FY3464 Quantum Field Theory
Problemset 7

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SUGGESTED SOLUTION

Problem 1
a) We showed in the beginning of this course that the Lorentz-transformation tensor has to satisfy:

$$\eta_{\mu\nu} = \Lambda^{\alpha}{}_{\nu}\eta_{\alpha\beta}\Lambda^{\beta}{}_{\nu}. \tag{1}$$

Inserting

$$\Delta^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - \varepsilon^{\mu}{}_{\nu} \tag{2}$$

into this equation gives us

$$\begin{aligned}\eta_{\mu\nu} &= (\delta^{\alpha}_{\mu} - \varepsilon^{\alpha}{}_{\mu})\eta_{\alpha\beta}(\delta^{\beta}_{\nu} - \varepsilon^{\beta}{}_{\nu}) \\ &= \eta_{\mu\nu} - (\varepsilon_{\mu\nu} + \varepsilon_{\mu\nu}) + O(\varepsilon^2).\end{aligned} \tag{3}$$

This shows that ε must be antisymmetric in its indices.

b) An infinitesimal Lorentz-transformation applied to a contravariant 4-vector reads:

$$x^{\mu} \to (x')^{\mu} = x^{\mu} + \varepsilon_{\nu}{}^{\mu}x^{\nu} = x^{\mu} - \frac{i}{2}\varepsilon_{\nu\lambda}J^{\nu\lambda}x^{\mu}. \tag{4}$$

Applying this transformation N times and taking the limit $N \to \infty$, we obtain

$$x^{\mu} \to (x')^{\mu} = \mathrm{e}^{-\mathrm{i}N\varepsilon_{\mu\nu}J^{\mu\nu}/2}x^{\mu}, \tag{5}$$

by using the formula

$$\lim_{N\to\infty}(1 - \mathrm{i}J/N)^N = \mathrm{e}^{-\mathrm{i}J}. \tag{6}$$

c) We see that it $\omega_{\nu\mu} = N\varepsilon_{\mu\nu}$. It is thus assumed that while ε is infinitesimal and $N \to \infty$, their product is finite.

Problem 2
First, we note that

$$\begin{aligned}[S^{\mu\lambda}, S^{\nu\sigma}] &= \left(\frac{\mathrm{i}}{4}\right)^2[(\gamma^{\mu}\gamma^{\lambda} - \gamma^{\lambda}\gamma^{\mu}), (\gamma^{\nu}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\nu})] \\ &= -\frac{1}{4}\Big([\gamma^{\mu}\gamma^{\lambda}, \gamma^{\nu}\gamma^{\sigma}] - [\gamma^{\mu}\gamma^{\lambda}, \gamma^{\sigma}\gamma^{\nu}] \\ &\quad - [\gamma^{\lambda}\gamma^{\mu}, \gamma^{\nu}\gamma^{\sigma}] + [\gamma^{\lambda}\gamma^{\mu}, \gamma^{\sigma}\gamma^{\nu}]\Big).\end{aligned} \tag{7}$$

Now use the matrix identity

$$[AB,CD] = A\{B,C\}D - \{A,C\}BD + CA\{B,D\} - C\{A,D\}B. \tag{8}$$

Using this on the first term of the second line of Eq. (7) gives

$$[\gamma^\mu\gamma^\lambda,\gamma^\nu\gamma^\sigma] = 2(\eta^{\lambda\nu}\gamma^\mu\gamma^\sigma - \eta^{\mu\nu}\gamma^\lambda\gamma^\sigma + \eta^{\lambda\sigma}\gamma^\nu\gamma^\mu - \eta^{\mu\sigma}\gamma^\nu\gamma^\lambda). \tag{9}$$

Applying the identity to the remaining terms in a similar way verifies that S satisfies the Lorentz algebra.