TFY4205 Quantum Mechanics II
Problemset 6 fall 2022

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SUGGESTED SOLUTION

Problem 1
The electron density in the hydrogen atom is

$$n(r) = |\psi_{100}(r)|^2 = \frac{1}{\pi a_0^3} \mathrm{e}^{-2r/a_0}, \tag{1}$$

where a_0 is the Bohr radius (this can be looked up in a textbook). The atomic form factor then becomes

$$F(\boldsymbol{q}) = \frac{1}{\pi a_0^3} \int \mathrm{e}^{-2r/a_0} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} d\boldsymbol{r}. \tag{2}$$

We integrate the angle Θ between $\boldsymbol{q}$ and $\boldsymbol{r}$:

$$\int_0^{\pi} \mathrm{e}^{-\mathrm{i}qr\cos\Theta} 2\pi \sin\Theta d\Theta = 2\pi \left[\frac{\mathrm{e}^{-\mathrm{i}qr\cos\Theta}}{\mathrm{i}qr} \right]_0^{\pi} = 2\pi \frac{\mathrm{e}^{\mathrm{i}qr} - \mathrm{e}^{-\mathrm{i}qr}}{\mathrm{i}qr}. \tag{3}$$

Since the radial part of the volume element is $r^2 dr$:

$$F(q) = \frac{2\pi}{\pi a_0^3 \mathrm{i}q} \int_0^{\infty} \mathrm{e}^{-2r/a_0} (\mathrm{e}^{\mathrm{i}qr} - \mathrm{e}^{-\mathrm{i}qr}) r dr. \tag{4}$$

Evaluating this integral (using Rottman if needed) we obtain

$$F(q) = \frac{16}{(4 + a_0^2 q^2)^2}. \tag{5}$$

As usual, $q = 2k\sin(\vartheta/2)$. The amplitude for scattering on the hydrogen atom is then

$$f(\vartheta) = \frac{e^2}{4\pi\varepsilon_0} \frac{2m}{\hbar^2} \frac{Z - F}{q^2}. \tag{6}$$

Inserting the given expression for F into $f(\vartheta)$, and after some manipulation, we get

$$f(\vartheta) = a_0 \frac{16 + 2a_0^2 q^2}{(4 + a_0^2 q^2)^2}. \tag{7}$$

Now, the differential scattering cross section is given by $d\sigma/d\Omega = |f(\vartheta)|^2$ and so the total scattering cross section may be found by integration over $d\Omega = 2\pi \sin\vartheta d\vartheta$. To do so, we introduce a new integration variable

$$s = 1 + a_0^2 k^2 \sin^2(\vartheta/2),\ ds = \frac{1}{2} a_0^2 k^2 \sin\vartheta d\vartheta. \tag{8}$$

Since the limits for ϑ are 0 and π, the limits for s are 1 and $1+a_0^2k^2$:

$$\sigma = \int_0^\pi \frac{d\sigma}{d\Omega} 2\pi \sin\vartheta d\vartheta = \frac{\pi}{k^2} \int_1^{1+a_0^2k^2} \frac{(s+1)^2}{s^4} ds. \tag{9}$$

Assuming that the scattered electrons are fast, we may replace the upper integration limit with ∞ (which more precisely means $k \gg 1/a_0$, or the energy $\hbar^2k^2/2m \gg \hbar^2/2ma_0^2 =$ the ionization energy for hydrogen). Thus:

$$\sigma \simeq \frac{\pi}{k^2} \int_1^\infty (s^{-2} + 2s^{-3} + s^{-4}) ds = \frac{7\pi}{3k^2}. \tag{10}$$

as was to be shown. This high energy expression is valid when the energy of the incoming electron is much larger than 13.6 eV, but not so high that relativistic effects enter the problem.