Exercise : Ensemble-based inversion

The problem described here is motivated by tomographic imaging using for instance seismic wave data, ultrasound technology or sonar measurements.

We consider a layered earth model with $n = 100$ subsurface layers of 1m thickness from 0m to 100m depth. Data are acquired by emitting sound waves from a source located at the surface (0m) and at lateral location 40m. The measurements consist of travel times from this source to receiver sensors. There are 50 receivers located at the bottom of layers $51, \ldots, 100$, at lateral location 0m.

The case is illustrated in Figure 1, where the seismic ray-paths are sketched in black, assuming straight lines. (In practice these paths would bend according to Snell's law giving a non-linear problem - not considered here.)

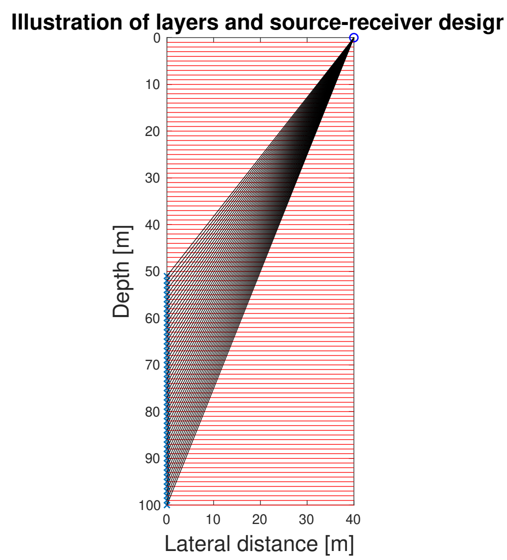

Figure 1: Layered earth model (red) is observed through seismic travel time data. The geometric acquisition design has a source (blue circle) at the surface and receivers in a borehole (blue crosses). The measurements are travel times [msec] from the source to receivers assuming straight ray-paths (black).

Figure 2 shows the travel time data of the seismic wave going from the source to each of the 50 receivers in the borehole. The goal is to learn the

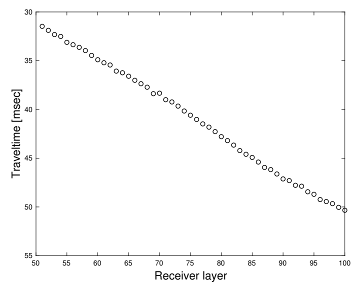

Figure 2: Traveltime data from source to receivers.

subsurface properties from these travel time data, and we use a statistical approach for this inverse problem.

The seismic wave slowness (the inverse of velocity) in layer i is denoted by x_i, and the length n vector of slownesses is $\boldsymbol{x} = (x_1, \ldots, x_n)$. The prior model for slownesses is Gaussian with mean $\mu_i = E(x_i) = 0.5 - 0.001i$, $i = 1, \ldots, n$, standard deviation $\sigma = 0.05$ and correlation $\mathrm{Corr}(x_i, x_{i'}) = (1 + \eta h_{i,i'}) \exp(-\eta h_{i,i'})$, $\eta = 0.1$, $h = |i - i'|$. Denote the joint model by $\boldsymbol{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Assuming straight ray-paths, the travel time data t_j, $j = 1, \ldots, 50$, are described by the following forward model

$$t_j = \frac{\sum_{i=1}^{j} x_i}{\cos(\theta_j)} + \epsilon_j, \;\; \theta_j = \arctan(40/d_j). \tag{1}$$

Here, d_j is the depth of receiver j ($d_j = 51, \ldots, 100$m), and the measurement noise terms $\epsilon_j \sim N(0, \tau^2)$ are assumed independent and $\tau = 0.1$.

(a)

We formulate an ensemble-based solution to the inverse problem by assimilating data t_j, $j = 1, \ldots, 50$, sequentially.

Generate $B = 200$ independent realizations from the prior model for slowness, denoted $\boldsymbol{x}^b$, $b = 1, \ldots, B$. Use for instance Cholesky factorization $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$ and then add the mean $\boldsymbol{\mu}$: $\boldsymbol{x}^b = \boldsymbol{\mu} + \boldsymbol{L}\boldsymbol{z}^b$, $\boldsymbol{z}^b \sim N(0, \boldsymbol{I})$.

Go recursively through all data in a loop from $j = 1$ to $j = 50$ (try also $j = 50$ to 1). At each step, use the following algorithm:

- Forecast ensemble members of travel time data t_j using eq (1), including the noise term. Each forecast data corresponds to one of the slowness ensemble members, so pairs are $(\boldsymbol{x}^b, t_j^b)$, $b = 1, \dots, B$.
- Form the empirical covariances $\boldsymbol{\Sigma}_{x,t_j}$ (size $n \times 1$) and $\sigma_{t_j}^2$ (size 1×1), and the Kalman gain $\boldsymbol{K} = \boldsymbol{\Sigma}_{x,t_j}\sigma_{t_j}^{-2}$.
- Assimilate each ensemble member for slowness using

$$\boldsymbol{x}^b = \boldsymbol{x}^b + \boldsymbol{K}(t_j - t_j^b).$$

Visualize the ensemble representation at an intermediate step ($j = 25$) and at the final data assimilation step. This can be done by plotting all ensembles or via the empirical 10 and 90 percentiles of slowness in each layer, obtained by sorting the ensemble values.

(b)

Repeat the procedure above, now with a different ensemble size. First, try reducing the ensemble size to $B = 50$. Then try increasing the ensemble size to $B = 400$. Run the algorithm a few times for the different ensemble sizes and comment on the results.

Vary the assimilation order in both: $j = 1$ to $j = 50$ or $j = 50$ to $j = 1$.

(c)

With the linear forward model, Gaussian prior and measurement noise assumptions, the posterior solution for slowness is Gaussian. This can hence be computed directly, or via a recursive formulation as in a Kalman filter. Here, this exact solution is compared with the ensemble-based approach.

For a Kalman filter solution, starting with $E(\boldsymbol{x}) = \boldsymbol{\mu} = \boldsymbol{\mu}_0$ and $\text{Var}(\boldsymbol{x}) = \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$, update the conditional mean $\boldsymbol{\mu}_j = E(\boldsymbol{x}|t_1, \dots, t_j)$ and covariance $\boldsymbol{\Sigma}_j = \text{Var}(\boldsymbol{x}|t_1, \dots, t_j)$ as follows:

- Pick j first indices for sum in eq (1) $\boldsymbol{g}_j = (1, \dots, 1, 0, \dots, 0)/\cos(\theta_j)$ (size $1 \times n$ vector)
- Kalman gain $\boldsymbol{K} = \boldsymbol{\Sigma}_{j-1}\boldsymbol{g}_j^T/(\boldsymbol{g}_j\boldsymbol{\Sigma}_{j-1}\boldsymbol{g}_j^T + \tau^2)$ (size $n \times 1$ vector)
- Update mean $\boldsymbol{\mu}_j = \boldsymbol{\mu}_{j-1} + \boldsymbol{K}(t_j - \boldsymbol{g}_j\boldsymbol{\mu}_{j-1})$
- Update covariance matrix $\boldsymbol{\Sigma}_j = \boldsymbol{\Sigma}_{j-1} - \boldsymbol{K}\boldsymbol{g}_j\boldsymbol{\Sigma}_{j-1}$

Visualize the solution by plotting the mean and the marginal 80 percent uncertainty intervals, and compare with the ensemble-based solutions in (a) and (b).