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Suggested solution for 2016 Exam in Quantum Mechanics II

NOTE: The solutions below are meant as guidelines for how the problems may be solved and do not necessarily contain all the detailed steps of the calculations.

Problem 1

- Time-independent perturbation theory: The student is here expected to know that the idea behind this method is to expand the eigenvalues and eigenstates by using a smallness parameter which we can call λ, e.g. $E_n = E_n^0 + \lambda E_n^{(1)} + \dots$ and similarly for $|\psi\rangle$. Inserting this into the SE, one obtains that if the Hamiltonian is perturbed with $\lambda\hat{H}_1$, the first correction to the eigenvalue is

$$\lambda E_n^{(1)} = \langle n|\lambda\hat{H}_1|n\rangle. \tag{1}$$

  This method is expected to be suitable for problems where the perturbation is small in magnitude, i.e. yielding only a minor quantitative change in the eigenvalues and eigenfunctions.

- Sudden approximation: In this case, H changes abruptly at a given time $t = t_0$ and changes from $H = H_0$ to $H = H_1$. By expanding the general solution of the wavefunctions before and after $t = t_0$ in stationary states, one can express the probability coefficients d_n after the sudden change via the probability coefficients c_k before the sudden change and scalar products of the type $\langle\phi_n|\psi_k\rangle$ where ϕ_n and ψ_k are eigenstates of H_1 and H_0, respectively. In reality, the change from H_0 to H_1 takes place over a short, but finite time interval τ. For a full score, the candidate should also discuss a quantitative condition (or at least how to obtain such a condition) required for this method to be valid.

- The expectation value of a Hamiltonian $\hat{H}$ in any state f will always be greater or equal to E_0, where E_0 is the lowest eigenvalue. In effect,

$$E[f] = \frac{\langle f|\hat{H}|f\rangle}{\langle f|f\rangle} \geq E_0. \tag{2}$$

  The student should be able to derive this equation following the procedure of the lecture notes. The result obtained for $E[f]$ will be closer to E_0 the more similar f is to the true ground-state.

- The main idea behind the WKB approximation is that if the potential $V(x)$ which a particle moves in is sufficiently slowly varying, then a reasonable ansatz for the wavefunction is

$$\psi(x) = e^{iS(x)/\hbar} \tag{3}$$

  where $S(x)$ is a slowly varying function. For a constant potential $V(x) = V_0$, this is indeed an exact solution with $S(x) = \pm\sqrt{2m(E-V_0)}x$. Inserting the above ansatz into the SE, one obtains a differential equation that determines $S(x)$. The validity of the approach relies on the second derivative of S is small in this differential equation compared to the main term, so that one can treat it as a perturbation, i.e.

$$|i\hbar S''| \ll |p^2|. \tag{4}$$

  One can derive (the student is not expected to have accomplished this) that this effectively means that the change in wavelength λ of the particle over a distance λ should be small compared to λ itself.

- Just as one assumed in the WKB-treatment that the potential varies slowly in space, in the adiabatic approximation one assumes that the potential varies slowly in time. The perturbation in $\hat{H}$ need not be small in magnitude. If this is satisfied, one can expect the approximate solutions of the time-dependent SE to be obtained in terms of the eigenfunctions $\psi_k(t)$ of the "instantaneous" Hamiltonian $H(t)$:

$$H(t)\psi_k(t) = E_k(t)\psi_k(t) \tag{5}$$

at any given time t, so that the total wavefunction Ψ satisfying the time-dependent SE can be expanded generally as:

$$\Psi(t) = \sum_k c_k(t)\psi_k(t)\mathrm{e}^{-\mathrm{i}\int_{t_0}^{t} E_k(t')dt'/\hbar}. \tag{6}$$

For a full score, the candidate should also discuss a quantitative condition (or at least how to obtain such a condition) required for this method to be valid.

Problem 2
This is explained in detail in the solution to Exercise #5, see the course webpage.

Problem 3
The energy levels of charged particles in a magnetic field is quantized in quantum mechanics, resulting in the particles only being allowed to occupy orbits with discrete energies known as Landau levels. These levels are highly degenerate with a number of electrons per level proportional to the external field B. The sawtooth pattern of E_F as a function of magnetic field B is directly related to the occupancy of the Landau levels: it occurs because both the energy level itself and the number of electrons occupying a given level change with B, so that abrupt jumps occur at specific threshold values of B when a given level becomes full. This is discussed in detail in the lecture notes. The SdH effect is the oscillation of the conductivity with B due to the above described behavior of the Landau levels, since the conductivity is determined by the electrons at the Fermi level.

Problem 4
A mixed state is not the same as a superposition of states, the latter being a pure state. A mixed state consists of a statistical mixture of different pure states: this results in different physical measurements, as illustrated by the box experiment in the lecture notes. A more mathematically stringent definition of the difference is that for a system which is in a superposition of states, $\mathrm{Tr}(\rho^2) = 1$ where ρ is the density matrix of the system. The expectation value of an operator A is given by $\mathrm{Tr}(\rho A)$, so that one finds $\langle J_z \rangle$ from this identity by using the ρ and $A = J_z$ from the problem text.

Problem 5
Flux quantization is the phenomenon that magnetic flux is allowed to pass through a given area only in discrete, quantized amounts. This phenomenon occurs in superconductors and can be shown by applying an external magnetic field to a superconducting ring, in which case the flux passing through the ring displays quantization. The Aharonov-Bohm effect is an example of how a geometrical (Berry) phase is manifested, as it shows that the wavefunction of electrons are manifested by the presence of a gauge field which arises due to a magnetic field in a region that is inaccessible to the electrons. The AB effect showed that the interference pattern of electrons passing through two slits could be tuned by altering the field B penetrating a region that was inaccessible to the electrons: the electrons thus "felt" the field B even though they did not move in it, precisely due to the geometrical phase picked up by the wavefunction. The derivation is shown in the lecture notes: the student is expected to be able to set up the total wavefunction ψ as a contribution from the two slits, and then show that ψ will depend on the flux Φ since the phases for the wavefunctions from the two slits depend on $\int A \cdot ds$ where the integral is taken over the paths of the electrons as they move through the slit and to the screen where they are detected.

Problem 6
The incident wave is moving in the forward-direction. Therefore, if scattering occurs, the part of the wave moving in the forward direction must change and it makes sense that the scattering amplitude is dependent on the forward-scattered part $f(0)$ of the amplitude. The Born approximation is expected to be good for weak scattering, meaning that the wavefunction has a small quantitative deviation from the incident plane-wave form, which can happen when the potential is small or when the particles are highly energetic. The Born approximation is a perturbative approach and in fact is inconsistent with the optical theorem: one needs to go to at least second order in the potential V to start to describe forward-scattering with $\theta \to 0$ in the Born-approximation.

Problem 7
The derivation is shown in detail in the lecture notes chapter 11. The final result is:

$$F_{nk}(t) = \frac{V_{nk}(t)}{\mathrm{i}\hbar},\ g_{nk} = \frac{E_n - E_k}{\hbar} \tag{7}$$

where

$$V_{nk}(t) = \int \psi_n^*(r)\hat{V}(r,t)\psi_k(r)dr. \tag{8}$$

Problem 8
To prove (1), we know that $H|\psi\rangle = E|\psi\rangle$. Now operate with T from the left on this equation and obtain, using that $TH = HT$, $H(T|\psi\rangle) = E(T|\psi\rangle)$. This shows that $T|\psi\rangle$ is an eigenstate of H with eigenvalue E. Next, we want to show that $\langle\psi|T|\psi\rangle = 0$ by using the antiunitarity propery:

$$\langle\alpha|\beta\rangle = \langle\beta|T^\dagger T|\alpha\rangle. \tag{9}$$

Let $|\alpha\rangle = T|\psi\rangle$ and $|\beta\rangle = |\psi\rangle$ above. We then get for the l.h.s. of Eq. (9):

$$\langle\alpha|\beta\rangle = \langle\psi|T^\dagger|\psi\rangle. \tag{10}$$

For the r.h.s. of Eq. (9), we obtain

$$\langle\psi|T^\dagger T^2|\psi\rangle = -\langle\psi|T^\dagger|\psi\rangle. \tag{11}$$

Equating the l.h.s. and r.h.s., we effectively have the equation $x = -x$ with $x = \langle\psi|T^\dagger|\psi\rangle$, and since

$$\langle\psi|T^\dagger|\psi\rangle = (\langle\psi|T|\psi\rangle)^* \tag{12}$$

it is clear that $x = 0$ means that $\langle\psi|T|\psi\rangle$=0 which completes the proof.

Problem 9
The force is attractive, because the phase-shift is positive: the sign of δ_0 is related to the sign of the potential $V(r)$. If the potential is attractive so that $V < 0$, the effective wavenumber of the particle inside this attractive region becomes larger than outside of it, meaning that its wavelength will be shorter. The wavefunction is thus "pulled" closer to the origio, corresponding to a positive phase-change.