Exercise :
Estimation and prediction in Gaussian random fields

Here we will study **Gaussian processes or random fields**. We simulate spatial data, perform parameter estimation and prediction (Kriging).

Consider a Matern covariance model for the spatial distinction of interest $x(\mathbf{s})$:

$$Cov\left(x(\mathbf{s}_i), x(\mathbf{s}_j)\right) = \Sigma_{ij}\left(|\mathbf{t}_{ij}|\right) = \sigma^2\left(1+\eta|\mathbf{t}_{ij}|\right)\exp\left(-\eta|\mathbf{t}_{ij}|\right),$$

and data $y(\mathbf{s}_j) = x(\mathbf{s}_j) + N(0,\tau^2), \quad j = 1,\ldots,m.$

Here $|\mathbf{t}_{ij}|$ is the Euclidean distance between the two sites $\mathbf{s}_i$ and $\mathbf{s}_j$, and the measurements $y_j = y(\mathbf{s}_j)$ are assumed to be conditionally independent.

The unit square is the domain of interest. We assume a mean increasing with east and north coordinates as follows: $\mu_i = \alpha\left((s_{i1}-0.5)+(s_{i2}-0.5)\right)$, for site $\mathbf{s}_i = (s_{i1}, s_{i2})$ on the unit square.

1. Simulate $m = 200$ random data sites within the unit square. This is done by simulating uniform numbers along each axis. Plot the data sites. Set parameter values $\sigma^2 = 1$, $\eta = 10$ and $\tau^2 = 0.05^2$. Form the spatial covariance entries from the Matern covariance function to build a $m \times m$ covariance matrix for the data y_j, $j = 1,\ldots,m$. Take its Cholesky factorization and simulate dependent zero-mean Gaussian data variables, then add the mean using $\alpha = 1$.
2. Use the data to estimate the model parameters $\alpha, \sigma^2, \tau^2, \eta$. Do so by maximum likelihood estimation, iterating between an update for the mean parameter, and updating the covariance parameters. Monitor the likelihood function at each step of the algorithm to check convergence.
3. Use the Gaussian model with the estimated parameters to perform Kriging, i.e. predict variables $x(\mathbf{s})$, where sites are on a regular grid of size 25×25 for the unit square. Visualize the Kriging surface and the prediction standard error.

(Part of Norwegian wood exercise in Chapter 7.1, Eidsvik, Mukerji and Bhattacharjya, 2015, Value of Information in the Earth Sciences, Cambridge Univ Press.)