FY3464 Quantum Field Theory

Problemset 6

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SUGGESTED SOLUTION

Problem 1

The key difference between a gauge and a global symmetry is that the former is in our theoretical description, while the latter is a property of the system. A gauge symmetry has no "physical" meaning: it is an artifact of our choice for the coordinates/fields with which we describe the system. For instance, there are infinitely many gauge choices for the vector potential that gives the same physical, magnetic field. The gauge choice does not change the equations of motion or the physical state, and all states related by a gauge transformation are physically the same state.

In contrast, while a global symmetry transformation leaves the equations of motion the same, the state of the system is *not* physically the same after such a transformation. The easiest way to visualize this is to consider a ferromagnetic material. It has a magnetization $\boldsymbol{m}$ that points in some direction. Assume that there is no magnetic anisotropy in the system that favors a particular magnetization direction. In that case, the equations of motion that describe the magnetization have to be identical regardless of which direction the magnetization points in: the Lagrangian describing the ferromagnet has a rotational symmetry. However, a material with a magnetization that points in the z-direction is not physically identical to a material where the magnetization points in the x-direction, even if two such states are energetically equivalent. In other words, the possible ground states of the system are physically distinct, despite that the equations of motion for $\boldsymbol{m}$ are rotationally invariant. This is different from gauge symmetries, where the ground states corresponding to different gauge choices are physically identical.

Problem 2

We start with

$$\frac{\mathrm{i}\lambda^2}{6}\int_0^1\int_0^1\int_0^1 dxdydz\delta(x+y+z-1)\int\int\frac{d^4k_1d^4k_2}{(2\pi)^8}\frac{2}{(\alpha k_1^2+\beta k_2^2+\gamma k^2-m^2+\mathrm{i}\varepsilon)^3}. \tag{1}$$

Let us now perform the Wick rotations. Consider first the integral over dk_1^0. The integrand has a pole when

$$\begin{aligned}(k_1^0)^2 &= -\frac{\beta}{\alpha}(k_2^0)^2+\boldsymbol{k}_1^2+\frac{\beta}{\alpha}\boldsymbol{k}_2^2-\frac{\gamma}{\alpha}k^2+m^2-\mathrm{i}\varepsilon\\ &\equiv R-\mathrm{i}\varepsilon.\end{aligned} \tag{2}$$

Here, R is a real number that can be both positive and negative. If we're to be allowed to Wick rotate k_1^0 in the usual way that we demonstrated in the lectures, its poles should never lie in the 1st or 3rd quadrant. Let's then check both $R>0$ and $R<0$. For $R>0$, we get the poles:

$$k_1^0 = \pm(\sqrt{R}-\mathrm{i}\varepsilon) \tag{3}$$

while for $R < 0$, we get the poles:

$$k_1^0 = \pm(\mathrm{i}\sqrt{|R|} - \varepsilon). \tag{4}$$

In both cases, the poles lie in the 2nd and 4th quadrant in the complex plane. Similarly, you can also verify that there is no problem for $R = 0$. We conclude that the Wick rotation for k_1^0 can be done. Thus, we obtain:

$$\int\int \frac{d^4k_1 d^4k_2}{(2\pi)^8} \frac{2}{(\alpha k_1^2 + \beta k_2^2 + \gamma k^2 - m^2 + \mathrm{i}\varepsilon)^3} = \mathrm{i}\int\int \frac{d^4k_{1E} d^4k_2}{(2\pi)^8} \frac{2}{(-\alpha k_{1E}^2 + \beta k_2^2 + \gamma k^2 - m^2 + \mathrm{i}\varepsilon)^3}. \tag{5}$$

To do the Wick rotation for k_2^0, we can use exactly the same reasoning and verify that the poles of the integrand on the rhs of Eq. (5) always lie in the 2nd and 4th quadrant. Thus, we have now proven that the double Wick rotation can be done for the $\int\int d^4k_1 d^4k_2 1/D^3$ integral for *any* value of k^2.

All that remains now is to show that the remaining integral is equal to the integral we obtained when we did Wick rotations first and then Feynman parametrization. In the present problem, we are left with the following integral after having done the Wick rotations (absorbing the $\mathrm{i}\varepsilon$ back into m^2):

$$\begin{aligned} &-\frac{\mathrm{i}\lambda^2}{6}\int_0^1\int_0^1\int_0^1 dxdydz\delta(x+y+z-1)\int\int \frac{d^4k_{1E} d^4k_{2E}}{(2\pi)^8} \frac{2}{(-\alpha k_{1E}^2 - \beta k_{2E}^2 + \gamma k^2 - m^2)^3} \\ &= \frac{\mathrm{i}\lambda^2}{6}\int_0^1\int_0^1\int_0^1 dxdydz\delta(x+y+z-1)\int\int \frac{d^4k_{1E} d^4k_{2E}}{(2\pi)^8} \frac{2}{(\alpha k_{1E}^2 + \beta k_{2E}^2 - \gamma k^2 + m^2)^3}. \end{aligned} \tag{6}$$

Now use that

$$\int_0^\infty \rho^2 \mathrm{e}^{-\rho b} = \frac{2}{b^3} \tag{7}$$

and that $d^4k_{1E}d^4k_{2E} = d^4l_{1E}d^4l_{2E}$. The latter is shown in the same way as in the lectures (using the Jacobian of the transformation matrix) where we proved that $d^4k_1d^4k_2 = d^4l_1d^4l_2$. We then obtain

$$\begin{aligned} &\frac{\mathrm{i}\lambda^2}{6}\int_0^1\int_0^1\int_0^1 dxdydz\delta(x+y+z-1)\int\int \frac{d^4k_{1E} d^4k_{2E}}{(2\pi)^8} \frac{2}{(\alpha k_{1E}^2 + \beta k_{2E}^2 - \gamma k^2 + m^2)^3} \\ &= \frac{\mathrm{i}\lambda^2}{6}\int_0^1\int_0^1\int_0^1 dxdydz\delta(x+y+z-1)\int\int \frac{d^4l_{1E} d^4l_{2E}}{(2\pi)^8} \int_0^\infty d\rho\rho^2 \mathrm{e}^{-\rho(\alpha k_{1E}^2 + \beta k_{2E}^2 - \gamma k^2 + m^2)}. \end{aligned} \tag{8}$$

In the lectures, we used that

$$\alpha k_1^2 + \beta k_2^2 + \gamma k^2 - m^2 = x l_1^2 + y l_2^2 + z(k - l_1 - l_2)^2 - m^2 \tag{9}$$

when we did the Feynman parametrization. Since $k_{iE}^2 = -k_i^2$, it follows that

$$\alpha k_{1E}^2 + \beta k_{2E}^2 - \gamma k^2 + m^2 = x l_{1E}^2 + y l_{2E}^2 + z(k_E - l_{1E} - l_{2E})^2 + m^2. \tag{10}$$

Inserting Eq. (10) into Eq. (8), we finally obtain

$$\frac{\mathrm{i}\lambda^2}{6}\int_0^1\int_0^1\int_0^1 dxdydz\delta(x+y+z-1)\int\int \frac{d^4l_{1E} d^4l_{2E}}{(2\pi)^8} \int_0^\infty d\rho\rho^2 \mathrm{e}^{-\rho(x l_{1E}^2 + y l_{2E}^2 + z(k_E - l_{1E} - l_{2E})^2 + m^2)}. \tag{11}$$