: sums along center lines.

VOI - Forestry example

High flexibility:
Free to select units.



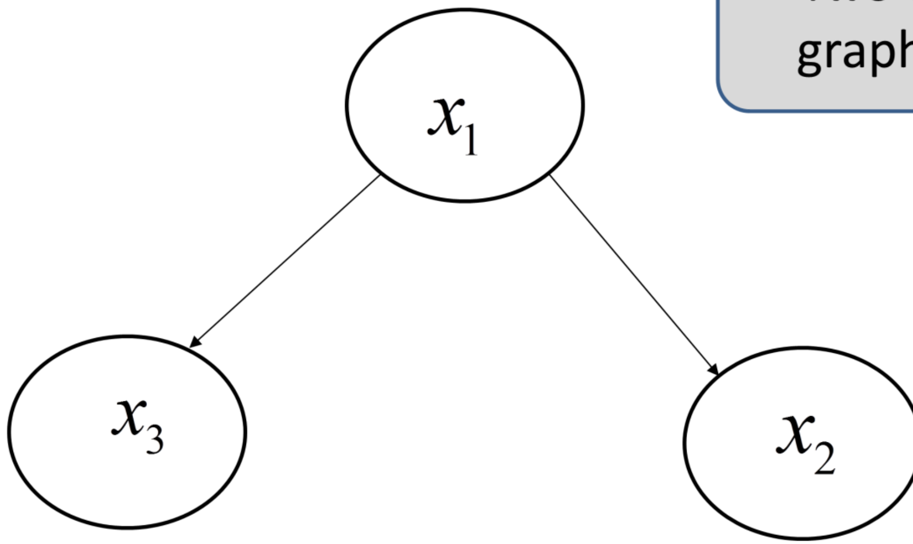
Total: all cells. Partial: Every cell along center lines. Aggregated partial: sums along center lines.

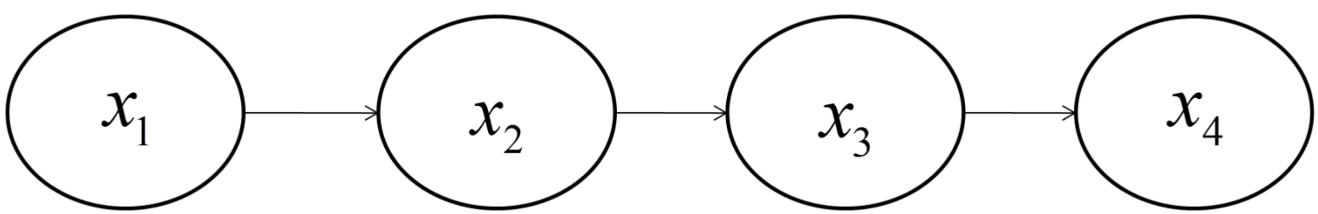
Insight in VOI – Gaussian example

- Higher decision flexibility gives larger VOI.
- Total test does not neccessarily give much higher VOI than a partial test. It depends on the spatial design of experiment as well as the prior model (mean and dependence).
- VOI increases with larger dependence in spatial uncertainties.
- VOI is largest when we are most indifferent in prior (mean near 0 and large prior uncertainty.
- VOI increases with higher accuracy of measurements.

Bayesian networks and Markov chains

Two examples of graphical models

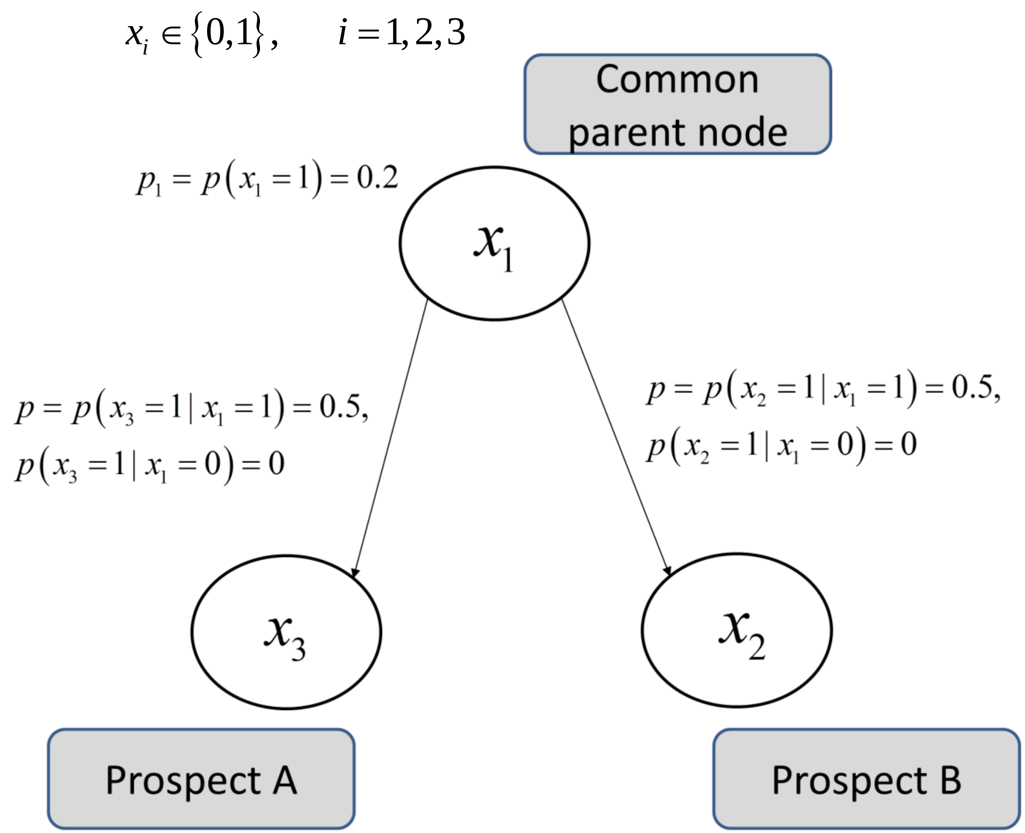




Bivariate petroleum prospects example

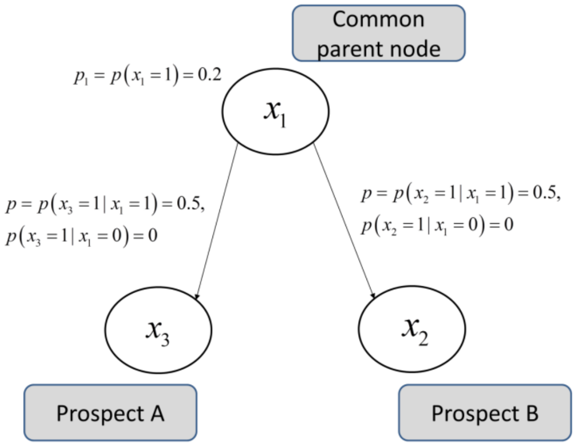
Conditional independence between prospect A and B, given outcome of parent!

Similar network models have been used in medicine/genetics, and testing for heritable diseases.

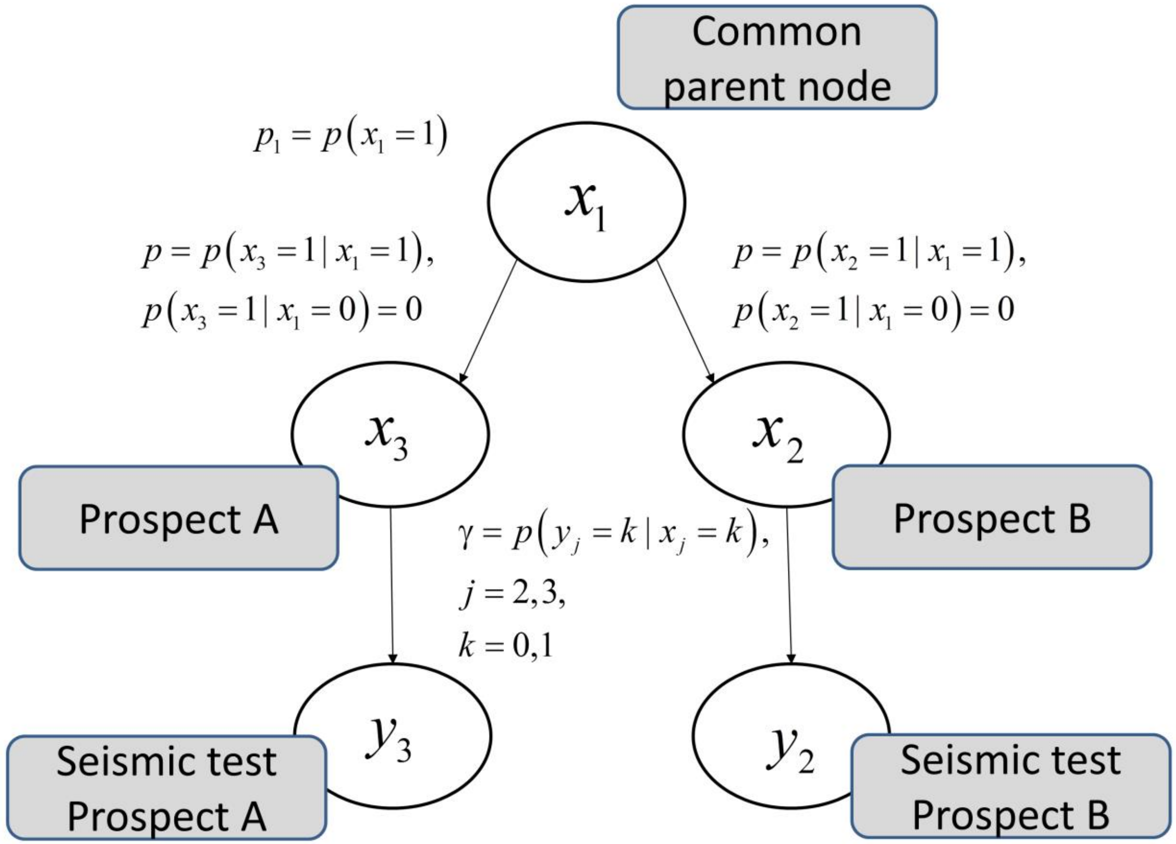


Bivariate petroleum prospects example

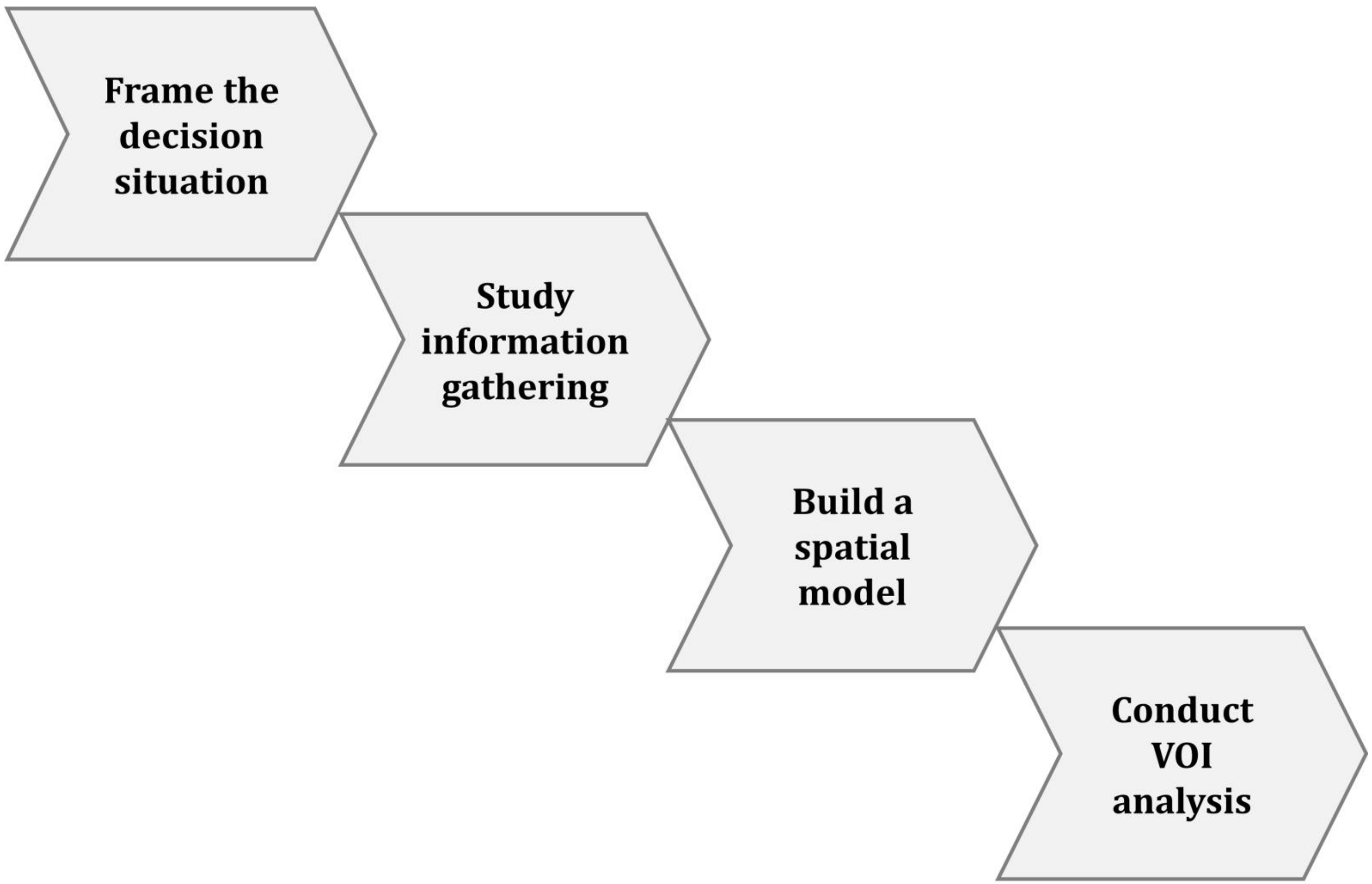
Joint	Failure prospect B	Success prospect B	Marginal probability
Failure prospect A	0.85	0.05	0.9
Success prospect A	0.05	0.05	0.1
Marginal probability	0.9	0.1	1



Example - Bivariate petroleum prospects



VOI workflow



Bivariate petroleum prospects

Need to frame the **decision situation**:

- Can one freely select (profitable) prospects, or must both be selected.
- Does value decouple?
- Can one do sequential selection?

Need to study **information gathering** options:

- Imperfect (seismic), or perfect (well data)?
- Can one test both prospects, or only one (total or partial)?
- Can one perform sequential testing?

Bivariate petroleum prospects

Need to frame the **decision situation**:

- Can one freely select (profitable) prospects, or must both be selected. Free selection.
- Does value decouple? Yes, no communication between prospects.
- Can one do sequential selection? Non-sequential.

Need to study **information gathering** options:

- Imperfect (seismic), or perfect (well data)? Study both.
- Can one test both prospects, or only one (total or partial)? Study both.
- Can one perform sequential testing? Not done here.

Bivariate prospects example - perfect

Assume we can freely select (develop) prospects, if profitable.

$$\text{Rev}_1 = \text{Rev}_2 = \text{Rev} = 3$$

$$PV = \sum_{i \in \{A,B\}} \max\left\{0, \text{Rev} \cdot p\left(x_i = 1\right) - \text{Cost}\right\}$$

$$= 2\max\left\{0, 0.3 - \text{Cost}\right\}$$

Total clairvoyant information

$$PoV\left(\boldsymbol{x}\right) = \sum_{i \in \{A,B\}} p\left(x_i = 1\right) \cdot \max\left\{0, \text{Rev} - \text{Cost}\right\}$$

$$= 0.2\max\left\{0, 3 - \text{Cost}\right\}$$

$$VOI\left(\boldsymbol{x}\right) = PoV\left(\boldsymbol{x}\right) - PV$$

Bivariate prospects example - perfect

Assume we can freely select (develop) prospects, if profitable.

$$\mathrm{Rev}_1 = \mathrm{Rev}_2 = \mathrm{Rev} = 3$$

$$PV = \sum_{i \in \{A,B\}} \max\left\{0, \mathrm{Rev} \cdot p\left(x_i = 1\right) - \mathrm{Cost}\right\}$$

$$= 2\max\left\{0, 0.3 - \mathrm{Cost}\right\}$$

Partial clairvoyant information

$$PoV\left(x_A\right) = p\left(x_A = 1\right) \cdot \max\left\{0, 3 - \mathrm{Cost}\right\}$$

$$+ \sum_l p\left(x_A = l\right) \cdot \max\left\{0, \mathrm{Rev} \cdot p\left(x_B = 1 \mid x_A = l\right) - \mathrm{Cost}\right\}$$

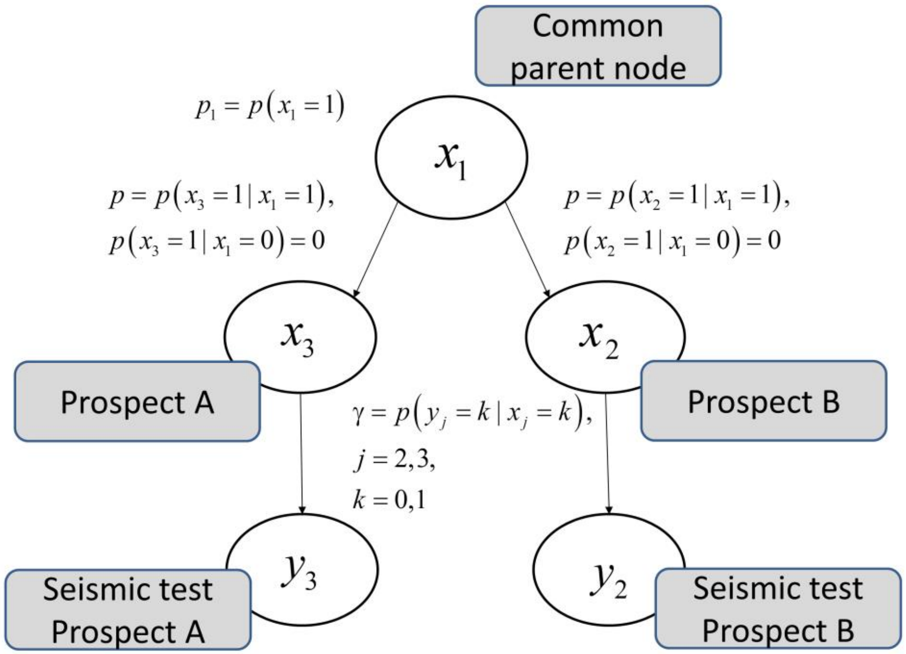
$$= 0.1 \cdot \max\left\{0, 3 - \mathrm{Cost}\right\} + 0.1 \cdot \max\left\{0, \mathrm{Rev} \cdot 0.5 - \mathrm{Cost}\right\}$$

$$+ 0.9 \cdot \max\left\{0, 3 \cdot 0.055 - \mathrm{Cost}\right\}$$

Bivariate prospects example - imperfect

Define sensitivity of seismic test (imperfect):

$$p\left(y_j = k \mid x_j = k\right) = \gamma = 0.9, \qquad k = 1,2$$



Bivariate prospects example - imperfect

Assume we can freely select (develop) prospects, if profitable.

$$\text{Rev}_1 = \text{Rev}_2 = \text{Rev} = 3$$

$$PV = \sum_{i \in \{A,B\}} \max\left\{0, \text{Rev} \cdot p\left(x_i = 1\right) - \text{Cost}\right\}$$

$$= 2 \max\left\{0, 0.3 - \text{Cost}\right\}$$

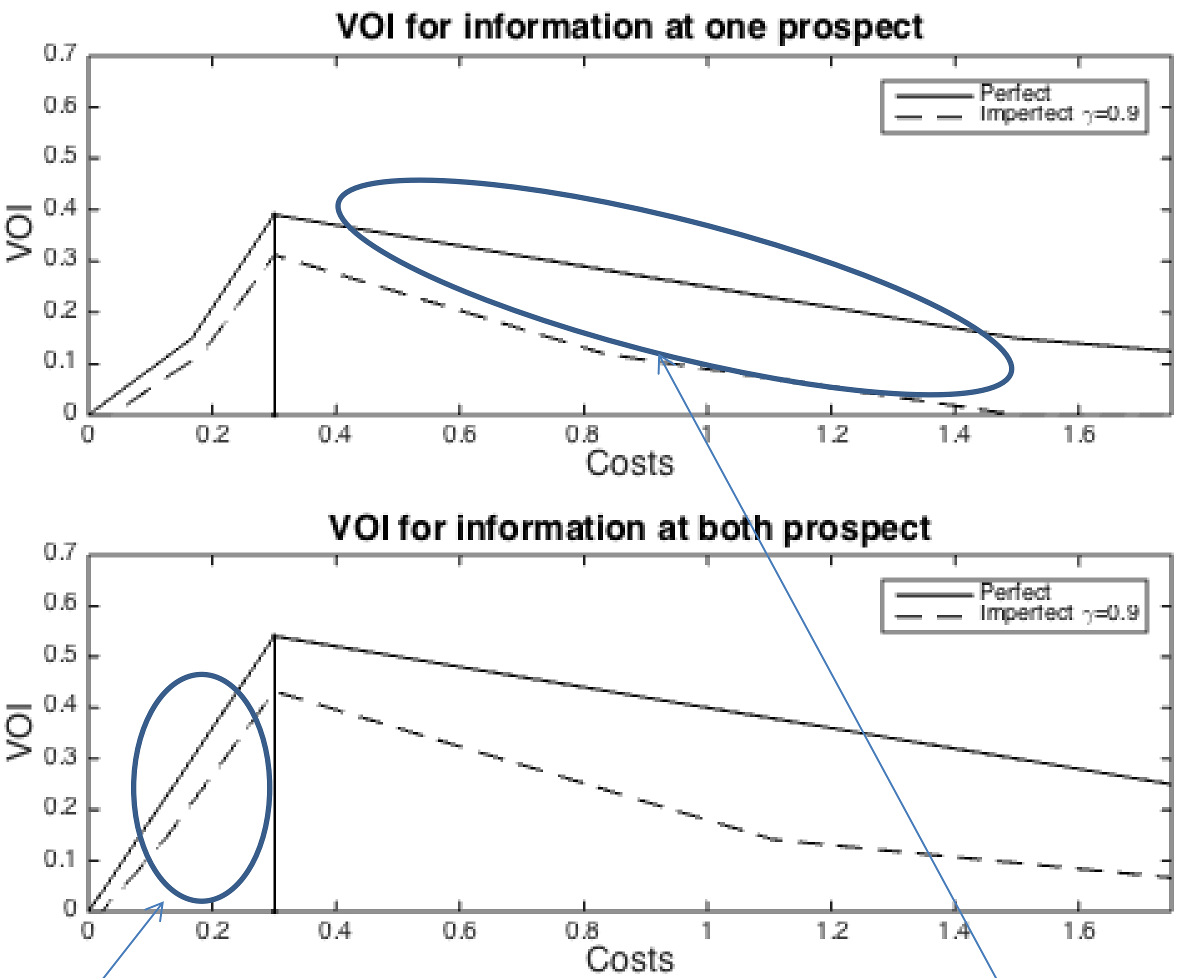
Total imperfect information

$$PoV\left(\boldsymbol{y}\right) = \sum_{\boldsymbol{y}} \sum_{i \in \{A,B\}} \max\left\{0, \text{Rev}\, p\left(x_i = 1 \mid \boldsymbol{y}\right) - \text{Cost}\right\} p\left(\boldsymbol{y}\right)$$

$$VOI\left(\boldsymbol{y}\right) = PoV\left(\boldsymbol{y}\right) - PV$$

Can also purchase imperfect partial information i.e. about one of the prospects?

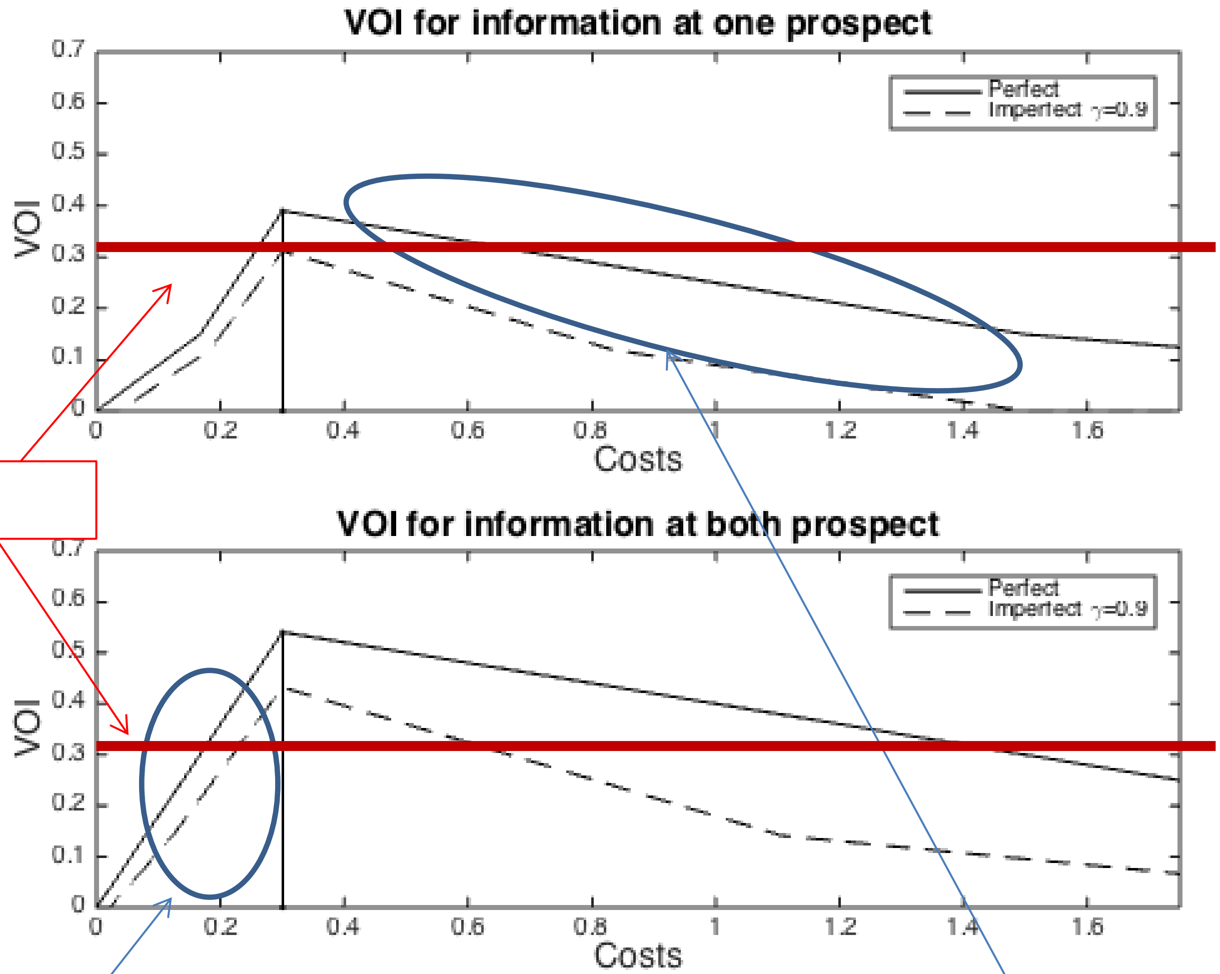
VOI for bivariate prospects example



Imperfect total better then partial perfect.

Partial perfect is better than imperfect total.

VOI for bivariate prospects example



Price of test is 0.3

Imperfect total better then partial perfect.

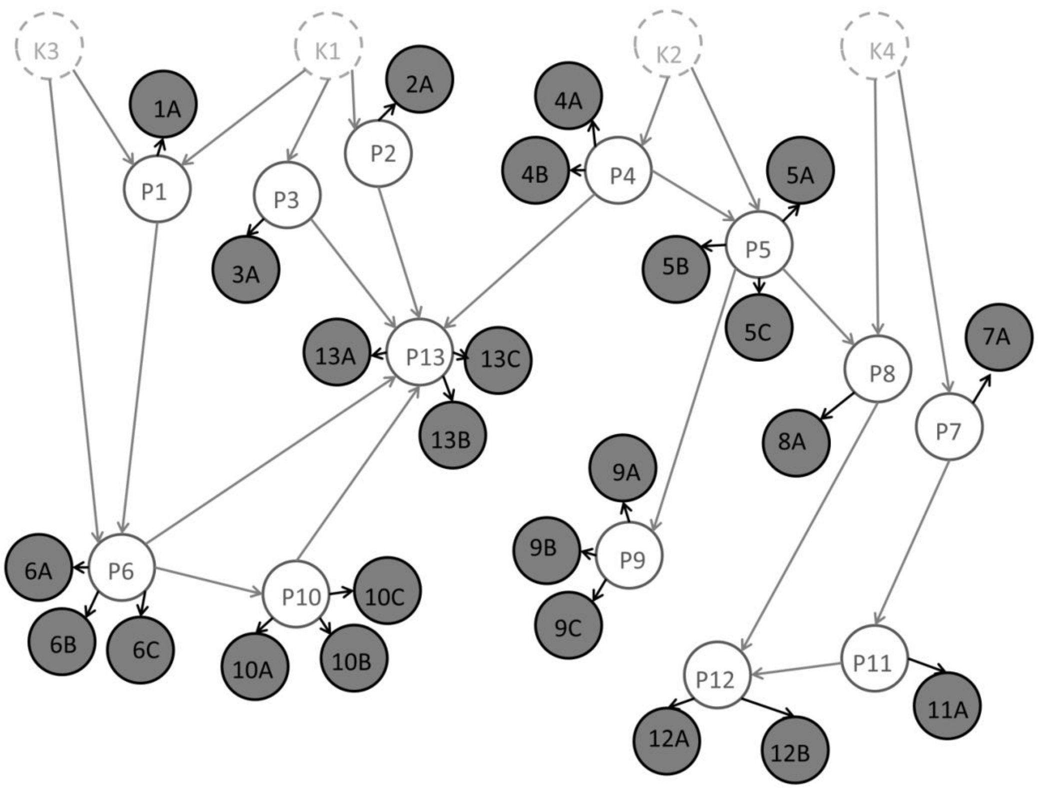
Partial perfect is better than imperfect total.

Insight in VOI – Bivariate prospects

- VOI of partial testing is always less than total testing, with same accuracy.
- Total imperfect test can give less VOI than a partial perfect test. Difference depends on the accuracy, prior mean for values, and correlation in spatial model.
- VOI is small for low costs (easy to start development) and for high cost (easy to avoid development). We do not need more data in these cases. We can make decisions right away.

Larger networks - computation

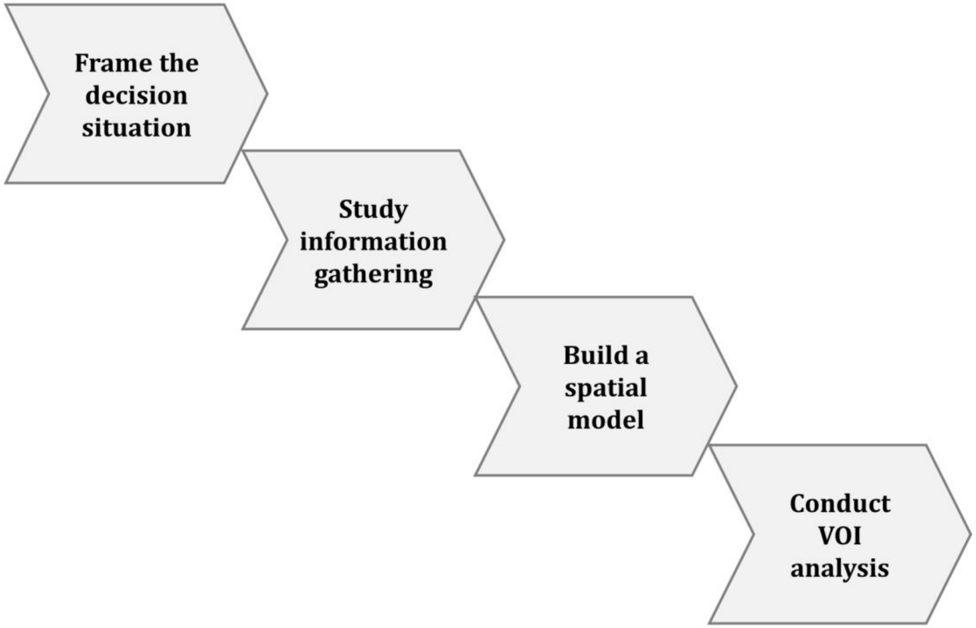
Algorithms have been developed for efficient marginalization, conditioning.



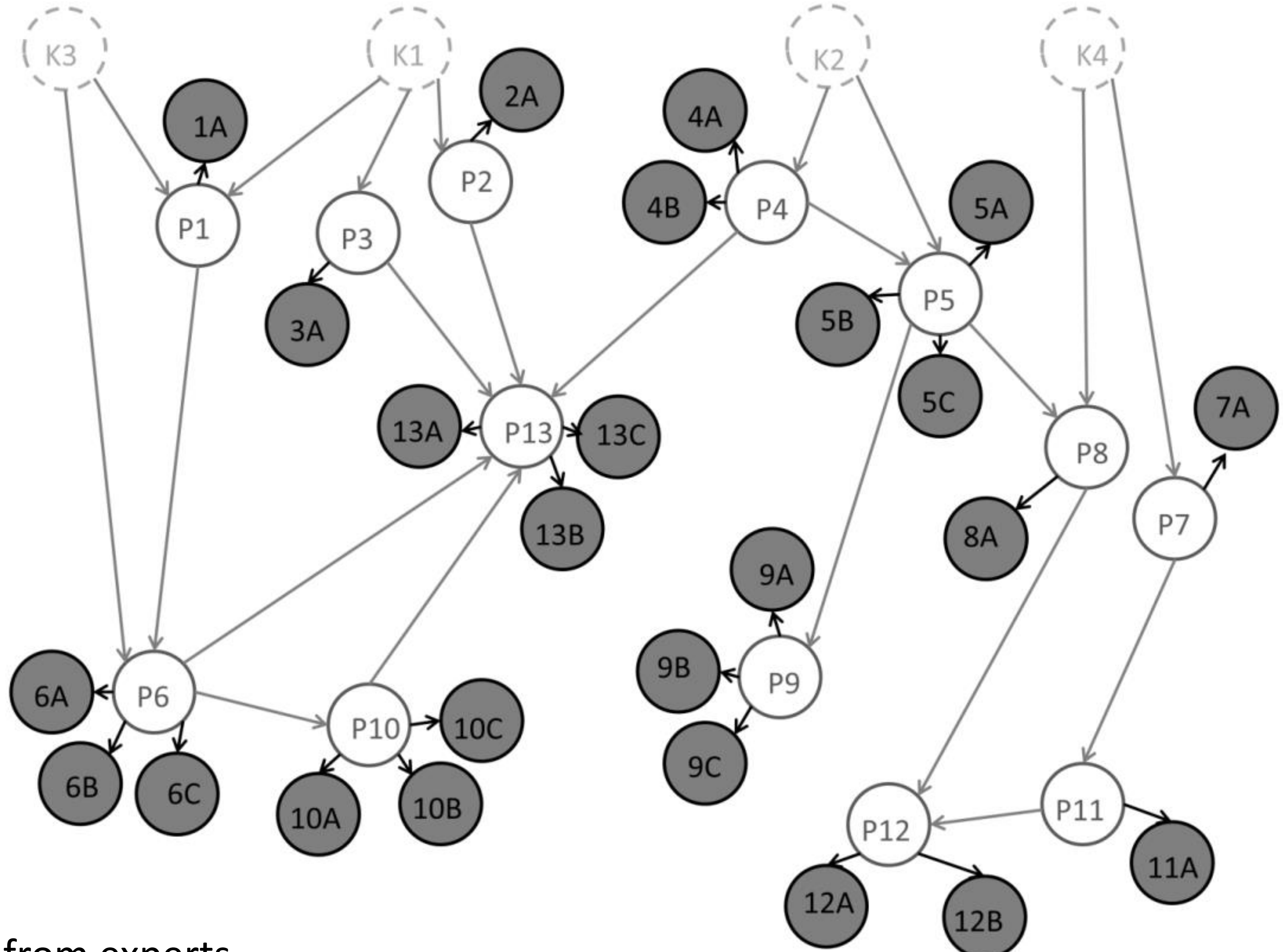
Martinelli, G., Eidsvik, J., Hauge, R., and Førland, M.D., 2011, Bayesian networks for prospect analysis in the North Sea, *AAPG Bulletin*, 95, 1423-1442.

VOI workflow

- Develop prospects separately. Shared costs for segments within one prospect.
- Gather information by exploration drilling. One or two wells. No opportunities for adaptive testing.
- Model is a Bayesian network model elicited from expert geologists in this area.
- VOI analysis done by exact computations for Bayesian networks (Junction tree algorithm – efficient marginalization and conditioning).



Bayesian network , Kitchens

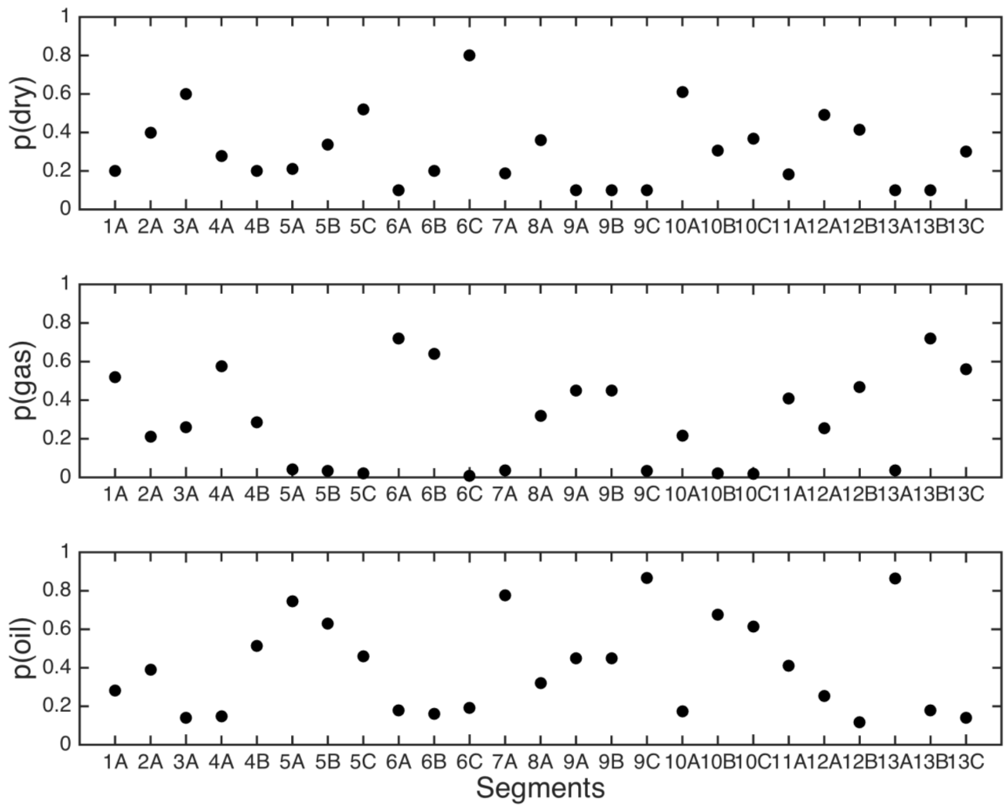


Model elicited from experts.
Migration from kitchens.
Local failure probability of migration.

Prior marginal probabilities

Three possible classes at all nodes:

- Dry
- Gas
- Oil



Prior values

Development fixed cost. Infrastructure at prospect r.

$$PV = \sum_{r=1}^{13} \max\left\{0, \sum_{i \in \Pr} IV(x_i) - DFC\right\}$$

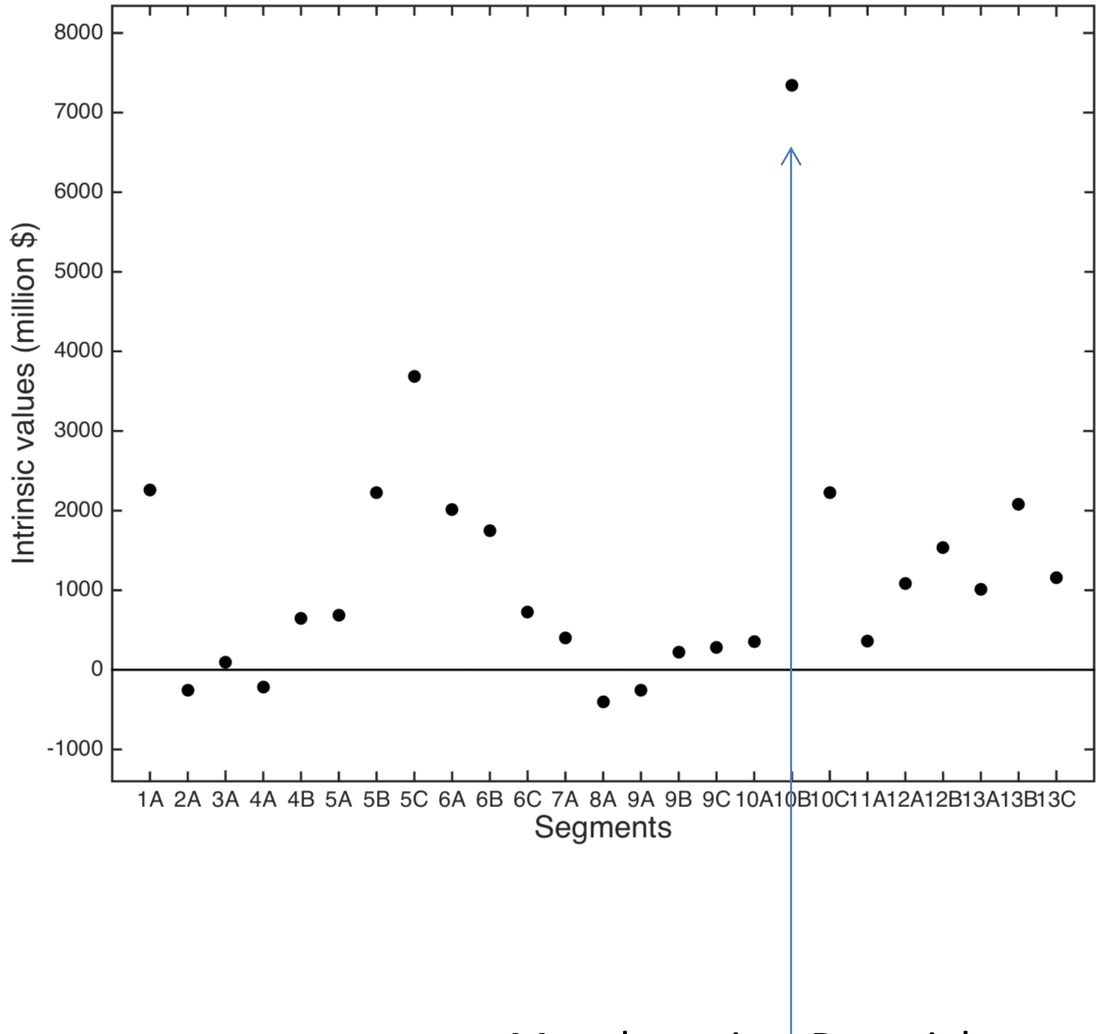
$$IV(x_i) = \sum_{k=1}^{3}\Big(\mathrm{Rev}_{i,k}\, p(x_i = k) - \mathrm{Cost}_{i,k}\, p(x_i = k)\Big) - \mathrm{Cost}_{i,0}$$

Revenues of oil/gas, 0 otherwise.

Cost if dry, 0 otherwise.

Cost of drilling segment i.

Values



Most lucrative. But might not be most informative.

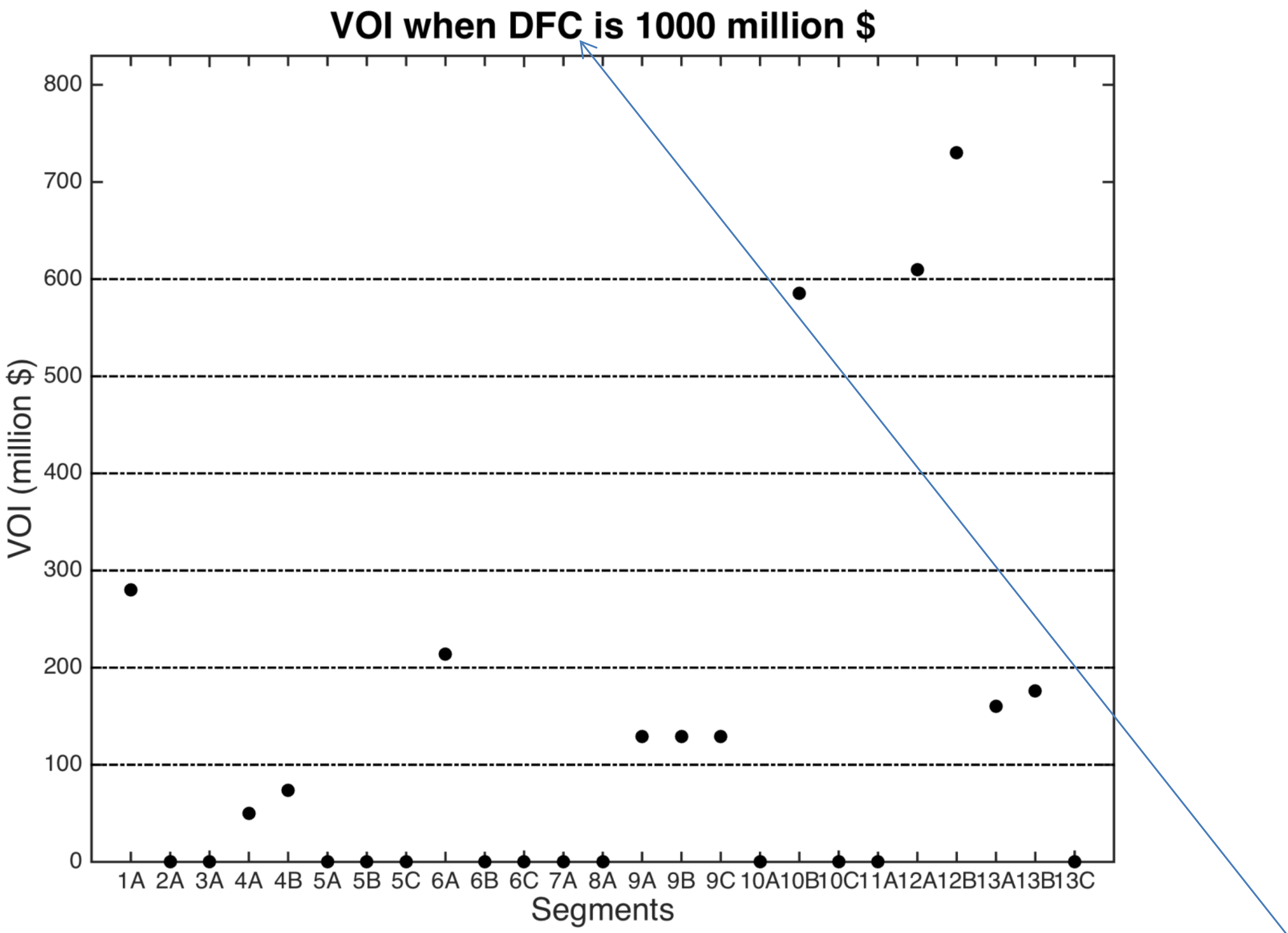
Posterior values and VOI

$$PoV\left(x_{\mathbb{K}}\right)=\sum_{l=1}^{3}\sum_{r=1}^{13}\max\left\{0,\sum_{i\in\Pr}IV\left(x_i \mid x_{\mathbb{K}}=l\right)-DFC\right\}p\left(x_{\mathbb{K}}=l\right)$$

$$VOI\left(x_{\mathbb{K}}\right)=PoV\left(x_{\mathbb{K}}\right)-PV$$

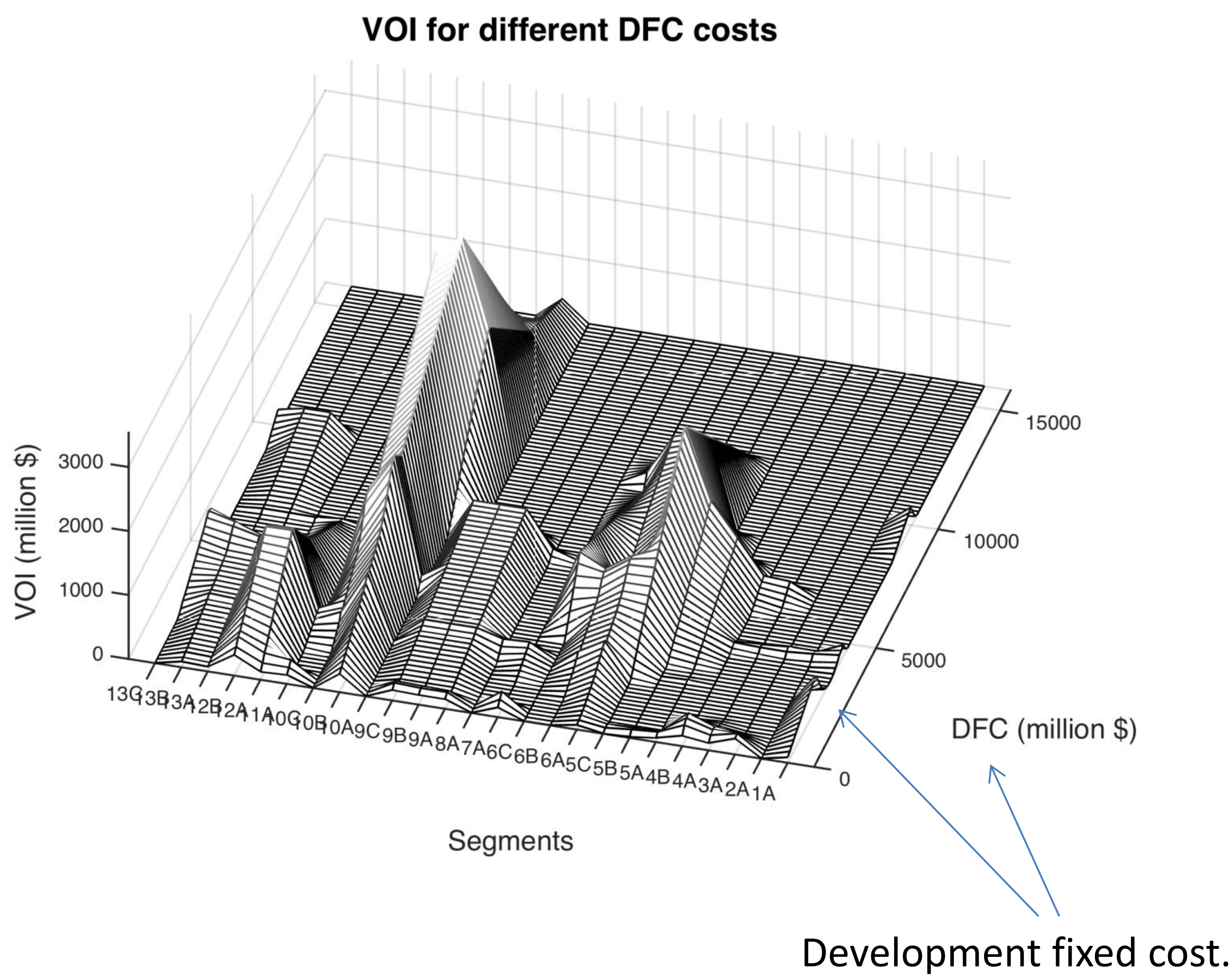
Data acquired at single well.

VOI single wells



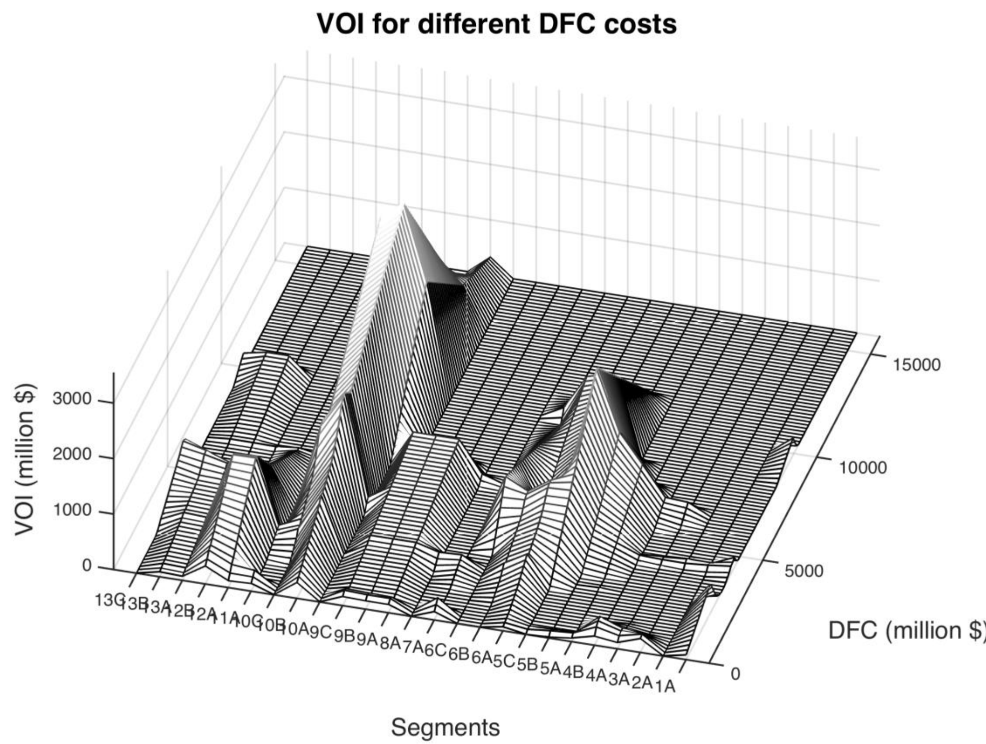
Development fixed cost.

VOI for different costs



VOI for different costs

- For each segment VOI starts at 0 (for small costs), grows to larger values, and decreases to 0 (for large costs).
- VOI is smooth for segments belonging to the same prospect. Correlation and shared costs.
- VOI can be multimodal as a function of cost, because the information influences neighboring segments, at which we are indifferent at other costs.



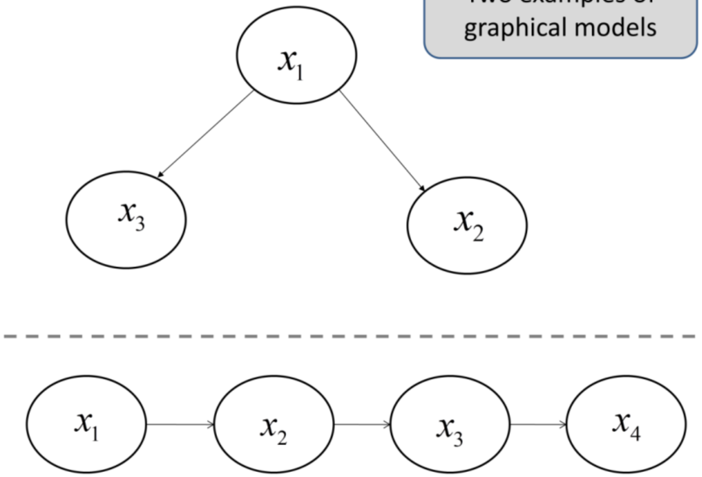
Insight from this example:

- VOI is not largest at the most lucrative prospects.
- VOI is largest where more data are likely to help us make better decisions.
- VOI also depends on whether the data gathering can influence neighboring segments – data propagate in the Bayesian network model.

- Compare with price? Or compare different data gathering opportunities, and provide a basis for discussion.

Markov chains

Two examples of graphical models



Markov chains are special graphs, defined by initial probabilities and transition matrices.

$$p(\mathbf{x}) = p(x_1, x_2, \ldots, x_n) = p(x_1)\, p(x_2 \mid x_1) \ldots p(x_n \mid x_{n-1})$$

$$p(x_1 = k), \qquad k = 1, \ldots, d$$

$$p(x_{i+1} = l \mid x_i = k) = P(k, l), \qquad k, l = 1, \ldots, d$$

$d = 2$

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \qquad P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \qquad P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} \qquad P = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 1 \end{bmatrix}$$

Independence

Absorbing

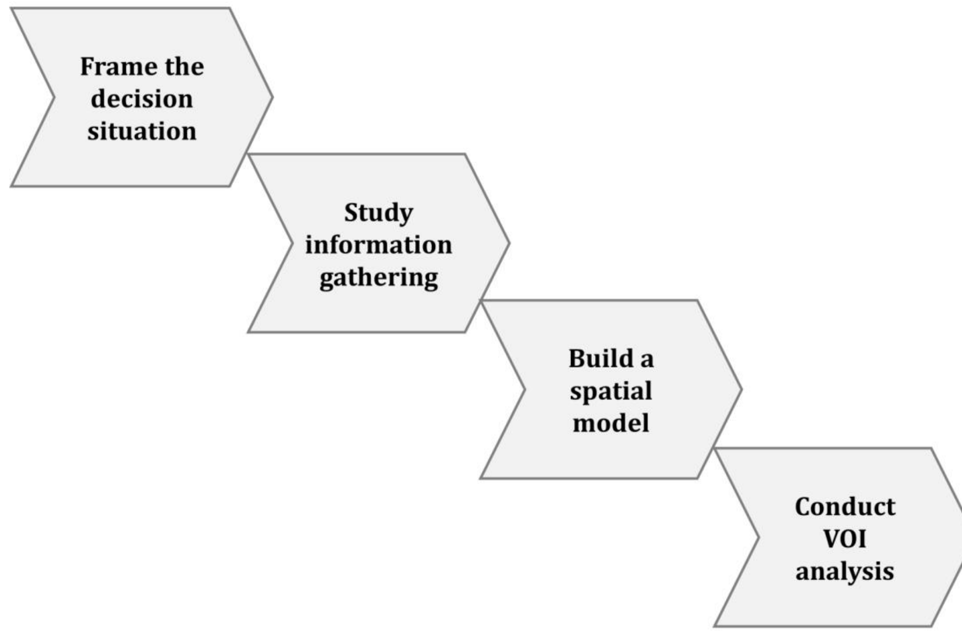
Avalanche decisions and sensors

Suppose that parts along a road are at risk of **avalanche**.
- One can remove risk by clearing roads, at a cost.
- Otherwise, the repair cost depends on the **unknown risk class: 1) low, 2) high**.

Data, sensor at a particular location, can help classify the risk class and hence improve the decisions made regarding cleaning / wait and see.

VOI workflow

- Clear entire road up front (fixed cost), or wait and see (uncertain cost at each location).
- Gather information by sensor at one location, perfect information about risk class at that location.
- Model is a Markov chain with increasing probability of high risk for later indeces (altitude).
- VOI analysis done by Markov chain calculations. Conducted for all possible sensor locations.



Avalanche decisions - risk analysis

n=50 identified locations along railroad track, at increasing altitude and risk of **avalanche**.
One can remove risk entirely by cost 100 000.
If it is not removed, the repair cost, at each location, depends on the unknown risk class:

$$C_j, \qquad j \in \{1,2\},$$

$$\mathrm{C}_1 = 0, \mathrm{C}_2 = 5000,$$

Decision maker must choose whether to

i) **clean tracks** up front, with fixed price.

ii) **wait and see**, with the uncertain price at each location.

The decision is based on the minimization of expected costs.

Prior value:

$$PV = \max\left\{-100000, -5000\sum_{i=1}^{50} p\left(x_i = 2\right)\right\}$$

Clean up front (−100000)

Expected value when wait and see ($-5000\sum_{i=1}^{50} p(x_i = 2)$).

Markovian model for risk of avalanche

Risk tends to start in lower class (1), and then move to higher class (2).
If risk class 2 is reached, it will stay there until location 50 (absorbing state).

$$x_i \in \{1,2\}, \qquad i = 1,\dots,50,$$

$$p(x_1 = 1) = 0.99,$$

$$p(x_1 = 2) = 0.01,$$

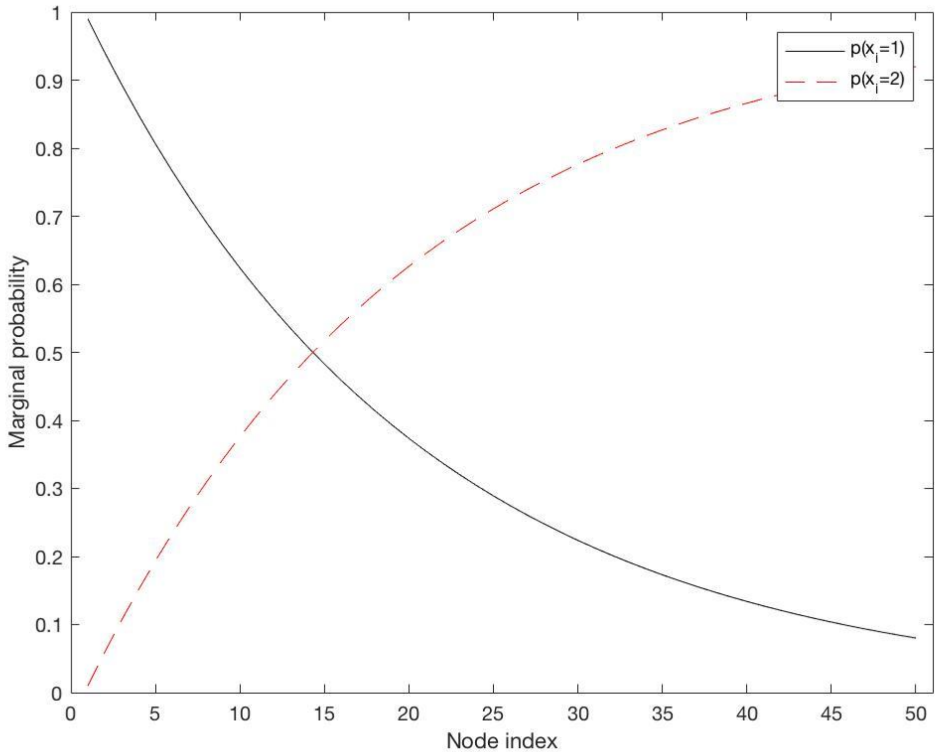
$$P = \begin{bmatrix} 0.95 & 0.05 \\ 0 & 1 \end{bmatrix}$$

Absorbing!

Results – marginals

$$p\left(x_i = l\right) = \sum_{k=1}^{2} p\left(x_{i-1} = k\right) p(x_i = l \mid x_{i-1} = k), \qquad l = 1, 2, \qquad i = 1, \ldots, n$$

$$p\left(x_i = 1\right) = p \cdot 0.99^{i-1} \qquad i = 1, \ldots, n$$



Sensor – perfect risk information at one location

- Install a sensor at one location, getting perfect information at that node.
- Compute conditional probabilities.

$$p\left(x_i = k \mid x_j = l\right), \qquad i = 1,\ldots,50$$

Results – conditionals (forward)

$$p\left(x_i = k \mid x_j = l\right) = \sum_{q=1}^{2} p\left(x_i = k, x_{i-1} = q \mid x_j = l\right) = \sum_{q=1}^{2} P(q,k)\, p\left(x_{i-1} = q \mid x_j = l\right)$$

$$p\left(x_i = 1 \mid x_j = 1\right) = 0.99^{i-j} \qquad i \geq j,$$

$$p\left(x_i = 2 \mid x_j = 2\right) = 1 \qquad i \geq j$$

Results – conditionals (backward)

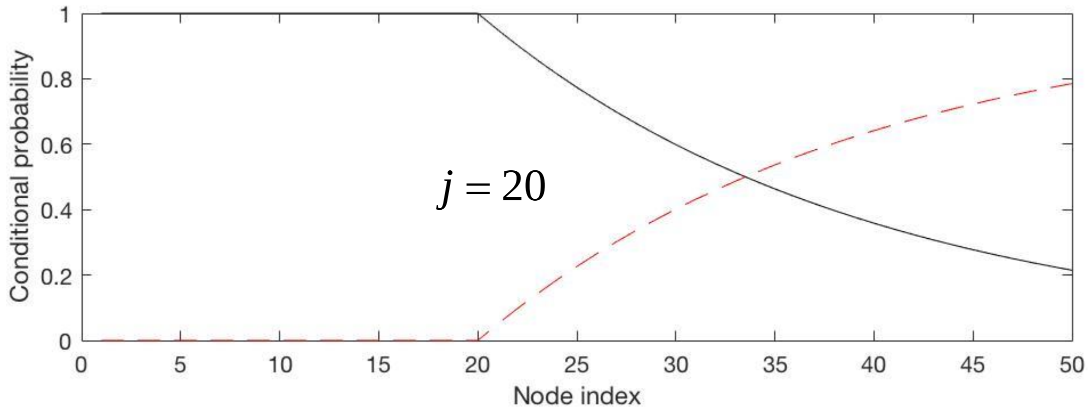
$$p\left(x_i = k \mid x_{i+1},\ldots,x_n\right) = p\left(x_i = k \mid x_{i+1} = l\right) = \frac{p\left(x_i = k, x_{i+1} = l\right)}{p\left(x_{i+1} = l\right)} = \frac{P\left(k,l\right)p\left(x_i = k\right)}{p\left(x_{i+1} = l\right)}$$

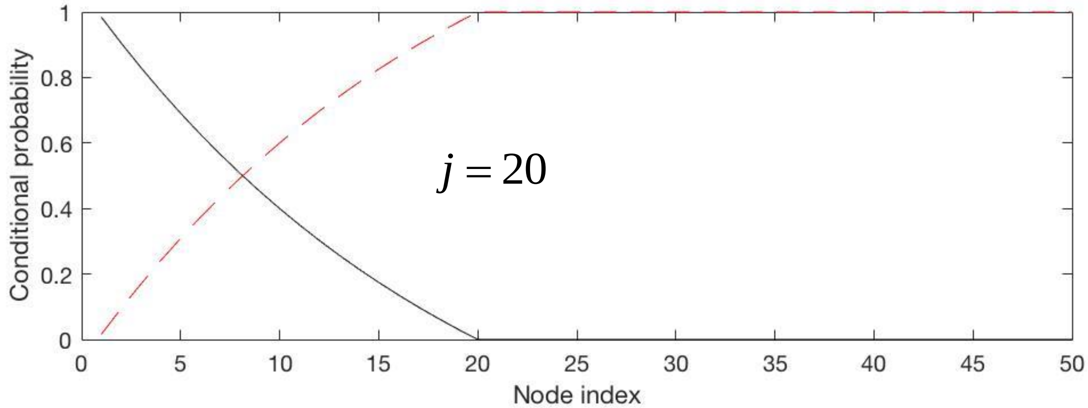
$$p\left(x_i = k \mid x_j = l\right) = \sum_{q=1}^{2} p\left(x_i = k \mid x_{i+1} = q\right) p\left(x_{i+1} = q \mid x_j = l\right)$$

$$p\left(x_i = 1 \mid x_j = 1\right) = 1 \qquad i < j,$$

$$p\left(x_i = 2 \mid x_j = 2\right) = \frac{p\left(x_i = 2\right)}{p\left(x_j = 2\right)} \qquad i < j$$

Results – conditional probabilities





Learning risk of avalanche

- Plan to install a sensor at one location, getting perfect information at that location.

$$j \in \{1, \ldots, 50\}$$

- Compute the posterior value, with sensor location at one location.
  Compute the VOI.
- What is the optimal sensor location, if the goal is to improve risk decisions?

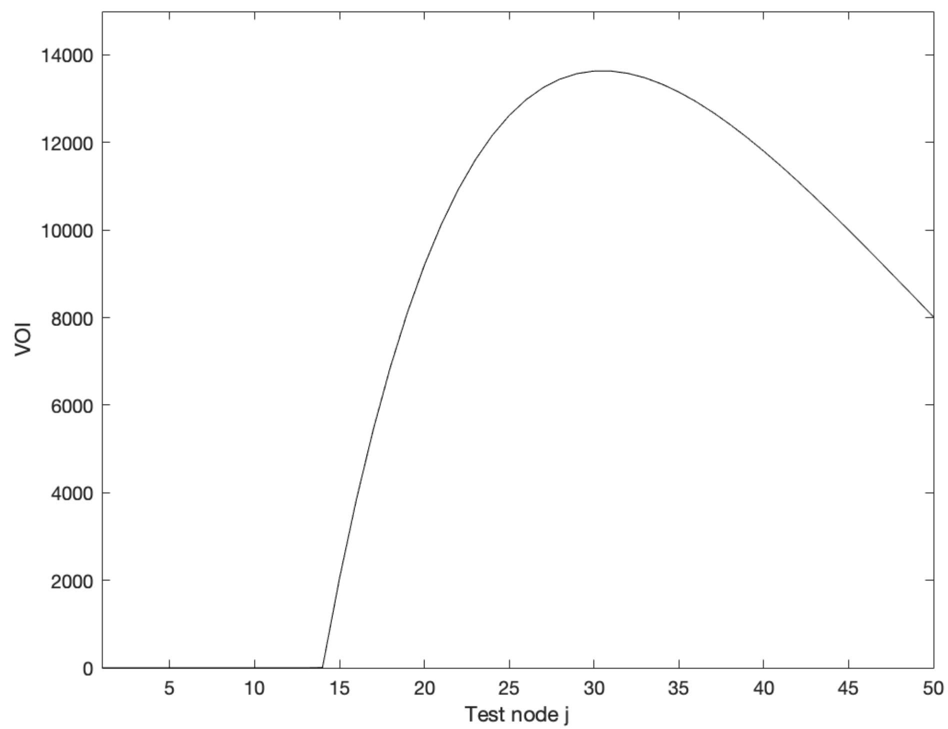
Results – VOI

$$PV = \max\left\{-100000, -5000\sum_{i=1}^{50} p\left(x_i = 2\right)\right\}$$

$$PoV\left(x_j\right) = \sum_{k=1}^{2} \max\left\{-100000, -5000\sum_{i=1}^{50} p\left(x_i = 2 \mid x_j = k\right)\right\} p\left(x_j = k\right)$$

$$VOI\left(x_j\right) = PoV\left(x_j\right) - PV$$

Best location near *j=30*.
The VOI is about 13000



Project : HMM (imperfect data)

$$PV = \max\left\{-100000, -5000\sum_{i=1}^{50} p\left(x_i = 2\right)\right\}$$

$$PoV\left(\boldsymbol{y}\right) = \int \max\left\{-100000, -5000\sum_{i=1}^{50} p\left(x_i = 2 \mid \boldsymbol{y}\right)\right\} p\left(\boldsymbol{y}\right) d\boldsymbol{y}$$

$$VOI\left(\boldsymbol{y}\right) = PoV\left(\boldsymbol{y}\right) - PV$$