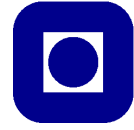


FY3464 Quantum Field Theory

Problemset 2

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SUGGESTED SOLUTION

Problem 1

In light of the fact that \sqrt{z} is known to have branch-points at $z = 0$ and $z \rightarrow \infty$, the candidates for branch points of $f(z)$ seems to be

- $z = i$
- $z = -i$
- $z \rightarrow \infty$.

Now we have

$$f(z) = \sqrt{z^2 + 1} = \sqrt{(z+i)(z-i)}. \quad (1)$$

Check $z = i$ first. We set $z - i = re^{i\theta}$ and plug it into $f(z)$:

$$f(z) = \sqrt{re^{i\theta}(re^{i\theta} + 2i)} = \sqrt{r}e^{i\theta/2}\sqrt{re^{i\theta} + 2i}. \quad (2)$$

The result is clearly not invariant under $\theta \rightarrow \theta + 2\pi$ due to the $e^{i\theta/2}$ factor. Same procedure shows that $z = -i$ is also a branch point.

What about $z \rightarrow \infty$? Rewrite $f(z)$ to

$$f(z) = g(\xi) = \sqrt{1/\xi^2 + 1} \quad (3)$$

where $\xi \equiv 1/z$. The question is now if $\xi = 0$ is a branch point. Set $\xi = re^{i\theta}$. As we let r be infinitesimal and perform $\theta \rightarrow \theta + 2\pi$, we see that the result is invariant. Thus, $z \rightarrow \infty$ is not a branch point.

Problem 2

With the assumption $x = y$ given in the problem text, we get

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{4\pi^2} \int_0^\infty \frac{|\mathbf{k}|^2 d|\mathbf{k}|}{\omega(|\mathbf{k}|)} e^{-i\omega(x^0 - y^0)}, \quad (4)$$

where $\omega(k) = \sqrt{|\mathbf{k}|^2 + m^2}$, and we used that there is no angular dependence in the integral over \mathbf{k} when $x = y$. Using

$$\frac{d\omega}{d|\mathbf{k}|} = \frac{|\mathbf{k}|}{\omega}, \quad (5)$$

we get:

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{4\pi^2} \int_m^\infty d\omega \sqrt{\omega^2 - m^2} e^{-i\omega(x^0 - y^0)}. \quad (6)$$

Since this expression should be Lorentz-invariant, we can replace $x^0 - y^0$ with $|x - y|$ (since we assumed $x^0 - y^0 > 0$).

To solve the above integral, we make the variable substitution $r \equiv \omega/m - 1$. This gives us

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{4\pi^2} m^2 e^{-im|x-y|} \int_0^\infty dr \sqrt{r^2 + 2r} e^{-irm|x-y|}. \quad (7)$$

To make this integral convergent, we need to give the mass a small imaginary part: $m \rightarrow m - i\epsilon$ where $\epsilon > 0$. This ensures that the integral vanishes in the limit $|x - y| \rightarrow \infty$, as is physically reasonable. We then end up with

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{4\pi^2} m^2 e^{-im|x-y|} \int_0^\infty dr \sqrt{r^2 + 2r} e^{-irm|x-y|} e^{-\epsilon r|x-y|}. \quad (8)$$

Since this integrand contains no poles in the fourth quadrant, integrating it in the complex plane over a closed contour consisting of $(\lim_{R \rightarrow \infty}) [0, R]$, a circular arc extending from $z = R$ to $Z = -iR$, and finally along $[-iR, 0]$ must yield zero. The contribution from the circular arc vanishes due to the ϵ -factor. Therefore, promoting r to be a complex variable ($r \rightarrow z$), we must have that

$$\int_0^\infty dz \dots = \int_0^{-i\infty} dz \dots \quad (9)$$

Shifting variables again to $\tau \equiv iz$, we thus find that

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{4\pi^2} m^2 e^{-im|x-y|} \int_0^\infty d\tau \sqrt{\tau^2 + 2i\tau} e^{-\tau m|x-y|}. \quad (10)$$

In the limit $m|x - y| \gg 1$ that we are asked to consider, the integrand decays rapidly unless τ is small. Therefore, the main contribution to the integral will come from small values of τ where the τ^2 inside the root can be neglected. We then obtain:

$$\begin{aligned} \langle \phi(x)\phi(y) \rangle &= \frac{1}{4\pi^2} m^2 e^{-im|x-y|} \sqrt{2i} \int_0^\infty d\tau \sqrt{\tau} e^{-\tau m|x-y|} \\ &= \frac{1}{4\pi^2} m^2 e^{-im|x-y|} \sqrt{2i} \frac{1}{2} \sqrt{\frac{\pi}{(m|x-y|)^3}} \\ &= \frac{1}{4\pi^2} \left(\frac{m\pi i}{2|x-y|^3} \right)^{1/2} e^{-im|x-y|}. \end{aligned} \quad (11)$$