# Exercise B: Hidden Markov models

We consider a first order Markov chain model where the variables are binary, $x_i \in \{0,1\}$, $i=1,\ldots,n$, with stationary conditional probabilities $p(x_{i+1}=l \mid x_i=k)=P(k,l)$. These one-step probabilities can be organized as a Markov transition matrix

$$P=\begin{pmatrix} P(0,0) & P(0,1) \\ P(1,0) & P(1,1) \end{pmatrix}, \qquad \sum_{l=0}^{1} P(0,l)=1, \sum_{l=0}^{1} P(1,l)=1.$$

The marginal pdf for $x_1$ must also be specified to complete the model formulation.

Data $\boldsymbol{y}=(y_1,\ldots,y_n)$ are conditionally independent, given the variable of interest, and the Gaussian likelihood is defined by $p(y_i \mid x_i)=N(x_i,\tau^2)$, $i=1,\ldots,n$.

a. Simulate a Markov chain of length $n=250$ with equal dependence: $P(0,0)=P(1,1)=p$, $p(x_1=1)=0.5$. We set $p=0.9$. For each node or time step, generate conditionally independent data $y_i=x_i+N(0,\tau^2), i=1,\ldots,n$. We set $\tau^2=0.4^2$. Plot the data $\boldsymbol{y}=(y_1,\ldots,y_n)$ and interpret.
b. Describe the forward recursion for evaluating the marginal likelihood $p(\boldsymbol{y})$. (This is based on marginalizing over one of the $x_i$ variables at a time to get $p(y_i \mid y_{i-1},\ldots,y_1)$.) Compute the marginal likelihood model for a grid of values for $p$ and $\tau^2$, given the data from a., and find the maximum likelihood estimate. Compare with the values used in the simulation.
c. Use the forward-backward recursions to find the marginal probabilities $p(x_i=1 \mid \boldsymbol{y})$ for all $i=1,\ldots,n$. Use similarly the forward-backward recursions to sample realizations $\boldsymbol{x}^b$ from $p(\boldsymbol{x} \mid \boldsymbol{y})$, $b=1,\ldots,100$.
d. Compare prediction results of states $x_i$, $i=1,\ldots,n$, assuming that there is no dependence in the Markov chain model. In this simpler model, the prediction is solely based on the data and the likelihood model at individual locations.

*(Part of Exercise 12, Chapter 7.1, Eidsvik, Mukerji and Bhattacharjya, 2015, Value of Information in the Earth Sciences, Cambridge Univ Press.)*