FY3464 Quantum Field Theory
Problemset 4

NTNU

Institutt for fysikk

SUGGESTED SOLUTION

Problem 1
To find probability of the system being in a state $|\psi_2\rangle$ at time t_2 when we know that it was in the state $|\psi_1\rangle$ at time t_1, i.e. $|\Psi, t_1\rangle = |\psi_1\rangle$, we need to compute the overlap amplitude $\langle\psi_2|\Psi, t_2\rangle$

$$\langle\psi_2|\Psi, t_2\rangle = \langle\psi_2|e^{-iH(t_2-t_1)}|\psi_1\rangle \equiv \langle\psi_2, t_2|\psi_1, t_1\rangle, \tag{1}$$

where we used that the state of the system $|\Psi, t_2\rangle$ at t_2 is obtained by applying the time-evolution operator to $|\psi_1\rangle$. Now, insert two completeness relations at times t_1 and time t_2, i.e. integrals over states with labels $|\boldsymbol{x}(t_1)\rangle$ and $|\boldsymbol{x}(t_2)\rangle$:

$$\langle\psi_2|\Psi, t_2\rangle = \int\int \langle\psi_2|\boldsymbol{x}(t_2)\rangle\langle\boldsymbol{x}(t_2)|e^{-iH(t_2-t_1)}|\boldsymbol{x}(t_1)\rangle\langle\boldsymbol{x}(t_1)|\psi_1\rangle d\boldsymbol{x}(t_2)d\boldsymbol{x}(t_1). \tag{2}$$

But we have already evaluated the quantity $\langle\boldsymbol{x}(t_2)|e^{-iH(t_2-t_1)}|\boldsymbol{x}(t_1)\rangle$ in the lectures [using the notation $\boldsymbol{x}_2 = \boldsymbol{x}(t_2)$]:

$$\langle\boldsymbol{x}(t_2)|e^{-iH(t_2-t_1)}|\boldsymbol{x}(t_1)\rangle = \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{3N/2}\int\prod_{t_1<t<t_2} d\boldsymbol{x}(t)e^{iS/\hbar}. \tag{3}$$

Inserting this into Eq. (2) gives us:

$$\langle\psi_2|\Psi, t_2\rangle = \langle\psi_2, t_2|\psi_1, t_1\rangle = \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{3N/2}\int\prod_{t_1\leq t\leq t_2}\Big(\psi]\boldsymbol{x}(t_2)]\Big)^*\psi[\boldsymbol{x}(t_1)]e^{iS/\hbar}d\boldsymbol{x}(t). \tag{4}$$

Problem 2
We start by considering the partition function

$$Z = \int\mathcal{D}\phi e^{iS} = \int\mathcal{D}\phi\exp\Big[i\int dt\frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}\omega_0^2\phi^2 + i\varepsilon\phi^2 + J\phi\Big] \tag{5}$$

where $\phi = \phi(t)$ and $J = J(t)$. Consider now the action S in the partition function and insert the Fourier-transform

$$\phi(t) = \frac{1}{2\pi}\int d\omega\tilde{\phi}(\omega)e^{-i\omega t} \tag{6}$$

and similarly for $J(t)$. After cleaning up the expression a bit, we end up with

$$S = \int\frac{d\omega}{2\pi}\Big(\frac{1}{2}(\omega^2 - \omega_0^2 + i\varepsilon)\tilde{\phi}(\omega)\tilde{\phi}(-\omega) + \tilde{J}(\omega)\tilde{\phi}(-\omega)\Big). \tag{7}$$

Now define new field variables

$$\tilde{\phi}'(\omega) \equiv \tilde{\phi}(\omega) + \frac{\tilde{J}(\omega)}{\omega^2 - \omega_0^2 + i\varepsilon}. \tag{8}$$

Now express S in terms of the new fields $\tilde{\phi}'$ and get:

$$S = \int \frac{d\omega}{2\pi}\Big(\frac{1}{2}(\omega^2 - \omega_0^2 + i\varepsilon)\tilde{\phi}'(\omega)\tilde{\phi}'(-\omega) - \frac{1}{2}\tilde{J}(\omega)\frac{1}{\omega^2 - \omega_0^2 + i\varepsilon}\tilde{J}(-\omega)\Big) \tag{9}$$

We can now Fourier-transform back, i.e. use

$$\tilde{\phi}'(\omega) = \int dt \phi'(t) e^{i\omega t} \tag{10}$$

and similarly for $\tilde{J}(\omega)$, to obtain

$$S = \int dt \Big(\frac{1}{2}\Big(\frac{1}{2}[\partial_0\phi'(t)]^2 - \frac{1}{2}\omega_0^2[\phi'(t)]^2 + i\varepsilon[\phi'(t)]^2\Big) - \frac{1}{2}\int\int dt dt' J(t)[G_F(t-t')/i]J(t'), \tag{11}$$

where

$$G_F(t-t') = \int \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega^2 - \omega_0^2 + i\varepsilon}. \tag{12}$$

Note that it follows from Eq. (8) that

$$\phi'(t) = \phi(t) + \int \frac{d\omega}{2\pi} e^{-i\omega t}\frac{\tilde{J}(\omega)}{\omega^2 - \omega_0^2 + i\varepsilon} = \phi(t) - i\int dt' J(t') G_F(t-t'), \tag{13}$$

since we have proven in the lectures that

$$\tilde{G}_F(\omega) = \frac{i}{\omega^2 - \omega_0^2 + i\varepsilon}. \tag{14}$$

We have thus shown that the partition function can be written as

$$Z = Z(J=0) \times e^{-\frac{1}{2}\int\int dt dt' J(t) G_F(t-t') J(t')}. \tag{15}$$