TFY4205 Quantum Mechanics II
Problemset mandatory exercise 2 fall 2022

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SUGGESTED SOLUTION

Problem 1

1. Since the given expression for δ_l does not depend on energy, the scattering amplitude

$$f(\vartheta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\vartheta) \tag{1}$$

will be proportional to k^{-1}, and the differential cross section will be inversely proportional to the energy:

$$\frac{d\sigma}{d\Omega} = |f^2| \propto 1/k^2 \propto 1/E. \tag{2}$$

2. For small g, we have

$$\delta_l = \frac{\pi}{2}\Big(l + \frac{1}{2} - \sqrt{(l+\frac{1}{2})^2 + g}\Big) \simeq -\frac{\pi g/2}{2l+1}. \tag{3}$$

The phase shifts are negative, as expected for a positive potential as discussed in the lectures. Inserting this phase-shift into the scattering amplitude and using that $|\delta_l| \ll 1$, we get:

$$f(\vartheta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\delta_l P_l(\cos\vartheta) = -\frac{\pi g}{2k}\sum_{l=0}^{\infty}P_l(\cos\vartheta). \tag{4}$$

With the help of the generating function for Legendre polynomials

$$\sum_{l=0}^{\infty}s^l P_l(\cos\vartheta) = (1 - 2s\cos\vartheta + s^2)^{-1/2}, \tag{5}$$

we get with $s = 1$ that

$$\sum_{l=0}^{\infty}P_l(\cos\vartheta) = \frac{1}{2\sin(\vartheta/2)}. \tag{6}$$

The scattering amplitude then becomes

$$f(\vartheta) = -\frac{\pi g}{4k\sin(\vartheta/2)} \tag{7}$$

and scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{\pi^2 g^2}{16k^2\sin^2(\vartheta/2)}. \tag{8}$$

3. The Born approximation for the scattering cross section is $d\sigma/d\Omega = |f|^2$ with

$$f(\vartheta) = -\frac{m}{2\pi\hbar^2}\int V(r)\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}d\boldsymbol{r} = -\frac{2m}{\hbar^2 q}\int_0^\infty V(r)\sin(qr)rdr \tag{9}$$

when integration over the angles is done. Here, $q = 2k\sin(\vartheta/2)$. For our potential we get

$$f(\vartheta) = -\frac{g}{q}\int_0^\infty \frac{\sin\xi}{\xi}d\xi = -\frac{g\pi}{2q}. \tag{10}$$

We have introduced $qr = \xi$ and used that the last integral is $\pi/2$. Inserting q, the scattering amplitude in the Born approximation becomes

$$f(\vartheta) = -\frac{\pi g}{4k\sin(\vartheta/2)}. \tag{11}$$

This is precisely the same expression as the one we got from the scattering phases.

4. The radial equation for this potential is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \Big[\frac{\hbar^2 l(l+1)}{2mr^2} + \frac{\hbar^2 g}{2mr^2}\Big]u = Eu. \tag{12}$$

This can be rewritten as

$$-\frac{d^2u}{dr^2} + \frac{\tilde{l}(\tilde{l}+1)}{r^2}u - k^2u = 0. \tag{13}$$

Here, $E = \hbar^2k^2/2m$ and

$$\tilde{l}(\tilde{l}+1) = l(l+1) + g. \tag{14}$$

The rewritten equation now looks like a radial equation without any potential, with the asymptotic solution

$$R_l(r) \propto \frac{\sin(kr - \tilde{l}\pi/2)}{r}. \tag{15}$$

The solution of the Schrodinger equation has the following form at large r:

$$R_l(r) \propto \frac{\sin(kr - l\pi/2 + \delta_l)}{r} \tag{16}$$

which is used to define the scattering phase shift δ_l. Comparing the last two equations gives

$$\delta_l = \pi(l - \tilde{l})/2. \tag{17}$$

The only remaining task is to solve the second order equation Eq. (14) with respect to $\tilde{l}$:

$$\tilde{l} = -\frac{1}{2} \pm \sqrt{(l+1/2)^2 + g}. \tag{18}$$

We must use the + sign in front of the square root to ensure that $\tilde{l} > 0$ (and when $g = 0$ we must have $\tilde{l} = l$). Inserted, we get

$$\delta_l = \frac{\pi}{2}\Big(l + \frac{1}{2} - \sqrt{(l+\frac{1}{2})^2 + g}\Big). \tag{19}$$

Problem 2
a) The wavefunction must be defined for $r = 0$, so only the sin term is allowed $(C = 0)$. Moreover, the wavefunction is continuous at $r = a$, whereas its derivative is not continuous due to the δ-function potential. Specifically, consider the radial equation for u (recall that $\psi = R(r) = u(r)/r$):

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \alpha\delta(x)u = Eu. \tag{20}$$

Integrating across an infinitesimal interval centered at $r = a$, i.e. from $r = a - \varepsilon$ to $r = a + \varepsilon$ and taking $\lim_{\varepsilon\to 0}$, we get

$$\lim_{\varepsilon\to 0}\left.\frac{du}{dr}\right|_{a-\varepsilon}^{a+\varepsilon} = \frac{2m\alpha}{\hbar^2}u(a). \tag{21}$$

Using now the continuity and derivative boundary condition to get rid of the remaining unknown constants A and B, we obtain the equation

$$c + ika_0y = (\beta s/a + kc)(a_0y/s + 1/k), \tag{22}$$

where we defined the quantities

$$\beta = 2m\alpha a/\hbar^2,\ s = \sin(ka),\ c = \cos(ka),\ y = e^{ika}. \tag{23}$$

Solving for a_0 and taking the limit $ka \ll 1$ gives

$$a_0 \simeq -\frac{\beta a}{1+\beta}. \tag{24}$$

b) The total scattering cross section is

$$\sigma = \int d\Omega |f|^2 = \int d\Omega |a_0|^2 = 4\pi\frac{\beta^2 a^2}{(1+\beta)^2}. \tag{25}$$

As $\alpha \to \infty$, we get $\beta \to \infty$, which in turn makes $\sigma \to 4\pi a^2$. This is the same result as the quantum mechanical total scattering cross section for low-energy scattering on a hard sphere potential of radius a.