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Suggested solution for 2021 spring exam in FY3464/FY8914 Quantum Field Theory

NOTE: The solutions below are meant as guidelines for how the problems may be solved and do not necessarily contain all the detailed steps of the calculations.

Problem 1

(a) The narration may differ from the student to student, but the following key points should be included and discussed in some detail for a full score:

- When evaluating Feyman diagrams using the Feynman rules arising from a perturbative treatment of a correlator via the path integral formalism, some of the resulting integrals diverge.
- This means that our original way of writing the Lagrangian with bare masses and interactions cannot be correct.
- To fix this, we first need to regularize the integrals so that the infinities become finite. This is done by some scheme, such as cutoff-regularization or dimensional regularization, which introduces a parameter ε that temporarily makes the integral finite whereas we can recover the original divergent results in a certain limit of ε.
- Once the diagrams are finite, we introduce finite counterterms that depend on ε which cancel the divergence.
- Once the divergences are gone, we can recover the original limit corresponding to divergent diagrams but without any divergences due to the counterterms. We have now renormalized the theory.
- The divergent diagrams in general can also contain finite corrections to both the mass, residue (amplitude) of the field, and the interaction in the theory. These effectively renormalize the bare mass, bare residue, and bare interaction strength used in the original Lagrangian. The counterterms in the on-shell renormalization scheme (which is the one we have used in the lectures) are chosen so that the renormalized quantities correspond to the physically measurable ones as determined by experimental measurements, *e.g.* the renormalized mass is the physical mass (pole of the propagator).

(b) When performing a dimensional regularization, a running coupling appears when the renormalization procedure of Feynman diagrams described above gives rise to a modified, effective interaction between the fields that depends explicitly on the so-called renormalization scale parameter μ. To make sure that any physically observable quantities are independent on this parameter μ, which is introduced by hand and thus arbitrary, the bare mass and interaction in the Lagrangian are promoted to "running" parameters which depend on μ in such a way that the physically measurable quantities become *independent* on μ.

(c) Mathematically, Green functions are objects which can be used to provide solutions for inhomogeneous partial differential equations characterized by a linear differential operator $\hat{L}$. They are used in many areas of physics to describe correlations between particles. In quantum field theory, Green functions are in fact often simply referred to as correlators as they provide information about the overlap (correlation) between states which have excitations of the field at different points in spacetime. For instance, a 2-point correlator of the form

$$\langle 0|\phi(x)\phi(y)|0\rangle \tag{1}$$

in scalar theory gives us the probability amplitude for a particle propagating between x and y if $x^0 > y^0$. The Feynman propagator $\langle 0|T\{\phi(x)\phi(y)\}|0\rangle$ can then also be referred to as a propagator as it is built up from such probability amplitudes. Higher-order correlators (with more fields inside the expectation value) are also used to describe the change in the effective interaction between fields and to compute experimentally measurable S-matrix elements via the LSZ-reduction formula.

Problem 2
(a) We use the Euler-Lagrange equations for ϕ:

$$d_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{2}$$

This gives the equation

$$\partial_\mu \partial^\mu \phi^* + m^2 \phi^* = 0 \tag{3}$$

which is the KG equation. The same equation is obtained for the ϕ-field, so both fields satisfy the KG equation.

(b) The Hamiltonian density is generally defined as

$$\mathcal{H} = \pi_\phi \partial_0 \phi + \pi_{\phi^*} \partial_0 \phi^* - \mathcal{L}. \tag{4}$$

The canonical momenta are

$$\begin{aligned} \pi_\phi &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^*, \\ \pi_{\phi^*} &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \partial_0 \phi. \end{aligned} \tag{5}$$

Therefore, we get

$$\mathcal{H} = \pi_\phi \pi_{\phi^*} + \nabla \phi^* \nabla \phi + m^2 \phi^* \phi. \tag{6}$$

(c) Consider a Lorentz-boost in the z-direction without loss of generality (since we can choose our coordinate-system as we like). The Lorentz-transformation reads

$$E' = \gamma(E - \beta p^z),\ (p^z)' = \gamma(p^z - \beta E), (p^{x,y})' = p^{x,y}. \tag{7}$$

The normalization condition written in the new frame is

$$\langle \mathbf{p}_1' | \mathbf{p}_2' \rangle = 2E_1'(2\pi)^3 \delta^3(\mathbf{p}_1' - \mathbf{p}_2'). \tag{8}$$

Inserting Eq. (7), we obtain

$$\langle \mathbf{p}_1' | \mathbf{p}_2' \rangle = 2\gamma(E_1 - \beta p_1^z)(2\pi)^3 [\gamma(p_1^z - \beta E_1) - (p_2^z)'] \delta[(p_1^x)' - (p_2^x)'] \delta[(p_1^y)' - (p_2^y)']. \tag{9}$$

We can simplify the first δ-function by using that

$$\delta[g(x)] = \frac{1}{\left| dg/dx|_{x=0} \right|} \delta(x), \tag{10}$$

which is valid when $g(x) = 0$. Inserting into Eq. (9):

$$\langle \mathbf{p}_1' | \mathbf{p}_2' \rangle = 2\gamma(E_1 - \beta p_1^z)(2\pi)^3 \delta(p_1^x - p_2^x)\delta(p_1^y - p_2^y) \frac{\delta(p_1^z - p_2^z)}{\gamma(1 - \beta dE_1/dp_1^z)}. \tag{11}$$

Due to the δ-functions, we used that everything is zero except if $p_1^i = p_2^i$ so that we may set $E_1 = E_2$. We obtain

$$1 - \beta dE_1/dp_1^z = (E_1 - \beta p_1^z)/E_1, \tag{12}$$

so that Eq. (11) ends up as

$$\langle \mathbf{p}_1' | \mathbf{p}_2' \rangle = 2E_1(2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{p}_2). \tag{13}$$

Problem 3
(a) A general n-point correlator in scalar ϕ^4-theory is written

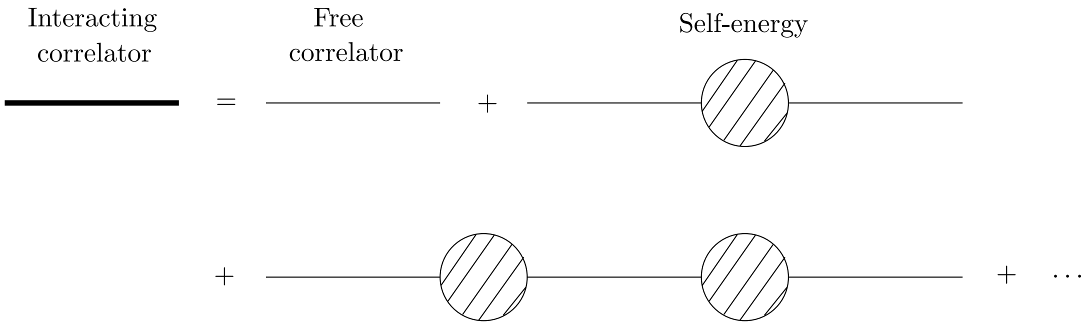
$$\langle T\{\phi(t_1)\phi(t_2)\ldots\phi(t_n)\}\tag{14}$$

In the course, we used the generating functional $Z(J)$ in the presence of an interaction term $\lambda\phi^4/4!$ to derive that this could be expressed as

$$\langle T\{\phi(t_1)\phi(t_2)\ldots\phi(t_n)\} = \frac{\langle T\{\phi(t_1)\phi(t_2)\ldots\phi(t_n)\mathrm{e}^{\mathrm{i}\int dt L_I(t)}\}\rangle_{\text{free}}}{\langle T\{\mathrm{e}^{\mathrm{i}\int dt L_I(t)}\}\rangle_{\text{free}}}.\tag{15}$$

The correlator may now be computed up to a desired order in λ by expanding $\mathrm{e}^{\mathrm{i}\int dt L_I(t)} = 1 + \mathrm{i}\int dt L_I(t) + \ldots$ and then evaluating the resulting correlator in both nominator and denominator of Eq. (15) in a free theory. This expansion of the exponential may be represented diagramatically in terms of Feynman diagrams where the belonging Feynman rules are derived by looking at the exact form of each term in the series expansion of $\mathrm{e}^{\mathrm{i}\int dt L_I(t)}$. For instance, vertices in a diagram are accounted for by an integration over time and a multiplicative factor of λ. The Feynman rules in Fourier-transformed momentum space gain an additional δ-function to account for energy-momentum conservation. In other words, the Feynman rules are derived from the perturbation expansion of the exponential of the interaction Lagrangian L_I in the correlator.

(b) The self-energy physically expresses the interaction that the field has with itself and its surroundings through virtual particles/excitations. Self-energies arise in theories where interactions between the field and other fields of the same type or interactions with other parts of the system exist. The 2-point correlator in an interacting ϕ^4-theory can be expressed by a combination of the 2-point correlator for a free theory and the self-energy for the field. Diagrammatically, we may express this relation as follows.

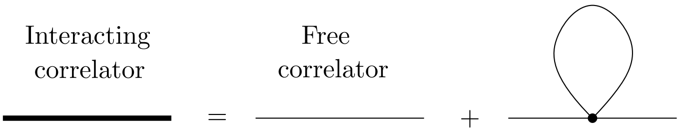


This relation is derived by expressing the interacting 2-point correlator as an expectation value in the free theory:

$$\langle T\{\phi(t_1)\phi(t_2)\} = \frac{\langle T\{\phi(t_1)\phi(t_2)\mathrm{e}^{\mathrm{i}\int dt L_I(t)}\}\rangle_{\text{free}}}{\langle T\{\mathrm{e}^{\mathrm{i}\int dt L_I(t)}\}\rangle_{\text{free}}}\tag{16}$$

and then expanding the exponential in the nominator and contracting fields via Wick's theorem, as mentioned in the previous point. We see from the figure that the interacting correlator is built up from non-interacting 2-point correlators and blobs which represent the self-energy. The self-energy, in turn, is built up from self-interaction diagrams of the field (see point c below).

(c) See figure below.



(d) This diagram does not contribute because of the denominator in Eq. (16) which cancels out all disconnected diagrams, such as the one considered in the present problem. The complex number associated with this diagram is then

$$G_F(t_1-t_2)\times\Big[-\mathrm{i}\lambda\frac{T}{8(2m_0)^2}\Big] \tag{17}$$

since it consists of a free propagator G_F and a vertex integral running from $-T/2$ to $+T/2$ multiplied with $[G_F(0)]^2$ and finally a symmetry factor of 8.

(e) Letting all 4-momenta in the figure have direction into their respective vertices and denoting the 4-momentum in one of the internal scalar field propagators as l, we obtain the following expression for the truncated diagram:

$$(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\frac{1}{2}(-\mathrm{i}\lambda)^2\int\frac{d^4l}{(2\pi)^4}\frac{\mathrm{i}}{l^2-m^2+\mathrm{i}\varepsilon}\frac{\mathrm{i}}{(k_1+k_4-l)^2-m^2+\mathrm{i}\varepsilon}. \tag{18}$$

The symmetry factor for the diagram is $2!=2$ due to having 2 internal propagators running between two vertices. Note that there is no symmetry factor for π rotation since the external lines have different momenta. The task is now to do a Feynman parametrization and subsequent Wick rotation. Thus, we use the formula

$$\frac{1}{ab}=\int_0^1\frac{dz}{[az+b(1-z)]^2} \tag{19}$$

with $a=l^2-m^2$ and $b=(l-q)^2-m^2$ where $q=k_1+k_4$ and we absorbed $+\mathrm{i}\varepsilon$ into m^2. We now "diagonalize" the momenta in the denominator by defining

$$l'=l-q(1-z) \tag{20}$$

so that the denominator on the rhs in Eq. (19) may be written

$$[(l')^2-m^2+q^2(1-z)z]^2. \tag{21}$$

The poles now lie at

$$[(l')^0]^2=R-\mathrm{i}\varepsilon \tag{22}$$

where R is a real number and we can thus always do the Wick rotation. Our Eq. (25) can now be written in terms of an integral over a Euclidean 4-dimensional vector l'_E:

$$\mathrm{i}(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\frac{1}{2}\lambda^2\int_0^1 dz\int\frac{d^4l'_E}{(2\pi)^4}\frac{1}{[(l'_E)^2+m^2-q^2z(1-z)]^2}. \tag{23}$$

after performing the Wick rotation (utilizing that $\int_{-\infty}^{\infty}dl^0=\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}dl^0$).

Problem 4

(a) The S-matrix element $\langle f|S|i\rangle$ expresses the transition probability amplitude from an initial state i at $t\to-\infty$ to a final state f at $t\to\infty$ for a particular theory. The S-matrix depends on both the interacting Hamiltonian of that theory and the Hamiltonian describing non-interacting fields in the initial and final state that are still "dressed" by their self-interactions (i.e. have renormalized masses and residues). The S-matrix element is related to any physical observable that depends on the probability of transition from an initial non-interacting state to a final non-interacting state, such as the scattering cross section for a particular scattering process.

(b) If the Dirac equation is to be Lorentz-covariant, it should have the form

$$(\mathrm{i}\gamma^\mu\partial'_\mu-m)\psi'(x')=0. \tag{24}$$

in a different inertial frame than the original one (where $\psi(x)$ is the solution). In the original frame, the Dirac equation reads:

$$(\mathrm{i}\gamma^\mu\partial_\mu-m)\psi(x)=0. \tag{25}$$

The question is now how $\psi'(x')$ is related to $\psi(x)$. First, we know that under a Lorentz-transformation, the differential operator transforms as

$$\partial_\mu = \Lambda^\nu{}_\mu \partial'_\nu. \tag{26}$$

Inserting this into Eq. (25), as well as the relation

$$\Lambda^\nu{}_\mu \gamma^\mu = U_\gamma^{-1} \gamma^\nu U_\gamma \tag{27}$$

which we derived in the lectures [where $U_\gamma = \mathrm{e}^{\mathrm{i}\omega_{\mu\nu} S^{\mu\nu}/2}$] gives us the equation

$$(\mathrm{i} U_\gamma^{-1} \gamma^\nu U_\gamma \partial'_\nu - m)\psi(x) = 0. \tag{28}$$

Since U_γ is x-independent, we may rewrite this to

$$U_\gamma^{-1}(\mathrm{i}\gamma^\nu \partial'_\nu - m) U_\gamma \psi(x) = 0. \tag{29}$$

Multiplying with U_γ from the left, we see that we achieve the equation

$$(\mathrm{i}\gamma^\nu \partial'_\nu - m)\psi'(x') = 0 \tag{30}$$

if we define

$$\psi'(x') = U_\gamma \psi(x). \tag{31}$$

This equation gives us the desired transformation rule.

(c) Parity invariance is a discrete symmetry and is not accompanied by any conserved current, since Noether's theorem only applies to continuous symmetries.