TFY4205 Quantum Mechanics II

Problemset 1 fall 2022

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SUGGESTED SOLUTION

Problem 1

The stationary Schrodinger equation in one dimension,

$$-\frac{\hbar^2}{2m}\psi''(x)+V(x)\psi(x)=E\psi(x), \tag{1}$$

has a complete set of real solutions. In fact, if $\psi(x)$ is a complex solution then we may complex conjugate the equation to prove that $\psi^*(x)$ is a solution with the same energy E. Hence the real and imaginary parts of $\psi(x)$,

$$\mathrm{Re}\{\psi\}=\frac{1}{2}(\psi+\psi^*),\ \mathrm{Im}\{\psi\}=\frac{1}{2\mathrm{i}}(\psi-\psi^*), \tag{2}$$

are real solutions with the same energy E. Thus, we may assume without loss of generality that $\psi(x)$ is real for all x. The equation may then be written as

$$\frac{\psi''}{\psi}=\frac{2m}{\hbar^2}[V(x)-E]. \tag{3}$$

Let us now investigate the cases $(i)E<V_{\min}$ and $(ii)E>V_+$ separately.

1. For $E<V_{\min}$ the right hand side of Eq. (3) is positive for all x, so that

$$\psi''/\psi>0. \tag{4}$$

   The function ψ will then turn away from the x-axis in both limits $x\to\pm\infty$. A function $|\psi(x)|^2$ that increases in the limits $x\to\pm\infty$ cannot have a finite integral.

2. For $E>V_+$, the right hand side of Eq. (3) can become negative for large values of x. We label this coefficient $-k^2$. This means that for large values of $|x|$, the stationary Schrodinger equation will become similar to the equation

$$\psi''+k^2\psi=0 \tag{5}$$

   which is solved by $\psi(x)=A\sin(kx)+B\cos(kx)$. The wavefunction will then oscillate and not decay towards zero when $x\to\infty$. The norm of the wavefunction becomes infinite.

The conclusion is that we cannot find eigenfunctions with finite norm for energies in the two intervals we have investigated.

Problem 2

To get a better grasp on Hilbert space in QM, consider first ordinary vectors.

- They exist in a N-dimensional space (we are used to $N = 3$).
- We have a set of basis vectors (N of them for an N-dimensional space). There are many possible choices: Cartesian coordinates, spherical coordinates, and so forth.
- The basis vectors span the N-dimensional space and form a complete set. The latter statement means that any vector $\boldsymbol{v} = \sum_i v_i \boldsymbol{e}_i$ is a unique linear combination of basis vectors $\boldsymbol{e}_i$.
- Each scalar v_i is the projection of the arbitrary vector $\boldsymbol{v}$ on to basis vector i.

All of these statements have analogies in the general formulation of QM. Consider the state vector $|\Psi\rangle$ which describes the system:

- It lives in Hilbert-space which can be both finite or infinite dimensional.
- We have a set of basis vectors which span Hilbert-space. For an infinitely dimensional space, the basis may be countable (discretely labelled basis $|n\rangle$) or uncountable (continuously labelled basis vectors $|x\rangle$).
- The basis for a Hilbert space is complete: any state-vector can be expanded in them in a unique way, *e.g.* $|\Psi\rangle = \sum_i v_i |i\rangle$ or $|\Psi\rangle = \int dp c_p |p\rangle$.
- The coefficients in the expansion are the projections of the state-vector on the basis vectors.

For instance: if we use position eigenvectors $|x\rangle$ as basis vectors, then $\langle x|\Psi\rangle$ tells us what the state $|\Psi\rangle$ looks like as a function of x. In effect, $|\langle x|\Psi\rangle|^2$ is the probability of finding the system in position x.

We see that $|\Psi\rangle$ can be thought of as an analogy to a vector $\boldsymbol{v}$ whereas $\psi(x)$ are the specific components v_x, v_y, v_z in a particular basis. Thus, we don't need to use $|x\rangle$ as basis vectors at all, just as little as we must use Cartesian instead of spherical coordinates!

Thus, a Hilbert space is formally an abstract vector space where the inner product operation is defined (so that we can measure lengths of vectors in that space). Moreover, it is a space which is complete, which essentially means that we can use the methods of vector algebra and calculus in Hilbert space.

What we have stated above is what we need in practice for calculations. There are some distinctions between a discrete and continuous basis which are good to be aware of, and we have discussed these in the lectures.

Problem 3

Generally, the dual vector to $c|a\rangle$ is $c^*\langle a|$ and using that $\langle a|a\rangle = 1$ and $\langle d|c\rangle = \langle c|d\rangle^*$, we obtain

1. $\langle b| = (1 - \mathrm{i})\langle a|$.
2. $\langle a|b\rangle = 1 + \mathrm{i}$.
3. $\langle b|a\rangle = 1 - \mathrm{i}$.
4. $\langle b|b\rangle = 2$.

Problem 4

1. Both states that make up $|\nu\rangle$ have $l = 1$: thus measuring $\boldsymbol{L}^2$ we will with certainty obtain the value $l(l+1)\hbar^2 = 2\hbar^2$.

2. The eigenvalues of L_z are $m\hbar$ where m is an integer, so a measurement must in general give one of these values. For $|\nu\rangle$, we can obtain the values $m = 0$ or $m = 1$ corresponding to the first and second term. The probability to obtain each of these values is obtained from the square of the absolute value of the coefficients, namely $1/3$ and $2/3$.

3. Eigenvalues for S_z are $m_s\hbar$ where $m_s = \pm 1/2$. We can thus measure either $+\hbar/2$ or $-\hbar/2$ and, as above, the probabilities are $1/3$ and $2/3$.

4. The eigenvalues of $\boldsymbol{L}^2$ are $\hbar^2 j(j+1)$. Since the value of l is 1 and the spin value of the electron is $s = 1/2$, j can take the values $l + s = 3/2$ and $l - s = 1/2$. So when measuring $\boldsymbol{L}^2$, we can obtain either $3\hbar^2/4$ or $15\hbar^2/4$. Denote the probabilities for measuring these values $p_{1/2}$ and $p_{3/2} = 1 - p_{1/2}$, respectively. Now, we can make use of the fact that the problem text revealed that $\langle \boldsymbol{L}^2 \rangle = 41\hbar^2/12$, because this must be equal to

$$\langle \boldsymbol{L}^2 \rangle = p_{1/2} 3\hbar^2/4 + (1 - p_{1/2}) 15\hbar^2/4 = 41\hbar^2/12. \tag{6}$$

   This equation can now be solved for $p_{1/2}$ and gives the result $1/9$. We immediately conclude that $p_{3/2} = 8/9$.

5. We apply $J_z = S_z + L_z$ to the state vector, which we for brevity now denote as $|\nu\rangle = A|1,0\rangle|\uparrow\rangle + B|1,1\rangle|\downarrow\rangle$. We obtain:

$$J_z|\nu\rangle = \frac{1}{2}\hbar A|1,0\rangle|\uparrow\rangle - \frac{1}{2}\hbar B|1,1\rangle|\downarrow\rangle + \hbar B|1,1\rangle|\downarrow\rangle = \frac{1}{2}\hbar|\nu\rangle. \tag{7}$$

   Consequently, the given state is an eigenstate of J_z with eigenvalue $\hbar/2$ and measuring J_z will with certainty provide that value.