FY3464 Quantum Field Theory
Problemset 9

NTNU

Institutt for fysikk

SUGGESTED SOLUTION

Problem 1
We wish to prove that

$$\bar{u}^s(-\boldsymbol{k})\gamma^0 v^r(\boldsymbol{k}) = 0. \tag{1}$$

Multiply the equation with $2m$ and use that since $u(\boldsymbol{k})$ satisfies

$$(\not{k} - m)u^s(\boldsymbol{k}) = 0, \tag{2}$$

then $u(-\boldsymbol{k})$ satisfies

$$(k_0\gamma^0 + \boldsymbol{k}\cdot\boldsymbol{\gamma} - m)u^s(-\boldsymbol{k}) = 0. \tag{3}$$

We then obtain from the lhs of Eq. (1) that

$$\begin{aligned} 2m\bar{u}^s(-\boldsymbol{k})\gamma^0 v^r(\boldsymbol{k}) &= 2m[u^s(-\boldsymbol{k})]^\dagger\gamma^0\gamma^0 v^r(\boldsymbol{k}) \\ &= 2m\Big([2m + k_0\gamma^0 + \boldsymbol{k}\cdot\boldsymbol{\gamma} - m]u^s(-\boldsymbol{k})\Big)^\dagger v^r(\boldsymbol{k}) \\ &= \Big((\not{k}+m)^\dagger\gamma^0\gamma^0 u^s(-\boldsymbol{k})]^\dagger v(\boldsymbol{k}) \\ &= \bar{u}^s(-\boldsymbol{k})\gamma^0(\not{k}+m)v^r(\boldsymbol{k}) = 0, \end{aligned} \tag{4}$$

since $(\not{k}+m)v^r(\boldsymbol{k}) = 0$.

Problem 2
We start with:

$$\begin{aligned} S_F(x-y)_{\alpha\beta} &\equiv \langle T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}\rangle \\ &= \int \frac{d^3q}{(2\pi)^3 2q^0}[\theta(x^0-y^0)\mathrm{e}^{-iq(x-y)}(\not{q}+m)_{\alpha\beta} - \theta(y^0-x^0)\mathrm{e}^{\mathrm{i}q(x-y)}(\not{q}-m)_{\alpha\beta}]. \end{aligned} \tag{5}$$

Thus, the Fourier-transform is (dropping the subscript indices for brevity of notation):

$$\begin{aligned} \tilde{S}_F(k) &= \int d^4x \mathrm{e}^{\mathrm{i}kx} S_F(x) \\ &= \int \frac{d^3q}{(2\pi)^3 2q^0}\int dx^0 \mathrm{e}^{\mathrm{i}k^0x^0}[\theta(x^0)\mathrm{e}^{-\mathrm{i}q^0x^0}(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{q})(\gamma^0 q^0 - \boldsymbol{\gamma}\cdot\boldsymbol{q} + m) \\ &- \theta(-x^0)\mathrm{e}^{\mathrm{i}q^0x^0}(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{q})(\gamma^0 q^0 - \boldsymbol{\gamma}\cdot\boldsymbol{q} - m)]. \end{aligned} \tag{6}$$

Here, we used the integral over $\int d\boldsymbol{x}$ to get some δ-functions in the integrand. The integration over $\boldsymbol{q}$ now forces $\boldsymbol{q} = \boldsymbol{k}$ and therefore we must have $q^0 = \omega(\boldsymbol{k})$ since

$$q^2 = m^2 = (q^0)^2 - \boldsymbol{q}^2 = (q^0)^2 - \boldsymbol{k}^2. \tag{7}$$

The fact that the integration variable $q^2 = m^2$ follows from the fact that the Dirac fields have to satisfy the Dirac equation. On the other hand, note that we have *not* used $k^2 = m^2$ anywhere.

Therefore, we have $(q^0)^2 = \boldsymbol{k}^2 + m^2$, and we also have written in general the dispersion relation as $[\omega(\boldsymbol{k})]^2 = \boldsymbol{k}^2 + m^2$. It then follows that $q^0 = \omega(\boldsymbol{k})$ upon doing the integral over the delta-functions. This gives

$$\tilde{S}_F(k) = \frac{1}{2\omega}\int dx^0 [\mathrm{e}^{\mathrm{i}x^0(k^0-\omega)}\theta(x^0)(\gamma^0\omega - \boldsymbol{\gamma}\cdot\boldsymbol{k} + m) - \mathrm{e}^{\mathrm{i}x^0(k^0+\omega)}(\gamma^0\omega + \boldsymbol{\gamma}\cdot\boldsymbol{k} + m)], \tag{8}$$

where $\omega = \omega(\boldsymbol{k})$. We can now perform the integration and obtain:

$$\tilde{S}_F(k) = \frac{1}{2\omega}(\gamma^0\omega - \boldsymbol{\gamma}\cdot\boldsymbol{k} + m)\frac{\mathrm{e}^{\mathrm{i}(k^0-\omega+\mathrm{i}\varepsilon)x^0}}{\mathrm{i}(k^0-\omega+\mathrm{i}\varepsilon)}\bigg|_0^\infty - \frac{1}{2\omega}(\gamma^0\omega + \boldsymbol{k}\cdot\boldsymbol{\gamma} - m)\frac{\mathrm{e}^{\mathrm{i}(k^0+\omega-\mathrm{i}\varepsilon)x^0}}{\mathrm{i}(k^0+\omega-\mathrm{i}\varepsilon)}\bigg|_{-\infty}^{o}. \tag{9}$$

As before, we let $\omega \to \omega - \mathrm{i}\varepsilon$ in the integral in order to make it well-defined at asymptotic times (think of this as as an effective boundary condition that makes the Green function vanish at infinite times, as is physically reasonable). Continuing, we have:

$$\begin{aligned}
\tilde{S}_F(k) &= \frac{\mathrm{i}}{2\omega}\Big[\frac{[\gamma^0(\omega-k^0) + \not{k} + m](k^0+\omega) + [\gamma^0(\omega+k^0) - \not{k} - m](k^0-\omega)}{(k^0)^2 - \omega + \mathrm{i}\varepsilon}\Big] \\
&= \frac{\mathrm{i}}{2\omega}\Big[\frac{2\omega(\not{k}+m) + \gamma^0(\omega-k^0)(\omega+k^0) + \gamma^0(\omega+k^0)(k^0-\omega)}{k^2 - m^2 + \mathrm{i}\varepsilon}\Big] \\
&= \frac{i(\not{k}+m)}{k^2 - m^2 + \mathrm{i}\varepsilon}.
\end{aligned} \tag{10}$$

In the transition from the first to second line in the equation above, we used that $\omega^2 + \boldsymbol{k}^2 + m^2$. Again, note that while $q^2 = m^2$ in the integral, we have not assumed anything about k^2 being on-shell ($k^2 = m^2$).