EXERCISE A: INVERSE PROBLEM OF DIFFUSION

We consider an inverse problem with a basis in the following differential equation

$$\frac{du(x,t)}{dt} = \frac{d^2u(x,t)}{dx^2}, \quad u(x,0) = h_0(x), \quad x \in (0,1), t \geq 0. \tag{1}$$

Data is $u(x,t) = h_t(x)$ for a given time $t > 0$. The aim of the inverse problem is $h_0(x)$.

The forward model can be written as

$$u(x,t) = h_t(x) = \frac{1}{\sqrt{4\pi t}} \int e^{-(x-y)^2/(4t)} h_0(y) dy, \quad t > 0. \tag{2}$$

Using discretization we get

$$\boldsymbol{h}_t = \begin{bmatrix} h_t(x_1) \\ h_t(x_2) \\ \vdots \\ h_t(x_N) \end{bmatrix} = A \begin{bmatrix} h_0(x_1) \\ h_0(x_2) \\ \vdots \\ h_0(x_N) \end{bmatrix} = A\boldsymbol{h}_0, \tag{3}$$

where a regular grid of $N = 100$ points is used, such that $x_1 = 0$, $x_2 = 0.01$, ..., $x_N = 0.99$. The interval $(0,1)$ is made into a circle, i.e. 1 corresponds to 0. The matrix A has elements

$$A(i,j) = \frac{0.01}{\sqrt{4\pi t}} e^{-|x_i - x_j|^2/(4t)}, \tag{4}$$

The distance $|x_i - x_j|$ is modular on the circle $(0,1)$.

Measurements $\boldsymbol{y} = (y_1, \ldots, y_N)'$are acquired at time $t = 0.001$ (1ms):

$$y_i = h_t(x_i) + \epsilon_i, \quad \epsilon_i \sim N(0, 0.25^2), \quad \text{iid.} \tag{5}$$

These observations are shown in Figure 1. Data can be downloaded at www.math.ntnu.no/∼joeid/emnemodul (OppgA.txt). On a matrix/vector form the equation is written $\boldsymbol{y} = A\boldsymbol{h}_0 + \boldsymbol{\epsilon}$.

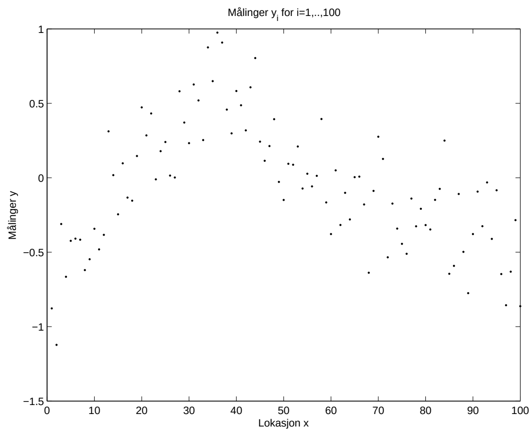

Figure 1: Observations $(y_1, \ldots, y_{100})'$ that are informative of the latent process $h_t(x)$ at time $t = 1$ms.

a)

Try to solve the inverse problem directly by $A^{-1}\boldsymbol{y}$. Compute the eigenvalues of matrix A, and display these in a plot. Show how a singular value decomposition of the matrix A is done, and how this leads to an approximate solution for $\boldsymbol{h}_0$ based on truncating the parts corresponding to small eigenvalues.

b)

Assume that we add prior information to $\boldsymbol{h}_0$ in the form of a Gaussian prior $\boldsymbol{h}_0 \sim N(0, I)$. Compute a posteriori expectation $\mathrm{E}(\boldsymbol{h}_0|\boldsymbol{y})$ and variance $\mathrm{Var}(\boldsymbol{h}_0|\boldsymbol{y})$. Display the Bayesian solution in a graph with uncertainty bounds.

c)

We will next use another prior; $\boldsymbol{h}_0 \sim N(0, \Sigma)$, where covariance matrix $\Sigma(i,j) = \exp(-|x_i - x_j|/0.1)$ and $|x_i - x_j|$ is defined as above. Compute a posteriori expectation $\mathrm{E}(\boldsymbol{h}_0|\boldsymbol{y})$ and variance $\mathrm{Var}(\boldsymbol{h}_0|\boldsymbol{y})$ for this model. Display the Bayesian solution in a graph with uncertainty bounds, and compare it with results in a) and b).