FY3464 Quantum Field Theory

Problemset 8

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SUGGESTED SOLUTION

Problem 1

Consider an infinitesimal Lorentz-transformation $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - \varepsilon^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \varepsilon_{\nu}{}^{\mu}$. By using the result in Eq. (1) of the problem text, we obtain

$$\begin{aligned}\Lambda^{\mu}{}_{\nu}\gamma^{\nu} &= \gamma^{\mu} - \frac{\mathrm{i}}{2}\varepsilon_{\nu\lambda}S^{\nu\lambda}\gamma^{\mu} + \frac{\mathrm{i}}{2}\gamma^{\mu}\varepsilon_{\nu\lambda}S^{\nu\lambda} \\ &\simeq (1 - \frac{\mathrm{i}\varepsilon_{\nu\lambda}}{2}S^{\nu\lambda})\gamma^{\mu}(1 + \frac{\mathrm{i}\varepsilon_{\nu\lambda}}{2}S^{\nu\lambda}).\end{aligned} \tag{1}$$

We could write the last step since $\varepsilon_{\nu\lambda}$ is infinitesimal and thus the term $O(\varepsilon^2)$ is negligible. Now, perform another infinitesimal Lorentz transformation of the same form:

$$\Lambda^{\sigma}{}_{\mu}\Lambda^{\mu}{}_{\nu}\gamma^{\nu} \simeq (1 - \frac{\mathrm{i}\varepsilon_{\nu\lambda}}{2}S^{\nu\lambda})^2\gamma^{\sigma}(1 + \frac{\mathrm{i}\varepsilon_{\nu\lambda}}{2}S^{\nu\lambda})^2. \tag{2}$$

Thus, repeating this N times, letting $N \to \infty$ and using the formula

$$\lim_{N\to\infty}(1 + \mathrm{i}x)^N = \mathrm{e}^{\mathrm{i}Nx}, \tag{3}$$

we arrive at

$$(\Lambda^{\kappa}{}_{\nu})_{\text{tot}}\gamma^{\nu} = \mathrm{e}^{-\mathrm{i}\omega_{\lambda\sigma}S^{\lambda\sigma}/2}\gamma^{\kappa}\mathrm{e}^{\mathrm{i}\omega_{\lambda\sigma}S^{\lambda\sigma}/2}. \tag{4}$$

In Eq. (4), we have $\omega_{\lambda\sigma} = N\varepsilon_{\lambda\sigma}$, assumed to be finite, while

$$(\Lambda^{\kappa}{}_{\nu})_{\text{tot}} = \Lambda^{\kappa}{}_{\delta}\Lambda^{\delta}{}_{\rho}\ldots\Lambda^{\mu}{}_{\nu} \tag{5}$$

now constitutes a finite Lorentz transformations.

Problem 2

Writing out the exponential, we have

$$u(\boldsymbol{k}) = \left[\cosh\frac{\eta}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \sinh\frac{\eta}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right]\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}. \tag{6}$$

Squaring the two block matrices no the diagonal gives

$$\begin{aligned}(\cosh\eta/2 \mp \sigma^3\sinh\eta/2)^2 &= \cosh\eta \mp \sigma^3\sinh\eta \\ &= (k^0 \mp k^3\sigma^3)/m \\ &= \begin{cases} k\cdot\sigma/m & \text{for } - \\ k\cdot\bar{\sigma}/m & \text{for } +\,. \end{cases}\end{aligned} \tag{7}$$

Here, we used that the η in our expressions should be $\eta\,\mathrm{sign}(k_3)$ if we allow for k_3 to have both positive and negative values. This is because the exponent in $U_\gamma(\Lambda)$ has $k_3/|k_3|$ if we allow for boosts along either $+\boldsymbol{z}$ or $-\boldsymbol{z}$, so that the two block matrices on the diagonal then become

$$\cosh\eta/2 \mp \mathrm{sign}(k_3)\sigma^3 \sinh\eta/2 \tag{8}$$

and we have $\mathrm{sign}(k_3)\sinh\eta = k_3/m$ from our definition of $m\sinh\eta = |\boldsymbol{k}|$. Since the direction of the $+\hat{\boldsymbol{z}}$-axis is arbitrary (we choose our coordinate system as we like), Eq. (8) should be valid for any direction of $\boldsymbol{k}$.

It is then clear that we may write

$$u(\boldsymbol{k}) = \begin{pmatrix} \sqrt{k\cdot\sigma}\xi \\ \sqrt{k\cdot\bar{\sigma}}\xi \end{pmatrix} \tag{9}$$

where $\sqrt{k\cdot\sigma}$ is a matrix that squares to $k\cdot\sigma$. It remains to show that a matrix with eigenvalues equal to the square root of the eigenvalues of $k\cdot\sigma$ satisfies this.

First, let $A = k\cdot\sigma$. We see that $A = A^\dagger$. Therefore, there exists a unitary matrix U which diagonalizes A according to $UAU^\dagger$. Now take the square root of every diagonal element in $UAU^\dagger$ to get a diagonal matrix with the square root of the eigenvalues of A. If we now rotate this matrix back, we can define this as a matrix $\sqrt{A}$ where:

$$\sqrt{A} \equiv U^\dagger \sqrt{UAU^\dagger}\, U. \tag{10}$$

The matrix $\sqrt{A} = \sqrt{k\cdot\sigma}$ has precisely the desired property: its eigenvalues are the square root of the eigenvalues of A and squaring it gives A:

$$\begin{aligned} \sqrt{A}^2 &= U^\dagger \sqrt{UAU^\dagger}\, U U^\dagger \sqrt{UAU^\dagger}\, U \\ &= U^\dagger \sqrt{UAU^\dagger}\sqrt{UAU^\dagger}\, U \\ &= U^\dagger UAU^\dagger U \\ &= A. \end{aligned} \tag{11}$$