EXERCISE : Value of information calculations for a Gaussian example

Part I:

Assume the profit of a project has **a univariate Gaussian** pdf,

$$p(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ -\infty < x < \infty .$$

A decision maker will invest in a project which has positive expected value.

a. Use a transformation of variables: $z = g(x) = \frac{x-\mu}{\sigma}$, $x = g^{-1}(z) = \mu + \sigma z$, and the transformation formula $p(z) = \left|\frac{dg^{-1}}{dz}\right| p\left(g^{-1}(z)\right)$ to show that the variable z is standard normal distributed.

b. Use integration by parts; i.e. $\left(\int u \cdot v' = u \cdot v | - \int u' \cdot v\right)$ to show that $\int_a^b z\phi(z)dz = \int_a^b 1 \cdot z\phi(z)dz = \phi(a) - \phi(b)$, where the standard normal pdf is $\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)$.

c. Suppose a decision maker is considering getting perfect information about the profit. The posterior value of perfect information is $PoV(x) = \int \max\{0, x\}\, p(x)dx = \int_0^\infty xp(x)dx$. Use the results from a. and b. along with symmetry properties of the standard normal pdf $\phi(z)$ and its cumulative distribution $\Phi(z)$ to show that $PoV(x) = \mu\Phi\left(\mu/\sigma\right) + \sigma\phi\left(\mu/\sigma\right)$.

d. Plot the analytical VOI as i) a function of the mean in the range $\mu \in (-2, 2)$ and ii) as a function of standard deviation $\sigma \in (0.1, 2)$. Interpret results.

Part II:

Consider **the bivariate Gaussian pdf** $p(\boldsymbol{x}) = N(\boldsymbol{0}, \boldsymbol{\Sigma})$ for profits $\boldsymbol{x} = (x_1, x_2)$ at two projects. Assume variance 1 for both projects, and correlation $-1 < \rho < 1$.

The decision maker selects a project (alternative $a_i = 1$, $i = 1,2$) if its expected profit is positive, otherwise avoids investment (alternative $a_i = 0$, $i = 1,2$). We consider a decision situation with a free selection of projects, without constraints. Assume the decision maker can purchase total imperfect information, $y_j = x_j + N(0, \tau^2)$, $j = 1,2$, i.e. conditional model for the data is then $p(\boldsymbol{y} \mid \boldsymbol{x}) = N(\boldsymbol{x}, \boldsymbol{T})$, $\boldsymbol{T} = \tau^2 \boldsymbol{I}$. It can be shown that the posterior model for project profits, conditional on observations $\boldsymbol{y}$, is $p(\boldsymbol{x} \mid \boldsymbol{y}) \propto p(\boldsymbol{x}) p(\boldsymbol{y} \mid \boldsymbol{x}) = N(\boldsymbol{\mu}_{x|y}, \boldsymbol{\Sigma}_{x|y})$, where $\boldsymbol{\mu}_{x|y} = \boldsymbol{\Sigma}(\boldsymbol{\Sigma}+\boldsymbol{T})^{-1} \boldsymbol{y}$ and $\boldsymbol{\Sigma}_{x|y} = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}(\boldsymbol{\Sigma}+\boldsymbol{T})^{-1} \boldsymbol{\Sigma}$.

a. Use the formula for linear combinations of Gaussian variables to show that $p(\boldsymbol{y}) = N(\boldsymbol{0}, \boldsymbol{\Sigma}+\boldsymbol{T})$ and $p(\boldsymbol{\mu}_{x|y}) = N\left(\boldsymbol{0}, \boldsymbol{\Sigma}(\boldsymbol{\Sigma}+\boldsymbol{T})^{-1} \boldsymbol{\Sigma}\right)$.

b. Use the result from a. and from Part I to compute the posterior value of total imperfect information: $PoV(\boldsymbol{y}) = \sum_{j=1}^{2} \int \max\{0, \mu_{x|y,j}\} \, p(\boldsymbol{y}) d\boldsymbol{y}$. (Note: the only part of the data which is informative of the decision is $\boldsymbol{\mu}_{x|y}$, so the expression simplifies to: $PoV(\boldsymbol{y}) = \sum_{j=1}^{2} \int \max\{0, \mu_{x|y,j}\} p(\mu_{x|y,j}) \, d\mu_{x|y,j}$.)

c. Cross-plot the VOI results as a function of the correlation ρ and for standard deviation $\tau = 0.5$ and $\tau = 1$ of the imperfect test. Interpret results.

d. Assume next that the decision maker can gather information about only one of the projects (partial testing), say $y_1 = x_1 + N(0, \tau^2) = \boldsymbol{F}\boldsymbol{x} + N(0, \tau^2)$, where $\boldsymbol{F} = (1, 0)$. Calculate the posterior value of this partial test. Note that the conditional mean now becomes $\boldsymbol{\mu}_{x|y} = \boldsymbol{\Sigma}\boldsymbol{F}^t \left(\boldsymbol{F}\boldsymbol{\Sigma}\boldsymbol{F}^t + \tau^2\right)^{-1} y_1$.
   Plot results as in c. and interpret and compare results.