TFY4205 Quantum Mechanics II
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SUGGESTED SOLUTION

Quantized radiation field

We follow here the derivation given on page 350 in the P. C. Hemmer book. By direct insertion of $\boldsymbol{A}$ we obtain

$$M = \langle f|\frac{e^2}{2m}\boldsymbol{A}^2|i\rangle = \frac{e^2}{2m}\langle f|\sum_{\boldsymbol{k}_1,\lambda_1}\boldsymbol{e}_{\boldsymbol{k}_1,\lambda_1}\sqrt{\frac{\hbar}{2V\varepsilon_0 ck_1}}\Big(a_{\boldsymbol{k}_1,\lambda_1}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}} + a^\dagger_{\boldsymbol{k}_1,\lambda_1}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}}\Big) \times \sum_{\boldsymbol{k}_2,\lambda_2}\boldsymbol{e}_{\boldsymbol{k}_2,\lambda_2}\sqrt{\frac{\hbar}{2V\varepsilon_0 ck_2}}\Big(a_{\boldsymbol{k}_2,\lambda_2}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}} + a^\dagger_{\boldsymbol{k}_2,\lambda_2}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}}\Big)|i\rangle. \tag{1}$$

With our $|i\rangle$ and $|f\rangle$, only two terms in the above double-summation will contribute: either $a_{\boldsymbol{k}_1,\lambda_1}$ has to annihilate the photon in the initial state while $a^\dagger_{\boldsymbol{k}_2,\lambda_2}$ creates the photon that is present in the final state (such that $\boldsymbol{k}_1 = \boldsymbol{k}$ and $\boldsymbol{k}_2 = \boldsymbol{k}'$), or vice versa. It can be verified that both terms contribute with equal magnitude. Therefore, using $a|n\rangle = \sqrt{n}\mathrm{e}^{-\mathrm{i}\omega t}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}\mathrm{e}^{\mathrm{i}\omega t}|n+1\rangle$, we obtain

$$M = 2\frac{e^2}{2m}\langle\psi_f|\boldsymbol{e}_{\boldsymbol{k},\lambda}\cdot\boldsymbol{e}_{\boldsymbol{k}',\lambda'}\frac{\hbar}{2V\varepsilon_0 c\sqrt{kk'}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega_k t}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}+\mathrm{i}\omega_{k'}t}\sqrt{n_{\boldsymbol{k},\lambda}(n_{\boldsymbol{k}',\lambda'}+1)}|\psi_i\rangle. \tag{2}$$

Introduce the quantity $r_0 = e^2/(4\pi\varepsilon_0 mc^2)$. This is the classical electron radius, defined as the distance where the Coulomb energy between two electrons is equal to the rest energy of one electron. When we use that

$$\langle\psi_f|\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}|\psi_i\rangle = \int_V \frac{d^3r}{V}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}_f)\cdot\boldsymbol{r}} = \delta_{\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{k}_f}, \tag{3}$$

since $\boldsymbol{k}_i = 0$, we finally obtain

$$|M|^2 = r_0^2(\boldsymbol{e}_{\boldsymbol{k},\lambda}\cdot\boldsymbol{e}_{\boldsymbol{k}',\lambda'})^2\frac{4\pi^2\hbar^2c^2}{V^2kk'}\delta_{(\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{k}_f)}n_{\boldsymbol{k},\lambda}(n_{\boldsymbol{k}',\lambda'}+1). \tag{4}$$