Problem 2.4.8

Gompertz’ equation for tumor growth reads

\[ \dot{N} = -aN \ln(bN) , \]  
(1)

where \( a, b > 0 \) are parameters. The fixed points \( N^* \) are given by

\[ f(N) = -aN \ln(bN) \]  
\[ = 0 . \]  
(2)

This yields \( N^* = 0 \) and \( N^* = 1/b \). The stability of the fixed point is given by the sign of \( f'(N) = -a \ln(bN) - a \). This yields

\[ f'(0) = \infty , \]  
(3)

\[ f'(1/b) = -a . \]  
(4)

Thus the origin is unstable and \( N^* = 1/b \) is stable.

Comments: The exact solution is

\[ N(t) = \frac{1}{b} e^{\ln(N_0 b)e^{-at}} . \]  
(5)

The solution satisfies \( N(0) = N_0 \) and

\[ \lim_{t \to \infty} = \frac{1}{b} . \]  
(6)

Fig. 1 shows the data points for tumor growth in a laboratory experiment at NTNU. The parameters \( a \) and \( b \) have been fitted to the data points. The agreement is very good.
Problem 2.5.1

a) The dynamics is governed by
\[ \dot{x} = -x^c. \]  

The origin is a fixed point only for \( c > 0 \). The stability is given by
\[ f'(x) = -cx^{c-1}. \]  

This implies that \( f'(0) = -\infty \) for \( 0 < c < 1 \). The flow is always towards the origin since \( f'(x) < 0 \) for \( x > 0 \) and so \( x = 0 \) is stable. For \( c = 1 \), \( f'(0) = -1 \) and for \( c > 1 \), \( f'(0) = 0 \). In the latter case \( f'(x) < 0 \) for \( x > 0 \) and the flow is towards the origin. Thus the origin is stable for all \( c > 0 \).
b) We can solve the differential equation exactly by separation of variables. This yields

\[ \int \frac{dx}{xe^c} = - \int dt . \]  

(9)

Integration yields

\[ \frac{x^{1-c}}{1-c} = -t + K , \quad c \neq 1 , \]  

(10)

where \( K \) is an integration constant. Using the initial condition \( x(0) = x_0 \), we can determine \( K \) and find

\[ x(t) = \left[ (c-1)t + x_0^{1-c} \right]^{\frac{1}{1-c}} . \]  

(11)

We must distinguish between two cases:

i) \( c > 1 \):
In this case the exponent \( 1/(1-c) < 0 \) and this tells us that it takes infinitely long to reach the origin.

ii) \( 0 < c < 1 \):
In this case the exponent \( 1/(1-c) > 0 \) and this tells us that it takes us a finite amount of time \( t^* \) to reach the origin. The equation for \( t^* \) is \( x(t^*) = 0 \) or

\[ (1-c)t^* = x_0^{1-c} . \]  

(12)

This yields

\[ t^* = \frac{x_0^{1-c}}{1-c} . \]  

(13)

For \( x_0 = 1 \), we find

\[ t^* = \frac{1}{1-c} . \]  

(14)

Finally, for \( c = 1 \), the solution is

\[ x(t) = x_0 e^{-t} , \]  

(15)

and so it takes infinitely long time to reach the origin.
Figure 2: The function \( g(x) \) for \( r = 1/4, r = 0, \) and \( r = -1/4. \) The number of fixed points depends on the parameter \( r. \) \( r_c = 0 \) is a bifurcation point.

**Problem 3.1.3**

The equation is

\[
\dot{x} = r + x - \ln(1 + x). \tag{16}
\]

In Fig. 2, we have plotted the function \( g(x) = r + x \) for three different values of \( r \) as well as the function \( h(x) = \ln(1 + x). \)

We note that \( g(x) \) crosses the \( y \)-axis at \( r \) and so there is one fix point for \( r = 0. \) For \( r > 0, \) there are no fixed points and for \( r < 0 \) there are two fixed points. Hence \( r = 0 \) is a bifurcation point. One of the fixed points \( x_1^* \) lies in the interval \((-1, 0]\) and the other \( x_2^* \) in the interval \([0, \infty). \) Since \( g(x) > h(x) \) for \( x < x_1^* \) and \( g(x) < h(x) \) for \( x < x_1^* \) and \( x_1^* < x < x_2^* , \) \( x_1^* \) is a stable fixed point. Since \( g(x) < h(x) \) for \( x_1^* < x < x_2^* \) and \( g(x) > h(x) \) for \( x > x_2^* , \) \( x_2^* \) is an unstable fixed point.

Finally, expanding the function around \( x = 0, \) we obtain

\[
\dot{x} \approx r + x - \left(x - \frac{1}{2} x^2\right) \\
= r + \frac{1}{2} x^2. \tag{17}
\]

After rescaling of \( x, \) this is the same function as in Example 3.1 in the textbook. Thus a saddle-point bifurcation takes place at \( r = 0. \)
The bifurcation diagram is shown in Fig. 3.

![Bifurcation diagram](image)

Figure 3: Bifurcation diagram.

**Problem 3.2.2**

In Fig. 4, we plot the function $g(x) = rx$ for three different values of $r$ as well as the function $h(x) = \ln(1 + x)$.

It is clear that $x = 0$ is a fixed point for all values of $r$. For $r < 1$ there is a second fixed point $x_2^* > 0$ and for $r > 1$ there is a second fixed point $x_1^* < 0$. Since $f'(x) = r - 1$, it follows that the origin is stable for $r < 1$ and unstable for $r > 1$. For $r = 1$, $g(x) > h(x)$ for all nonzero $x$ and so $x = 0$ is half stable. Moreover, for $r < 1$, the fixed point $x_2^*$ is unstable since $g(x) > h(x)$ for $x > x_2^*$ and $g(x) < h(x)$ for $0 < x < x_2^*$. Similar arguments show that $x_1^*$ is a stable fixed point for $r > 1$. Finally, expanding the function $f(x)$ around the origin yields

$$f(x) \approx rx - \left(x - \frac{1}{2}x^2\right)$$

$$= (r - 1)x + \frac{1}{2}x^2.$$  \hspace{1cm} (18)

After rescaling this is of the same form as Eq. (1) in Sec. 3.2 in the textbook and shows that $r = 1$ is a transcritical bifurcation. The bifurcation diagram is shown in Fig. 5.
Figure 4: The function $g(x)$ for $r = 0.7$, $r = 1$, and $r = 1.3$. Transcritical bifurcation for $r_c = 1$.

Figure 5: Bifurcation diagram.