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Contact during the exam:
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Exam in TFY4205 Quantum Mechanics II
Thursday, December 10, 2015
09:00–13:00

Allowed help: Alternativ C

This problem set consists of 3 pages, plus an Appendix of one page.

Problem 1

We consider a harmonic oscillator with hamiltonian

$$H = \frac{1}{2m}\, p^2 + \frac{m\omega^2}{2}\, q^2 \,, \tag{1}$$

where the position q and momentum p obey the commutator

$$[q, p] = i\hbar \,. \tag{2}$$

We introduce the annihilation and creation operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, q + \frac{i}{\sqrt{2m\hbar\omega}}\, p \,, \tag{3}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, q - \frac{i}{\sqrt{2m\hbar\omega}}\, p \,. \tag{4}$$

a) Show that a and $a^\dagger$ obey the commutator

$$[a, a^\dagger] = 1 \,. \tag{5}$$

b) Show that the hamiltonian expressed in terms of a and $a^\dagger$ takes the form

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) \,. \tag{6}$$

c) The Schrödinger equation is

$$H \; |n(t)\rangle = i\hbar \; \frac{d}{dt} \; |n(t)\rangle \; . \tag{7}$$

In the following, we will not write the time dependency of the states explicity.

Show that the energy eigenstates are

$$|n\rangle = \frac{e^{-in\omega t}}{\sqrt{n!}} \; (a^\dagger)^n |0\rangle \; , \tag{8}$$

where

$$a \; |0\rangle = 0 \tag{9}$$

and

$$|0\rangle \propto e^{-i\omega t/2} \; . \tag{10}$$

d) Show that

$$a \; |n\rangle = e^{-i\omega t} \; \sqrt{n} \; |n-1\rangle \; . \tag{11}$$

e) We now construct the *coherent state* $|\alpha\rangle$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \; \sum_{n=0}^{\infty} \; \frac{\alpha^n}{\sqrt{n!}} \; |n\rangle \; . \tag{12}$$

Show that

$$\langle\alpha| \; a \; |\alpha\rangle = e^{-i\omega t}\alpha \; . \tag{13}$$

and

$$\langle\alpha| \; a^\dagger \; |\alpha\rangle = e^{i\omega t}\alpha^* \; . \tag{14}$$

f) Show that

$$\langle\alpha| \; q \; |\alpha\rangle = q_0 \; \cos(\omega t - \Theta) \tag{15}$$

and

$$\langle\alpha| \; p \; |\alpha\rangle = p_0 \; \sin(\omega t - \Theta) \; , \tag{16}$$

where $\alpha = |\alpha| e^{i\Theta}$.

g) The coherent state $|\alpha\rangle$ is an eigenstate for the annihilation operator a: $a\;|\alpha\rangle = \exp(-i\omega t)\;\alpha\;|\alpha\rangle$. Show that the creation operator $a^\dagger$ has *no* eigenstate.

h) We have that $\langle\alpha_1|\alpha_2\rangle \neq 0$ when $\alpha_1 \neq \alpha_2$ when $|\alpha_1\rangle$ and $|\alpha_2\rangle$ are coherent states. (Do not show this.) We are used to eigenstates for operators with different eigenvalues being orthogoal. Which condition on the operator is broken allowing the eigenstates not to be orthogonal? Why is this condition sufficient to ensure orthogonality when not broken?

Problem 2

We will here study some aspects of *mixed states.* We assume a particle moving in a potential $V_j(x)$ in one dimension. The potential is given by

$$V_j(x) = \begin{cases} \infty & \text{if } x \leq 0 \;, \\ 0 & \text{if } 0 < x < L_j \;, \\ \infty & \text{if } L_j \leq x \;. \end{cases} \tag{17}$$

Here j can take two values, $j = 1$ and 2. We have that $L_1 < L_2$.
The energy spectrum for a particle moving in potential V_j is

$$E_{j,n} = \frac{\pi^2\hbar^2}{2mL_j^2}\, n^2 \;, \tag{18}$$

where $n = 1,\ 2,\ \ldots$. The corresponding wave functions are

$$\langle x|n,j\rangle = \psi_{n,j}(x) = \sqrt{\frac{2}{L_j}}\, \sin\left(\frac{n\pi x}{L_j}\right) \tag{19}$$

inside the boxes. Outside the wave functions are zero.
Suppose we do not know whether we have a box of size L_1 or of size L_2. In either case, the particle is in the ground state.

a) Explain why the density operator in this case is given by

$$\rho = \frac{1}{2}\left(|1,1\rangle\langle 1,1| + |1,2\rangle\langle 1,2|\right) \;. \tag{20}$$

b) We will work in the position basis, $|x\rangle$, which forms a complete set,

$$\int_{-\infty}^{+\infty} dx |x\rangle\langle x| = 1 \;. \tag{21}$$

Show that the density matrix in this representation becomes

$$\rho(x,x') = \frac{1}{2}\left(\psi_{1,1}(x)\psi_{1,1}^*(x') + \psi_{1,2}(x)\psi_{1,2}^*(x')\right) \;. \tag{22}$$

c) The average energy in this system is given by

$$\langle H\rangle = \mathrm{Trace}(\rho H) \;. \tag{23}$$

Use this equation and Equation (22) to calculate $\langle H\rangle$.

The following information may be of some use:

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \ , \tag{24}$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \ , \tag{25}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ , \tag{26}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \ , \tag{27}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \ , \tag{28}$$