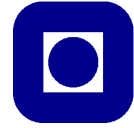


TFY4205 Quantum Mechanics II

NTNU

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Institutt for fysikk

SUGGESTED SOLUTION

Problem 1

Unpolarized light means that it is an equal mixture of the two polarization states $|1\rangle$ and $|2\rangle$, so that the density operator becomes

$$\hat{\rho} = \frac{1}{2}(|1\rangle\langle 1| + |2\rangle\langle 2|). \quad (1)$$

To check whether this is a pure or mixed state, we recall that $\text{Tr}(\rho) = \text{Tr}(\rho^2) = 1$ for a pure state. Since the above operator $\hat{\rho}$ satisfies

$$\hat{\rho}^2 = \hat{\rho}/2, \quad (2)$$

we see that $\text{Tr}(\rho^2) = \frac{1}{2}$ and hence it is a mixed state.

If the light-beam passes through a polarizer, we are left with only one of the states, say $|1\rangle$ for concreteness. Formally, this can be seen by applying the projection operator $\hat{P} = |1\rangle\langle 1|$ on $\hat{\rho}$, which gives

$$\hat{\rho}_P = \hat{P} \cdot \hat{\rho} = \frac{|1\rangle\langle 1|}{2}. \quad (3)$$

After doing so, we also have to normalize this density operator since we must always have $\text{Tr}(\rho)=1$, which means removing the factor $\frac{1}{2}$ from $\hat{\rho}_P$.

We are then left with the properly normalized

$$\hat{\rho}_{\text{polarized}} = |1\rangle\langle 1| \quad (4)$$

which does satisfy $\hat{\rho}_{\text{polarized}} = (\hat{\rho}_{\text{polarized}})^2$, so that this is a pure state.

Problem 2

[a.] Suppose $|S\rangle$ were separable. Then, there would exist single qubit states $|\phi\rangle$ and $|\psi\rangle$ such that $|S\rangle = |\phi\rangle \otimes |\psi\rangle$. Without loss of generality, let

$$|\phi\rangle = a|0\rangle + b|1\rangle \quad (5)$$

$$|\psi\rangle = c|0\rangle + d|1\rangle \quad (6)$$

for some undetermined coefficients. Then:

$$|\phi\rangle \otimes |\psi\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \quad (7)$$

But since this must equal $|S\rangle$, we must have $ac = 0$ and $bd = 0$ since $|S\rangle$ has no component of $|00\rangle$ or $|11\rangle$. Then, either $a = 0$ or $c = 0$. If $a = 0$, we have

$$|\phi\rangle \otimes |\psi\rangle = bc|10\rangle + bd|11\rangle \quad (8)$$

If $c = 0$, we have

$$|\phi\rangle \otimes |\psi\rangle = ad|01\rangle + bd|11\rangle \quad (9)$$

It is now clear by inspection that in either case, no matter the choice of the remaining coefficients, we cannot make this equal to $|S\rangle$. This proof is equivalent for $|T, 0\rangle$. For any single qubit operator U , let

$$U|0\rangle = a|0\rangle + b|1\rangle \quad (10)$$

$$U|1\rangle = c|0\rangle + d|1\rangle \quad (11)$$

In matrix-form,

$$U \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}. \quad (12)$$

Since U is a unitary operator, U must have $|ad - bc| = 1$ since $|\det(U)| = 1$ for a unitary matrix U . Now

$$(U \otimes U)|S\rangle = \frac{1}{\sqrt{2}}(U \otimes U)[|01\rangle - b|10\rangle] \quad (13)$$

$$= \frac{1}{\sqrt{2}}[(a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) - (c|0\rangle + d|1\rangle)(a|0\rangle + b|1\rangle)] \quad (14)$$

$$= \frac{1}{\sqrt{2}}[(ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle) - ac|00\rangle - bc|01\rangle - ad|10\rangle - bd|11\rangle] \quad (15)$$

$$= \frac{1}{\sqrt{2}}[((ad - bc)|01\rangle - (ad - bc)|10\rangle] \quad (16)$$

$$= \frac{ad - bc}{\sqrt{2}}[|01\rangle - |10\rangle] \quad (17)$$

We are almost there! Now, note that acting on a state with a unitary operator preserves the norm of the state since $U^{-1}U = U^\dagger U = 1$. For this to be true, $ad - bc$ must equal a $U(1)$ number $e^{i\theta}$. This is consistent with the fact that for a unitary matrix U with elements as we defined earlier, a general property is that $|ad - bc|^2 = 1$. Thus, our final state is $e^{i\theta}|S\rangle$, which is just an overall phase times $|S\rangle$. This overall phase does not change the physical of the system and we have shown that the system is physically invariant upon acting on it with a unitary operator.