TFY4205 Quantum Mechanics II
Problemset 10 fall 2022

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SUGGESTED SOLUTION

Problem 1

Unpolarized light means that it is an equal mixure of the two polarization states $|1\rangle$ and $|2\rangle$, so that the density operator becomes

$$\hat{\rho} = \frac{1}{2}(|1\rangle\langle 1| + |2\rangle\langle 2|). \tag{1}$$

To check whether this is a pure or mixed state, we recall that Tr(ρ) = Tr(ρ^2) = 1 for a pure state. Since the above operator $\hat{\rho}$ satisfies

$$\hat{\rho}^2 = \hat{\rho}/2, \tag{2}$$

we see that Tr(ρ^2) = $\frac{1}{2}$ and hence it is a mixed state.

If the light-beam passes through a polarizer, we are left with only one of the states, say $|1\rangle$ for concreteness. Formally, this can be seen by applying the projection operator $\hat{\mathcal{P}} = |1\rangle\langle 1|$ on $\hat{\rho}$, which gives

$$\hat{\rho}_{\mathcal{P}} = \hat{\mathcal{P}} \cdot \hat{\rho} = \frac{|1\rangle\langle 1|}{2}. \tag{3}$$

After doing so, we also have to normalize this density operator since we must always have Tr(ρ)=1, which means removing the factor $\frac{1}{2}$ from $\hat{\rho}_{\mathcal{P}}$.

We are then left with the properly normalized

$$\hat{\rho}_{\text{polarized}} = |1\rangle\langle 1| \tag{4}$$

which does satisfy $\hat{\rho}_{\text{polarized}} = (\hat{\rho}_{\text{polarized}})^2$, so that this is a pure state.

Problem 2

[a.]Suppose $|S\rangle$ were separable. Then, there would exist single qubit states $|\phi\rangle$ and $|\psi\rangle$ such that $|S\rangle = |\phi\rangle \otimes |\psi\rangle$. Without loss of generality, let

$$|\phi\rangle = a|0\rangle + b|1\rangle \tag{5}$$
$$|\psi\rangle = c|0\rangle + d|1\rangle \tag{6}$$

for some undetermined coefficients. Then:

$$|\phi\rangle \otimes |\psi\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \tag{7}$$

But since this must equal $|S\rangle$, we must have $ac = 0$ and $bd = 0$ since $|S\rangle$ has no component of $|00\rangle$ or $|11\rangle$. Then, either $a = 0$ or $c = 0$. If $a = 0$, we have

$$|\phi\rangle \otimes |\psi\rangle = bc|10\rangle + bd|11\rangle \tag{8}$$

If $c = 0$, we have

$$|\phi\rangle \otimes |\psi\rangle = ad|01\rangle + bd|11\rangle \tag{9}$$

It is now clear by inspection that in either case, no matter the choice of the remaining coefficients, we cannot make this equal to $|S\rangle$. This proof is equivalent for $|T,0\rangle$. For any single qubit operator U, let

$$U|0\rangle = a|0\rangle + b|1\rangle \tag{10}$$

$$U|1\rangle = c|0\rangle + d|1\rangle \tag{11}$$

In matrix-form,

$$U \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}. \tag{12}$$

Since U is a unitary operator, U must have $|ad - bc| = 1$ since $|\det(U)|$ = 1 for a unitary matrix U. Now

$$(U \otimes U)|S\rangle = \frac{1}{\sqrt{2}}(U \otimes U)\big[|01\rangle - b|10\rangle\big] \tag{13}$$

$$= \frac{1}{\sqrt{2}}\big[(a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) - (c|0\rangle + d|1\rangle)(a|0\rangle + b|1\rangle)\big] \tag{14}$$

$$= \frac{1}{\sqrt{2}}\big[(ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle - ac|00\rangle - bc|01\rangle - ad|10\rangle - bd|11\rangle\big] \tag{15}$$

$$= \frac{1}{\sqrt{2}}\big[\big((ad - bc)|01\rangle - (ad - bc)\big)|10\rangle\big] \tag{16}$$

$$= \frac{ad - bc}{\sqrt{2}}\big[|01\rangle - |10\rangle\big] \tag{17}$$

We are almost there! Now, note that acting on a state with a unitary operator preserves the norm of the state since $U^{-1}U = U^{\dagger}U = 1$. For this to be true, $ad - bc$ must equal a U(1) number $\mathrm{e}^{\mathrm{i}\theta}$. This is consistent with the fact that for a unitary matrix U with elements as we defined earlier, a general property is that $|ad - bc|^2 = 1$. Thus, our final state is $e^{i\theta}|S\rangle$, which is just an overall phase times $|S\rangle$. This overall phase does not change the physical of the system and we have shown that the system is physically invariant upon acting on it with a unitary operator.