TFY4205 Quantum Mechanics II
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SUGGESTED SOLUTION

Problem 1
In this case, the name of the phases give a clue about the solution right from the start: the dynamical phase does depends on the dynamics of the Hamiltonian, such as precisely the rate at which it changes between the two times $t=0$ and $t=t_1$. The geometrical phase (Berry phase), however, does not: it depends only on the path taken between the two times, not how fast the Hamiltonian changes on that path.

This is just a curious remark, of course, and so let us get to proving these statements rigorously in a mathematical way.

The eigenfunctions for $\hat{\tilde{H}}(t) = \hat{H}[g(t)]$ are seen to be $\psi_n[g(t)]$. The dynamical phase acquired at time $t=t_1$ when the system evolves according to $\hat{\tilde{H}}(t)$ is thus

$$\begin{aligned}\tilde{\theta}(t_1) &= -\frac{1}{\hbar}\int_0^{t_1} dt' E_i[g(t')] \\ &= -\frac{1}{\hbar}\int_{g(0)}^{g(t_1)} dg \frac{dt'}{dg} E_i(g) \\ &= -\frac{1}{\hbar}\int_0^{t_1} dg \frac{dt'}{dg} E_i(g). \end{aligned} \tag{1}$$

We see that the extra factor $\left(\frac{dg(t')}{dt'}\right)^{-1}$ inside the integral ensures that $\tilde{\theta}(t_1) \neq \theta(t_1)$ in general, so that the acquired dynamical phase does depend on the time rate of change of the Hamiltonian between the two points.

As for the geometrical (Berry) phase, we obtain in the same way

$$\begin{aligned}\tilde{\gamma}(t_1) &= \mathrm{i}\int_0^{t_1} dt' \langle \psi_i[g(t')] | \dot{\psi}_i[g(t')] \rangle \\ &= \mathrm{i}\int_{g(0)}^{g(t_1)} dg \frac{dt'}{dg} \langle \psi_i | \frac{d\psi}{dg}\frac{dg}{dt'} \rangle. \end{aligned} \tag{2}$$

We see that the two factors dt'/dg and dg/dt' cancel each other (the latter can be moved out of the expectation value since it only depends on time and not position). Using again that $g(t_1) = t_1$ and $g(0) = 0$ in the integral limits, we obtain

$$\tilde{\gamma}(t_1) = \mathrm{i}\int_0^{t_1} dg \langle \psi_i(g) | \psi_i'(g) \rangle. \tag{3}$$

where $'$ means d/dg. It is clear that this is equal to the original Berry phase just by changing the variable name $g \to t'$. Thus, we have

$$\gamma(t_1) = \tilde{\gamma}(t_1) \tag{4}$$

which shows that the Berry phase indeed does not depend on the rate of change of the Hamiltonian between the two points.

Problem 2

The eigenstates of the initial system ($t < 0$) are known and can be looked up in a textbook (see e.g. section 3.2 in the P. C. Hemmer book).

$$\psi_n^L(x) = \sqrt{2/L}\sin(n\pi x/L),\ n = 1, 2, \ldots \tag{5}$$

Since we know that the system was in the ground-state $n = 1$ at $t < 0$, it follows from the sudden approximation treatment that the probabilitity coefficient d_n that the system is in eigenstate n of the new well with width $2L$ at $t > 0$ is:

$$d_n = \langle \psi_n^{2L} | \psi_1^L \rangle \tag{6}$$

The probability to find the system in the nth state at $t > 0$ is then $P_n = |d_n|^2$ where:

$$d_n = \int_0^L \sqrt{2/(2L)}\sin[n\pi x/(2L)] \times \sqrt{2/L}\sin(\pi x/L)dx \tag{7}$$

The integral may be evaluated analytically.