FY3464 Quantum Field Theory
Problemset 2

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SUGGESTED SOLUTION

Problem 1

In light of the fact that $\sqrt{z}$ is known to have branch-points at $z = 0$ and $z \to \infty$, the candidates for branch points of $f(z)$ seems to be

- $z = \mathrm{i}$
- $z = -\mathrm{i}$
- $z \to \infty$.

Nowwe have

$$f(z) = \sqrt{z^2+1} = \sqrt{(z+\mathrm{i})(z-\mathrm{i})}. \tag{1}$$

Check $z = \mathrm{i}$ first. We set $z - \mathrm{i} = r\mathrm{e}^{\mathrm{i}\theta}$ and plug it into $f(z)$:

$$f(z) = \sqrt{r\mathrm{e}^{\mathrm{i}\theta}(r\mathrm{e}^{\mathrm{i}\theta}+2\mathrm{i})} = \sqrt{r}\mathrm{e}^{\mathrm{i}\theta/2}\sqrt{r\mathrm{e}^{\mathrm{i}\theta}+2\mathrm{i}}. \tag{2}$$

The result is clearly not invariant under $\theta \to \theta + 2\pi$ due to the $\mathrm{e}^{\mathrm{i}\theta/2}$ factor. Same procedure shows that $z = -\mathrm{i}$ is also a branch point.

What about $z \to \infty$? Rewrite $f(z)$ to

$$f(z) = g(\xi) = \sqrt{1/\xi^2+1} \tag{3}$$

where $\xi \equiv 1/z$. The question is now if $\xi = 0$ is a branch point. Set $\xi = r\mathrm{e}^{\mathrm{i}\theta}$. As we let r be infinitesimal and perform $\theta \to \theta + 2\pi$, we see that the result is invariant. Thus, $z \to \infty$ is not a branch point.

Problem 2

With the assumption $\boldsymbol{x} = \boldsymbol{y}$ given in the problem text, we get

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{4\pi^2}\int_0^\infty \frac{|\boldsymbol{k}|^2 d|\boldsymbol{k}|}{\omega(|\boldsymbol{k}|)}\mathrm{e}^{-\mathrm{i}\omega(x^0-y^0)}, \tag{4}$$

where $\omega(k) = \sqrt{|\boldsymbol{k}|^2+m^2}$, and we used that there is no angular dependence in the integral over $\boldsymbol{k}$ when $\boldsymbol{x} = \boldsymbol{y}$. Using

$$\frac{d\omega}{d|\boldsymbol{k}|} = \frac{|\boldsymbol{k}|}{\omega}, \tag{5}$$

we get:

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{4\pi^2}\int_m^\infty d\omega\sqrt{\omega^2-m^2}\mathrm{e}^{-\mathrm{i}\omega(x^0-y^0)}. \tag{6}$$

Since this expression should be Lorentz-invariant, we can replace x^0-y^0 with $|x-y|$ (since we assumed $x^0-y^0>0$).

To solve the above integral, we make the variable substitution $r\equiv\omega/m-1$. This gives us

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{4\pi^2}m^2\mathrm{e}^{-\mathrm{i}m|x-y|}\int_0^\infty dr\sqrt{r^2+2r}\mathrm{e}^{-\mathrm{i}rm|x-y|}. \tag{7}$$

To make this integral convergent, we need to give the mass a small imaginary part: $m\to m-\mathrm{i}\varepsilon$ where $\varepsilon>0$. This ensures that the integral vanishes in the limit $|x-y|\to\infty$, as is physically reasonable. We then end up with

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{4\pi^2}m^2\mathrm{e}^{-\mathrm{i}m|x-y|}\int_0^\infty dr\sqrt{r^2+2r}\mathrm{e}^{-\mathrm{i}rm|x-y|}\mathrm{e}^{-\varepsilon r|x-y|}. \tag{8}$$

Since this integrand contains no poles in the fourth quadrant, integrating it in the complex plane over a closed contour consisting of ($\lim_{R\to\infty}$) $[0,R]$, a circular arc extending from $z=R$ to $Z=-\mathrm{i}R$, and finally along $[-\mathrm{i}R,0]$ must yield zero. The contribution from the circular arc vanishes due to the ε-factor. Therefore, promoting r to be a complex variable ($r\to z$), we must have that

$$\int_0^\infty dz\ldots = \int_0^{-\mathrm{i}\infty} dz\ldots. \tag{9}$$

Shifting variables again to $\tau\equiv\mathrm{i}z$, we thus find that

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{4\pi^2}m^2\mathrm{e}^{-\mathrm{i}m|x-y|}\int_0^\infty d\tau\sqrt{\tau^2+2\mathrm{i}\tau}\mathrm{e}^{-\tau m|x-y|}. \tag{10}$$

In the limit $m|x-y|\gg 1$ that we are asked to consider, the integrand decays rapidly unless τ is small. Therefore, the main contribution to the integral will come from small values of τ where the τ^2 inside the root can be neglected. We then obtain:

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle &= \frac{1}{4\pi^2}m^2\mathrm{e}^{-\mathrm{i}m|x-y|}\sqrt{2\mathrm{i}}\int_0^\infty d\tau\sqrt{\tau}\mathrm{e}^{-\tau m|x-y|}\\ &= \frac{1}{4\pi^2}m^2\mathrm{e}^{-\mathrm{i}m|x-y|}\sqrt{2\mathrm{i}}\frac{1}{2}\sqrt{\frac{\pi}{(m|x-y|)^3}}\\ &= \frac{1}{4\pi^2}\left(\frac{m\pi\mathrm{i}}{2|x-y|^3}\right)^{1/2}\mathrm{e}^{-\mathrm{i}m|x-y|}. \end{aligned}\tag{11}$$