TFY4205 Quantum Mechanics II
Problemset 8 fall 2022

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SUGGESTED SOLUTION

Landau levels

From the outset, we know that the eigenvalues are in this new gauge must be the same as the eigenvalues we derived for the different gauge used in the lectures. The reason is the well-known fact that our gauge-choice cannot influence the physics.

From our previous treatment using a different gauge, we know that the energy eigenvalues take the same form as for a harmonic oscillator (+ the kinetic energy of a plane-wave in the z-direction). If we can show that the Hamiltonian in the gauge $\boldsymbol{A} = (B/2)(-y, x, 0)$ is equivalent to a harmonic oscillator, then we know what the eigenvalues will be (just like we did for the gauge-choice in the lectures). This will be our strategy.

First, let us write out the Hamiltonian explicitly in our gauge-choice:

$$H = (\boldsymbol{p} - q\boldsymbol{A})^2/2m = \frac{1}{2m}[(p_x - m\omega y/2)^2 + (p_y + m\omega x/2)^2] \tag{1}$$

where we used that the electron charge is $q = -e$ and also defined $\omega \equiv eB/m$. From now on, let $m = \omega = \hbar = 1$ for brevity of notation so we don't have to drag along all these coefficients in the calculations. We can simply reinstate them in the end. The solution procedure below will be clear even if we omit these constant prefactors. We thus have:

$$H = \frac{1}{2}(-\mathrm{i}\partial_x - y/2)^2 + \frac{1}{2}(-\mathrm{i}\partial_y + x/2)^2. \tag{2}$$

Now, let us define the complex variable $z = x - \mathrm{i}y$. This has the opposite sign-convention for the imaginary component compared to what we may be used to, but we are certainly free to define a variable z in this way if we like. We also define the variable $\bar{z} = z^* = x + \mathrm{i}y$. Observe that since

$$\partial f/\partial z = (\partial x/\partial z)(\partial f/\partial x) + (\partial y/\partial z)(\partial f/\partial y) \tag{3}$$

and $x = (z + \bar{z})/2$, it follows that

$$\partial_z = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y) \tag{4}$$

since $\partial x/\partial_z = 1/2$ and $\partial y/\partial z = \mathrm{i}/2$. Similarly, it follows that

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y). \tag{5}$$

It is useful at this point to establish that the following commutation rules are true:

$$[z, \partial_z] = [\bar{z}, \partial_{\bar{z}}] = -1, \tag{6}$$

and

$$[z, \partial_{\bar{z}}] = [\bar{z}, \partial_z] = 0 \tag{7}$$

Let us prove this for one of these relations explicitly. The rest follow in a similar manner. We have that

$$[z, \partial_z] f = z\partial_z f - \partial_z(zf) = -f \tag{8}$$

for some function f, which proves the first relation in Eq. (6).

The point is now that we can use our variables z and $\bar{z}$ to introduce a set of creation and annihilation operators $\{a, a^\dagger\}$ and these can, in turn, be used to express the Hamilton-operator precisely as a harmonic oscillator. Let us now show this. If we define the annihilation operator according to

$$a = \sqrt{2}(\partial_{\bar{z}} + z/4) \tag{9}$$

then it follows that we must have the creation operator

$$a^\dagger = \sqrt{2}(-\partial_z + \bar{z}/4). \tag{10}$$

To see this, first note that

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y). \tag{11}$$

We see that

$$[\frac{1}{2}(\partial_x - \mathrm{i}\partial_y)]^\dagger = \frac{1}{2}(-\partial_x - \mathrm{i}\partial_y) \tag{12}$$

where it was used that

$$\partial_j^\dagger = -\partial_j,\ j = x, y. \tag{13}$$

To verify that $\{a, a^\dagger\}$ in fact act as creation and annihilation operators, we must verify that $[a, a^\dagger] = 1$ and that $[a, a] = [a^\dagger, a^\dagger] = 0$. Let us prove the first relation explicitly. We have

$$[a, a^\dagger] = 2[\partial_{\bar{z}}, z/4] - 2[z/4, \partial_z] \tag{14}$$

where we used that $[z, \bar{z}] = 0$ (since they are scalars and clearly commute) and also that

$$[\partial_{\bar{z}}, \partial_z] = 0 \tag{15}$$

The last equation follows since the order of differentiation operators do not matter so long as the partial derivatives of the function they act on exist and are continuous. In effect, we have used that $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$. Going back to Eq. (14) now, we can use Eq. (6) and obtain the result $[a, a^\dagger] = 1$.

Having verified that $\{a, a^\dagger\}$ are annihilation and creation operators, just as in the harmonic oscillator problem, it remains to see if we can express our Hamilton-operator in terms of them. We see that

$$
\begin{aligned}
a^\dagger a &= 2(-\partial_z + \bar{z}/4)(\partial_{\bar{z}} + z/4) \\
&= -2\partial_z\partial_{\bar{z}} - \frac{1}{2}\partial_z z + \frac{1}{2}\bar{z}\partial_{\bar{z}} + \frac{1}{8}(x^2+y^2) \\
&= -\frac{1}{2}(\partial_x^2 + \partial_y^2) - \frac{1}{2}z\partial_z - \frac{1}{2} + \frac{1}{2}\bar{z}\partial_{\bar{z}} + \frac{1}{8}(x^2+y^2) \\
&= -\frac{1}{2}(\partial_x^2 + \partial_y^2) - \frac{1}{2}(\mathrm{i}x\partial_y - \mathrm{i}y\partial_x) - \frac{1}{2} + \frac{1}{8}(x^2+y^2). \qquad (16)
\end{aligned}
$$

Comparing this with Eq. (2), we see that the equations are identical except for a summative factor $\frac{1}{2}$. Therefore, we have shown that

$$
H = a^\dagger a + \frac{1}{2} \qquad (17)
$$

which proves that the Hamiltonian can be written in harmonic oscillator form with creation and annihilation operators. Reinstating the variables we set to 1, it follows that the eigenvalues are exactly the same as we obtained in the lectures.