FY3464 Quantum Field Theory

Problemset 5

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SUGGESTED SOLUTION

Problem 1

The symmetry factor is $S = 2 \times 2 = 4$: factor 2 for two propagators going between two vertices and factor 2 for the top propagator starting and ending at the same point. Note that there is no factor 2 for reflection along an axis running upwards through the loops since the original, non-truncated diagram has different endpoints t_1 and t_2.

The truncated diagram expression is obtained from the non-truncated diagram expression by getting rid of the operators associated with the end-points (exponentials), getting rid of the integral over the frequency coming out of/going into the end-points, and finally dividing on a propagator for each end-point. The non-truncated diagram, starting at t_1 and ending at t_2, is (dropped all internal exponential factors corresponding to energy flow out of and into vertices, which anyway cancel each other out in the end):

$$\begin{aligned}
&\int \frac{d\omega d\omega_1 d\omega_2 d\omega_3 d\omega_4}{(2\pi)^5} \frac{(-\mathrm{i}\lambda)^2}{S} \mathrm{e}^{-\mathrm{i}\omega t_1 + \mathrm{i}\omega_4 t_2} (2\pi)^2 \delta(\omega + \omega_3 - \omega_1 - \omega_4)\delta(\omega_1 - \omega_3) \\
&\times \tilde{G}(\omega)\tilde{G}(\omega_1)\tilde{G}(\omega_2)\tilde{G}(\omega_3)\tilde{G}(\omega_4) \\
&= \frac{(-\mathrm{i}\lambda)^2}{S} \int \frac{d\omega d\omega_1 d\omega_2}{(2\pi)^3} \Big(\frac{\mathrm{i}}{\omega^2 - m_0^2 + \mathrm{i}\varepsilon}\Big)^2 \Big(\frac{\mathrm{i}}{\omega_1^2 - m_0^2 + \mathrm{i}\varepsilon}\Big)^2 \frac{\mathrm{i}}{\omega_2^2 - m_0^2 + \mathrm{i}\varepsilon} \mathrm{e}^{-\mathrm{i}\omega(t_1 - t_2)}.
\end{aligned} \tag{1}$$

Therefore, the truncated diagram is:

$$\frac{(-\mathrm{i}\lambda)^2}{S} \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \Big(\frac{\mathrm{i}}{\omega_1^2 - m_0^2 + \mathrm{i}\varepsilon}\Big)^2 \frac{\mathrm{i}}{\omega_2^2 - m_0^2 + \mathrm{i}\varepsilon}. \tag{2}$$

We perform the integration over ω_2 first by using

$$\int_C d\omega_2 \frac{\mathrm{i}}{\omega_2^2 - m_0^2 + \mathrm{i}\varepsilon} = \int_C d\omega_2 \frac{\mathrm{i}}{(\omega_2 - m_0 + \mathrm{i}\varepsilon)(\omega_2 + m_0 - \mathrm{i}\varepsilon)} = 2\pi\mathrm{i} \sum_i \mathrm{Res} f(\omega = \omega_i). \tag{3}$$

where ω_i are the poles of

$$f(\omega) = \frac{\mathrm{i}}{\omega^2 - m_0^2 + \mathrm{i}\varepsilon} \tag{4}$$

contained within the closed contour C. Taking this contour to be the real axis and closing it in the upper half-plane, so that the pole $\omega_+ = -m_0 + \mathrm{i}\varepsilon$ is enclosed, we have that $\int_C = \int_{-\infty}^{\infty}$ since the integrand goes to zero on the semicircle with radius $R \to \infty$ closing the contour.

Now in general, the residue of a function $f(\omega)$which has a pole of n'th order in ω_0 is given by

$$\text{Res}f(\omega=\omega_0) = \frac{1}{(n-1)}\lim_{z\to\infty}\frac{d^{n-1}}{d\omega^{n-1}}[(\omega-\omega_0)^n f(\omega)]. \tag{5}$$

Therefore, for f given in Eq. (4), we have:

$$\text{Res}f(\omega=\omega_+) = -\frac{\text{i}}{2m_0}. \tag{6}$$

The expression for the diagram is then, after performin the ω_2-integral, reduced to

$$\frac{(-\text{i}\lambda)^2}{S}\frac{1}{2m_0}\int\frac{d\omega_1}{2\pi}\Big(\frac{\text{i}}{\omega_1^2-m_0^2+\text{i}\varepsilon}\Big)^2. \tag{7}$$

We can now perform the ω_1-integral using the residue-theorem again. Closing the contour in the upper half-plane and using that the pole at $\omega_+ = -m_0+\text{i}\varepsilon$ is now of second order for the function

$$f(\omega) = \Big(\frac{\text{i}}{\omega_1^2-m_0^2+\text{i}\varepsilon}\Big)^2, \tag{8}$$

we find that the residue for f given by Eq. (8) is:

$$\text{Res}f(\omega=\omega_+) = -\frac{2}{8m_0^3}. \tag{9}$$

Using finally $S=4$, we then have the final amplitude for the diagram:

$$\frac{(-\text{i}\lambda)^2}{4}\frac{1}{2m_0}\frac{-2\text{i}}{8m_0^3} = \frac{\text{i}\lambda^2}{32m_0^4}. \tag{10}$$

Problem 2

We found in the lectures that up to $O(\lambda^2)$, we had generally (without assuming anything about ω):

$$\omega^2-m_0^2-\Sigma_0(\omega) \simeq \omega^2-m_0^2-\frac{\lambda}{4m_0}+\frac{\lambda^2}{32m_0^4}-\frac{\lambda^2}{8m_0^2}\frac{1}{\omega^2-9m_0^2+\text{i}\varepsilon}. \tag{11}$$

Rewrite this as:

$$\omega^2-m_0^2-\Sigma_0(\omega) \simeq \omega^2-9m_0^2+8m_0^2-\frac{\lambda}{4m_0}+\frac{\lambda^2}{32m_0^4}-\frac{\lambda^2}{8m_0^2}\frac{1}{\omega^2-9m_0^2+\text{i}\varepsilon}. \tag{12}$$

We then observe that for $\omega^2 \simeq 9m_0^2$, the dominant terms will be:

$$\omega^2-m_0^2-\Sigma_0(\omega) \simeq 8m_0^2-\frac{\lambda^2}{8m_0^2}\frac{1}{\omega^2-9m_0^2+\text{i}\varepsilon}. \tag{13}$$

We now omit $\text{i}\varepsilon$ as it will have no consequence in what follows. Using the above result for $\omega^2-m_0^2-\Sigma_0(\omega)$, the propagator now takes the form:

$$\frac{\text{i}}{\omega^2-m_0^2-\Sigma_0(\omega)} \simeq \frac{\text{i}(\omega^2-9m_0^2)/8m_0^2}{(\omega^2-9m_0^2)-\frac{\lambda^2}{64m_0^4}}. \tag{14}$$

Clearly, this has a pole at

$$\omega^2 = 9m_0^2 + \frac{\lambda^2}{64m_0^4}. \tag{15}$$

But be careful: setting $\lambda = 0$ does *not* mean that the propagator has a pole at $\omega^2 = 9m_0^2$ as is obvious by setting $\lambda = 0$ in the original equation Eq. (11). Thus, our derived result is only valid for $\lambda \neq 0$ and for $\omega^2 \simeq 9m_0^2$.

With the identified pole, we can now identify the residue Z at this pole. Writing in general:

$$\frac{\mathrm{i}}{\omega^2 - m_0^2 - \Sigma_0(\omega)} \overset{\omega^2 \simeq 9m_0^2}{=} \frac{\mathrm{i}Z_3}{\omega^2 - (9m_0^2 + \frac{\lambda^2}{64m_0^4})}, \tag{16}$$

we see that the residue Z_3 at the 3−particle pole Eq. (15) is

$$Z = \left.\frac{\omega^2 - 9m_0^2}{8m_0^2}\right|_{\omega^2 = 9m_0^2 + \frac{\lambda^2}{64m_0^4}} = \frac{\lambda^2}{512m_0^6}. \tag{17}$$