TFY4205 Quantum Mechanics II
Problemset 2 fall 2022

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SUGGESTED SOLUTION

Problem 1

1. The average value can be expressed through the raising and lowering operators:
$$\langle q^5 \rangle = \langle n|\hat{q}^5|n\rangle \propto \langle n|(a+a^\dagger)^5|n\rangle. \tag{1}$$
It expands into a sum of terms with products of 5 operators. It is impossible to go 5 steps up and/or down in a ladder and end up exactly where one started: hence, the expectation value is zero since $(a+a^\dagger)^5|n\rangle$ is orthogonal to $|n\rangle$.

2. Using that $\hat{q} = q_0(a+a^\dagger)$ with $q_0 = \sqrt{\hbar/(2m\omega)}$ we obtain
$$\langle 0|\hat{q}^4|0\rangle = q_0^4\langle 0|(a+a^\dagger)^4|0\rangle. \tag{2}$$
Only terms carrying two lowering and two raising operators will contribute (ladder argument). Using that $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$, we obtain
$$\langle q^4 \rangle = 3q_0^4. \tag{3}$$

3. We have that
$$\langle 0|\hat{q}^4|0\rangle = \langle 0|\hat{q}^2\cdot\hat{q}^2|0\rangle \tag{4}$$
which is the norm squared of $\hat{q}^2|0\rangle$. We compute that
$$\hat{q}^2|0\rangle = q_0^2(a+a^\dagger)|1\rangle = q_0^2(|0\rangle+\sqrt{2}|2\rangle). \tag{5}$$
The norm of this state is $3q_0^4$, just as before.

4. We have
$$\langle k|\hat{q}|l\rangle = q_0\langle k|(\sqrt{l}|l-1\rangle + \sqrt{l+1}|l+1\rangle) = q_0(\sqrt{l}\delta_{k,l-1}+\sqrt{l+1}\delta_{k,l+1}). \tag{6}$$

5. Using the result from 4., we have
$$\langle\Psi|\hat{q}|\Psi\rangle = \sqrt{3}/4(\langle 0|\hat{q}|1\rangle + \langle 1|\hat{q}|0\rangle) = \sqrt{3}q_0/2. \tag{7}$$

6. The state is a superposition of two stationary states, so that the solution of the time dependent Schrodinger equation is a superposition of the same two states, but with each of them carrying a time-dependent factor $\mathrm{e}^{-\mathrm{i}E_nt/\hbar}$ where $E_n = (n+1/2)\hbar\omega$. Thus:
$$|\Psi(t)\rangle = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\omega t/2}|0\rangle + \frac{\sqrt{3}}{2}\mathrm{e}^{-3\mathrm{i}\omega t/2}|1\rangle. \tag{8}$$
The average position is then computed to be
$$\langle\Psi|\hat{q}|\Psi\rangle = \frac{\sqrt{3}}{2}q_0\cos(\omega t). \tag{9}$$

Problem 2

1. Let us set $\lambda = 1$ (the book-keeping parameter in perturbation theory), so that effectively F may be thought of as the small parameter. Using non-degenerate perturbation theory, we find to lowest order that

$$E_n^{(1)} = \langle n| - Fx|n\rangle = -F\int_{-\infty}^{\infty}[\psi_n^0(x)]^* x\psi_n^0(x)dx = 0 \tag{10}$$

since the integrand is antisymmetric. To second order we find that

$$E_n^{(2)} = \sum_{k\neq n}\frac{|\langle k| - Fx|n\rangle|^2}{E_n^0 - E_k^0}. \tag{11}$$

We showed in problem 1 that

$$\langle k|x|n\rangle = \sqrt{\hbar/(2m\omega)}(\sqrt{n}\delta_{k,n-1} + \sqrt{n+1}\delta_{k,n+1}). \tag{12}$$

Using this in Eq. (11), it follows that

$$E_n^{(2)} = -F^2\frac{1}{2m\omega^2}. \tag{13}$$

As for the first order correction to the state vector, we obtain

$$\begin{aligned} |\psi_n\rangle &\simeq |n\rangle + \sum_{k\neq n}\frac{\langle k| - Fx|n\rangle}{E_n^0 - E_k^0}|k\rangle \\ &= |n\rangle - F\sqrt{\frac{\hbar}{2m\omega}}\frac{1}{\hbar\omega}(a - a^\dagger)|n\rangle. \end{aligned} \tag{14}$$

2. Let us rewrite the Hamiltonian as follows:

$$\begin{aligned} H = H_0 - Fx &= \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2\Big(x^2 - \frac{2F}{m\omega^2}x + \frac{F^2}{m^2\omega^4}\Big) - \frac{F^2}{2m\omega^2} \\ &= \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2(x-x_0)^2 - \frac{F^2}{2m\omega^2}. \end{aligned} \tag{15}$$

The perturbed Hamiltonian thus describes a harmonic oscillator with equilibrium position at the point $x_0 \equiv F/m\omega^2$ and energy levels

$$E_n(F) = E_n^0 - \frac{F^2}{2m\omega^2}. \tag{16}$$

Problem 3

1. At $\lambda = 0$, the Hamiltonian is the sum of two one dimensional harmonic oscillators: one in the x-direction and one in the y-direction. The wavefunction is then a product of two wavefunctions: one for each Cartesian coordinate. Moreover, the energy eigenvalue is the sum of the energy eigenvalues for the two separate problems. In effect, we have:

$$E_{n_x,n_y} = (n_x + 1/2)\hbar\omega + (n_y + 1/2)\hbar\omega,\ n_x, n_y = 0, 1, 2, \dots \tag{17}$$

$$\psi_{n_x,n_y}(x,y) = \psi_{n_x}(x)\psi_{n_y}(y). \tag{18}$$

The three states with lowest energy are $|0,0\rangle$ with energy $\hbar\omega$, then $|0,1\rangle$ and $|1,0\rangle$ both with energy $2\hbar\omega$.

2. We are now taking into account the perturbation

$$\lambda H_1 = (\lambda/2)m\omega^2(x^2 + y^2)xy = \lambda k(x^3y + xy^3), \tag{19}$$

where we defined $k = m\omega^2/2$. Consider first the ground state. It is non-degenerate, and the first order perturbation theory gives the energy correction

$$\int\int \psi_{00}(x,y)\lambda H_1 \psi_{00}(x,y)dxdy. \tag{20}$$

Since H_1 contains both x and y to odd powers, the integral vanishes by symmetry arguments. In effect: $E_{00} = E_{00}^0 = \hbar\omega$.

Consider now the first excited state for the unperturbed system. The energy of this level is $2\hbar\omega$, and it is degenerate, with degree of degeneracy 2. Using the technique we have discussed in the lectures for calculating the first order correction $E^{(1)}$ to the energy level of a degenerate state, the relevant determinant to evaluate is

$$\begin{vmatrix} \langle 0,1|H_1|0,1\rangle - E^{(1)} & \langle 0,1|H_1|1,0\rangle \\ \langle 1,0|H_1|0,1\rangle & \langle 1,0|H_1|1,0\rangle - E^{(1)} \end{vmatrix} = 0 \tag{21}$$

We have to compute all the matrix elements in this determinant. This is most easily done using the energy eigenvector basis $|n\rangle$. Since $q = q_0(a + a^\dagger)$ (where q is x or y) with $q_0 = \sqrt{\hbar/(2m\omega)}$, we have that

$$\langle 0|q|1\rangle = \langle 1|q|0\rangle = q_0,$$

$$\langle 0|q^3|1\rangle = \langle 1|q^3|0\rangle = q_0^3\langle 1|(aa^\dagger + a^\dagger a)|1\rangle = 3q_0^3. \tag{22}$$

Substituting this into the determinant, we find the equation

$$[E^{(1)}]^2 - (6kq_0^4)^2 = 0. \tag{23}$$

Therefore, we have

$$E^{(1)} = \pm\frac{3\hbar^2}{4m}. \tag{24}$$

The degeneracy is lifted by the perturbation, so that the former degenerate first excited states now has energies $2\hbar\omega \pm \frac{3\hbar^2\lambda}{4m}$.