TFY4205 Quantum Mechanics II
Problemset 12 fall 2022

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SUGGESTED SOLUTION

Problem 1

1. They will get at least one white card $3/4$ of the time, so if they just always pick the same door, lets say the left one, they have a winning probability of 75%.

2. There are four possible combinations of cards:

   - Both cards white: They win if A_1 and B_1 give the same result. This means that $A_1B_1 = +1 \rightarrow$ win, and $A_1B_1 = -1 \rightarrow$ lose.

   $$\langle A_1B_1\rangle = P(\text{W})\cdot 1 + P(\text{L})\cdot(-1) = P(\text{W}) - P(\text{L})$$

   - The same logic holds for when each of them gets white card. The same measurement means a win, and different means they lose.

   $$\langle A_1B_2\rangle = P(\text{W}) - P(\text{L}) = \langle A_2B_1\rangle$$

   - Both cards black: They now win if A_2 and B_2 give different results. This means that $A_2B_2 = -1 \rightarrow$ win, and $A_2B_2 = +1 \rightarrow$ lose.

   $$\langle A_2B_2\rangle = P(\text{W})\cdot(-1) + P(\text{L})\cdot 1 = -(P(\text{W}) - P(\text{L}))$$

   Adding these four exepectation values together (subtracting the last), we get

   $$4(P(\text{W}) - P(\text{L})) = \langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle = \langle A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2\rangle$$

   We can now recognize the rhs as C in eq. (5.8) in the ”Additional notes relevant for curriculum” regarding quantum entanglement (http://www.damtp.cam.ac.uk/user/tong/aqm/topics5.pdf). The CHSH inequality then gives

   $$P(\text{W}) - P(\text{L}) \leq \frac{1}{4}2\sqrt{2}$$

   If we do measurements of the EPR-pair (which essentially are spins), in angles that differ by $45°$, we get an equality instead of an inequality. We also know that $P(\text{W}) + P(\text{L}) = 1$. Combining these, we get that

   $$P(\text{W}) = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + 1\right) \approx 0.854$$

   which beats the classical strategy.

Problem 2
Alices four options:

- Alice does nothing; I. The entangled pair remains $|\chi^-\rangle$
- Alice acts with σ_x. This changes the state to $-|\phi^-\rangle$, where

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle - |\downarrow\rangle|\downarrow\rangle)$$

- Alice acts with σ_y. This changes the state to $i|\phi^+\rangle$, where

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle)$$

- Alice acts with σ_z. This changes the state to $|\chi^+\rangle$, where

$$|\chi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)$$

This was computed by i.e using that

$$\sigma_y|\downarrow\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i|\uparrow\rangle$$

Now it is Bobs turn to do a measurement. The possible results he can measure are

$$\sigma_x \otimes \sigma_x |\phi^\pm\rangle = \pm|\phi^\pm\rangle \qquad \sigma_x \otimes \sigma_x |\chi^\pm\rangle = \pm|\chi^\pm\rangle$$

and

$$\sigma_z \otimes \sigma_z |\phi^\pm\rangle = +|\phi^\pm\rangle \qquad \sigma_z \otimes \sigma_z |\chi^\pm\rangle = -|\chi^\pm\rangle$$

- If Bob measures $\sigma_x \otimes \sigma_x = +1$ and $\sigma_z \otimes \sigma_z = +1$ then he knows he has the state $|\phi^+\rangle$
- If Bob measures $\sigma_x \otimes \sigma_x = +1$ and $\sigma_z \otimes \sigma_z = -1$ then he knows he has the state $|\chi^+\rangle$
- If Bob measures $\sigma_x \otimes \sigma_x = -1$ and $\sigma_z \otimes \sigma_z = +1$ then he knows he has the state $|\phi^-\rangle$
- If Bob measures $\sigma_x \otimes \sigma_x = -1$ and $\sigma_z \otimes \sigma_z = -1$ then he knows he has the state $|\chi^-\rangle$

From this, Bob now knows which measurement Alice did. One out of four possibilities means a transfer of two bits.