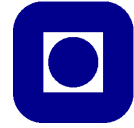


FY3464 Quantum Field Theory

Problemset 7

NTNU



Institutt for fysikk

SUGGESTED SOLUTION

Problem 1

a) We showed in the beginning of this course that the Lorentz-transformation tensor has to satisfy:

$$\eta_{\mu\nu} = \Lambda_{\nu}^{\alpha} \eta_{\alpha\beta} \Lambda_{\nu}^{\beta}. \quad (1)$$

Inserting

$$\Delta_{\nu}^{\mu} = \delta_{\nu}^{\mu} - \varepsilon_{\nu}^{\mu} \quad (2)$$

into this equation gives us

$$\begin{aligned} \eta_{\mu\nu} &= (\delta_{\mu}^{\alpha} - \varepsilon_{\mu}^{\alpha}) \eta_{\alpha\beta} (\delta_{\nu}^{\beta} - \varepsilon_{\nu}^{\beta}) \\ &= \eta_{\mu\nu} - (\varepsilon_{\mu\nu} + \varepsilon_{\nu\mu}) + O(\varepsilon^2). \end{aligned} \quad (3)$$

This shows that ε must be antisymmetric in its indices.

b) An infinitesimal Lorentz-transformation applied to a contravariant 4-vector reads:

$$x^{\mu} \rightarrow (x')^{\mu} = x^{\mu} + \varepsilon_{\nu}^{\mu} x^{\nu} = x^{\mu} - \frac{i}{2} \varepsilon_{\nu\lambda} J^{\nu\lambda} x^{\mu}. \quad (4)$$

Applying this transformation N times and taking the limit $N \rightarrow \infty$, we obtain

$$x^{\mu} \rightarrow (x')^{\mu} = e^{-iN\varepsilon_{\mu\nu}J^{\mu\nu}/2} x^{\mu}, \quad (5)$$

by using the formula

$$\lim_{N \rightarrow \infty} (1 - iJ/N)^N = e^{-iJ}. \quad (6)$$

c) We see that it $\omega_{\nu\mu} = N\varepsilon_{\mu\nu}$. It is thus assumed that while ε is infinitesimal and $N \rightarrow \infty$, their product is finite.

Problem 2

First, we note that

$$\begin{aligned} [S^{\mu\lambda}, S^{\nu\sigma}] &= \left(\frac{i}{4}\right)^2 [(\gamma^{\mu}\gamma^{\lambda} - \gamma^{\lambda}\gamma^{\mu}), (\gamma^{\nu}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\nu})] \\ &= -\frac{1}{4} \left([\gamma^{\mu}\gamma^{\lambda}, \gamma^{\nu}\gamma^{\sigma}] - [\gamma^{\mu}\gamma^{\lambda}, \gamma^{\sigma}\gamma^{\nu}] \right. \\ &\quad \left. - [\gamma^{\lambda}\gamma^{\mu}, \gamma^{\nu}\gamma^{\sigma}] + [\gamma^{\lambda}\gamma^{\mu}, \gamma^{\sigma}\gamma^{\nu}] \right). \end{aligned} \quad (7)$$

Now use the matrix identity

$$[AB, CD] = A\{B, C\}D - \{A, C\}BD + CA\{B, D\} - C\{A, D\}B. \quad (8)$$

Using this on the first term of the second line of Eq. (7) gives

$$[\gamma^\mu \gamma^\lambda, \gamma^\nu \gamma^\sigma] = 2(\eta^{\lambda\nu} \gamma^\mu \gamma^\sigma - \eta^{\mu\nu} \gamma^\lambda \gamma^\sigma + \eta^{\lambda\sigma} \gamma^\nu \gamma^\mu - \eta^{\mu\sigma} \gamma^\nu \gamma^\lambda). \quad (9)$$

Applying the identity to the remaining terms in a similar way verifies that S satisfies the Lorentz algebra.