FY3464 Quantum Field Theory
Problemset 3

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SUGGESTED SOLUTION

Problem 1
Using the definition time-ordering operator, we see that

$$G_F(x-y) = \langle T\{\phi(x)\phi(y)\}\rangle = \int \frac{d\boldsymbol{k}}{(2\pi)^3 2\omega(\boldsymbol{k})}[\Theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}k(x-y)} + \Theta(y^0-x^0)\mathrm{e}^{\mathrm{i}k(x-y)}]. \quad (1)$$

Acting with the operator ∂^2+m^2 on G_F ($\partial^2 = \partial_\mu\partial^\mu$), we obtain:

$$\begin{aligned}
(\partial^2+m^2)G_F(x-y) = &\int \frac{d\boldsymbol{k}}{(2\pi)^3 2\omega(\boldsymbol{k})}\Big[\partial_{x^0}\Big(\delta(x^0-y^0)\mathrm{e}^{-\mathrm{i}k(x-y)} - \mathrm{i}\omega\Theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}k(x-y)} - \delta(y^0-x^0)\mathrm{e}^{\mathrm{i}k(x-y)} \\
&+ \mathrm{i}\omega\Theta(y^0-x^0)\mathrm{e}^{\mathrm{i}k(x-y)}\Big) \\
&+ (\boldsymbol{k}^2+m^2)[\Theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}k(x-y)} + \Theta(y^0-x^0)\mathrm{e}^{\mathrm{i}k(x-y)}\Big] \\
&= \partial_{x^0}\int \frac{d\boldsymbol{k}}{(2\pi)^3 2\omega(\boldsymbol{k})}[\delta(x^0-y^0)\mathrm{e}^{-\mathrm{i}k(x-y)} - \delta(x^0-y^0)\mathrm{e}^{\mathrm{i}k(x-y)}] \\
&+ \int \frac{d\boldsymbol{k}}{(2\pi)^3 2\omega(\boldsymbol{k})}(-\mathrm{i}\omega)[\delta(x^0-y^0)\mathrm{e}^{-\mathrm{i}k(x-y)} - \delta(y^0-x^0)\mathrm{e}^{\mathrm{i}k(x-y)}\Big] \\
&+ \int \frac{d\boldsymbol{k}}{(2\pi)^3 2\omega(\boldsymbol{k})}(-\omega^2+\boldsymbol{k}^2+m^2)[\Theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}k(x-y)} + \Theta(y^0-x^0)\mathrm{e}^{\mathrm{i}k(x-y)}].
\end{aligned} \quad (2)$$

We used above that $\delta(x) = \delta(-x)$. Recall that $k^2 = \omega^2 - \boldsymbol{k}^2 = m^2$ and $k^0 = \omega$. This is required in order for $\phi(x)$ to satisfy its equation of motion. Because of this, the term $(-\omega^2+\boldsymbol{k}^2+m^2)$ is zero. The first term on the rhs of the second equality sign in the equation above is also zero because:

$$\begin{aligned}
&\int \frac{d\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega}[\delta(x^0-y^0)\mathrm{e}^{-\mathrm{i}k^0(x^0-y^0)}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} - \delta(x^0-y^0)\mathrm{e}^{\mathrm{i}k^0(x^0-y^0)}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}] \\
&\quad = \int \frac{d\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega}\delta(x^0-y^0)[\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} - \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}]
\end{aligned} \quad (3)$$

by using that $\delta(x)f(x) = \delta(x)f(0)$. Thus, the above is equal to

$$\delta(x^0-y^0)\int \frac{d\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega}[\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} - \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}] \quad (4)$$

and this expression is equal to zero, as can be seen by performing a variable shift $\boldsymbol{k} \to -\boldsymbol{k}$ in the second term and using that $\omega(\boldsymbol{k}) = \omega(-\boldsymbol{k})$.

Therefore, all that remains of the original expression $(\partial^2+m^2)G_F$ is

$$\begin{aligned}(\partial^2+m^2)G_F(x-y) &= -\mathrm{i}\int\frac{d\boldsymbol{k}}{(2\pi)^3}\frac{1}{2}\delta(x^0-y^0)[\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}+\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}] \\ &= -\mathrm{i}\delta(x^0-y^0)\int\frac{d\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\ &= -\mathrm{i}\delta^4(x-y). \end{aligned} \tag{5}$$

Problem 2

We start by noting that the total derivative of $\mathcal{L}$ with respect to coordinate can be rewritten using the chain rule:

$$\frac{d\mathcal{L}}{dx_\nu}\equiv d^\nu\mathcal{L}=\frac{\partial\mathcal{L}}{\partial x_\nu}+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\frac{\partial(\partial_\mu\phi_\alpha)}{\partial x_\nu}+\frac{\partial\mathcal{L}}{\partial\phi_\alpha}\frac{\partial\phi_\alpha}{\partial x_\nu}. \tag{6}$$

The term $\frac{\partial\mathcal{L}}{\partial x_\nu}\equiv\partial^\nu\mathcal{L}$ is zero since we have translational invariance of the Lagrangian. Use now the Euler-Lagrange equation:

$$d_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\Big)=\frac{\partial\mathcal{L}}{\partial\phi_\alpha} \tag{7}$$

to rewrite Eq. (6) as

$$d^\nu\mathcal{L}=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\partial_\mu\partial^\nu\phi_\alpha+d_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\Big)\partial^\nu\phi_\alpha. \tag{8}$$

Staring a little bit on the right hand side, we see that we can write it as the derivative of a product as follows:

$$d^\nu\mathcal{L}=d_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\partial^\nu\phi_\alpha\Big), \tag{9}$$

where we used that $d_\mu\phi_\alpha=\partial_\mu\phi_\alpha$, as seen from the definition of a total derivative:

$$d_\mu=\partial_\mu+\sum_\alpha\partial_\mu\phi_\alpha\frac{\partial}{\partial\phi_\alpha}+\sum_{\alpha\nu}(\partial_\mu\partial_\nu\phi_\alpha)\frac{\partial}{\partial(\partial_\nu\phi_\alpha)}. \tag{10}$$

Since $d^\nu\mathcal{L}$ can be written as $\eta^{\mu\nu}d_\mu\mathcal{L}$, we can write the equation above as:

$$d_\mu\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\partial^\nu\phi_\alpha-\eta^{\mu\nu}\mathcal{L}\Big]=0. \tag{11}$$

The divergence of the expression inside the brackets is thus zero and we have identified the stress-energy tensor $T^{\mu\nu}$ satisfying $d_\mu T^{\mu\nu}=0$:

$$T^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\partial^\nu\phi_\alpha-\eta^{\mu\nu}\mathcal{L}. \tag{12}$$