FY3464 Quantum Field Theory
Problemset 11

NTNU
Institutt for fysikk

SUGGESTED SOLUTION

Problem 1
We want to compute $CP(\mathrm{i}\bar{\psi}\gamma^5\psi)PC$.

Consider first $P(\mathrm{i}\bar{\psi}\gamma^5\psi)P$. We know that $P\bar{\psi}P = \bar{\psi}\gamma^0$ and $P\psi P = \gamma^0\psi$. Therefore,

$$P(\mathrm{i}\bar{\psi}\gamma^5\psi)P = \mathrm{i}\bar{\psi}\gamma^0\gamma^5\gamma^0\psi = -\mathrm{i}\bar{\psi}\gamma^5\psi. \tag{1}$$

We are left with $C(-\mathrm{i}\bar{\psi}\gamma^5\psi)C$ and we know that $C\psi C = \mathrm{i}\gamma^2\psi^*$, $C\bar{\psi}C = \mathrm{i}\psi^T\gamma^2\gamma^0$. The latter follows by using that $(\gamma^2)^\dagger = -\gamma^2$. We thus obtain

$$C(-\mathrm{i}\bar{\psi}\gamma^5\psi)C = -\mathrm{i}[\mathrm{i}\psi^T\gamma^2\gamma^0]\gamma^5[\mathrm{i}\gamma^2\psi^*] = \psi^T\gamma^2\gamma^0\gamma^5\gamma^2\psi^*. \tag{2}$$

Now use that $(\gamma^2)^2 = -1$ and that γ^2 anticommutes with γ^0 and γ^5:

$$C(-\mathrm{i}\bar{\psi}\gamma^5\psi)C = -\mathrm{i}\psi^T\gamma^0\gamma^5\psi = -\mathrm{i}\psi_\alpha(\gamma^0\gamma^5)_{\alpha\beta}\psi^*_\beta = \mathrm{i}\psi^*_\beta(\gamma^0\gamma^5)_{\alpha\beta}\psi_\alpha \tag{3}$$

were we used the anticommutativity of the Dirac spinor elements. Now we use that

$$(\gamma^0\gamma^5)_{\alpha\beta} = [(\gamma^0\gamma^5)^T]_{\beta\alpha} = (\gamma^5\gamma^0)_{\beta\alpha} = -(\gamma^0\gamma^5)_{\beta\alpha} \tag{4}$$

and thus finally obtain

$$CP(\mathrm{i}\bar{\psi}\gamma^5\psi)PC = C(-\mathrm{i}\bar{\psi}\gamma^5\psi)C = -\mathrm{i}\psi^*_\beta(\gamma^0\gamma^5)_{\beta\alpha}\psi_\alpha = -\mathrm{i}\bar{\psi}\gamma^5\psi. \tag{5}$$

Problem 2
Start by considering the action in the problem:

$$S = \int dt\Big(\bar{\psi}(t)(\mathrm{i}\partial_0 - m + \mathrm{i}\varepsilon)\psi(t) + \bar{\eta}(t)\psi(t) + \bar{\psi}(t)\eta(t)\Big). \tag{6}$$

The Fourier-transformations of the field ψ and the source η is:

$$\psi(t) = \frac{1}{2\pi}\int\tilde{\psi}(\omega)\mathrm{e}^{-\mathrm{i}\omega t},\ \eta(t) = \frac{1}{2\pi}\int\tilde{\eta}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}. \tag{7}$$

It then follows that

$$\bar{\psi}(t) = \frac{1}{2\pi}\int\tilde{\bar{\psi}}(\omega)\mathrm{e}^{\mathrm{i}\omega t},\ \bar{\eta}(t) = \frac{1}{2\pi}\int\tilde{\bar{\eta}}(\omega)\mathrm{e}^{\mathrm{i}\omega t}. \tag{8}$$

Inserting these into S, we obtain:

$$\int\frac{d\omega}{2\pi}\Big(\tilde{\bar{\psi}}(\omega)(\omega - m + \mathrm{i}\varepsilon)\tilde{\psi}(\omega) + \tilde{\bar{\eta}}(\omega)\tilde{\psi}(\omega) = \tilde{\bar{\psi}}(\omega)\tilde{\eta}(\omega)\Big). \tag{9}$$

Now define the new fields:

$$\tilde{\psi}'(\omega) \equiv \tilde{\psi}(\omega) + \frac{\eta(\omega)}{\omega - m + i\varepsilon}, \; \tilde{\bar{\psi}}''(\omega) \equiv \tilde{\bar{\psi}}(\omega) + \frac{\bar{\eta}(\omega)}{\omega - m + i\varepsilon}. \tag{10}$$

Expressing S in terms of these new fields, we obtain:

$$\int \frac{d\omega}{2\pi}\Big(\tilde{\bar{\psi}}''(\omega)(\omega - m + i\varepsilon)\tilde{\psi}'(\omega) - \frac{\tilde{\bar{\eta}}(\omega)\eta(\omega)}{\omega - m + i\varepsilon}\Big). \tag{11}$$

If we now Fourier-transform back to time-space, we obtain

$$\int dt \bar{\psi}''(t)(i\partial_0 - m + i\varepsilon)\psi'(t) - \int\int dt dt' \bar{\eta}(t)\Big(\frac{1}{2\pi}\int d\omega \frac{e^{-i\omega(t-t')}}{\omega - m + i\varepsilon}\Big)\eta(t') \tag{12}$$

Note that

$$\begin{aligned}\psi'(t) &= \frac{1}{2\pi}\int e^{-i\omega t}\Big(\tilde{\psi}(\omega) + \frac{\eta(\omega)}{\omega - m + i\varepsilon}\Big)d\omega \\ &= \psi(t) + \int dt' \eta(t') \overbrace{\Big(\frac{1}{2\pi}\int d\omega \frac{e^{-i\omega(t-t')}}{\omega - m + i\varepsilon}\Big)}^{-iS_F(t-t')} \\ &= \psi(t) - iS_F\eta(t),\end{aligned} \tag{13}$$

just like we wrote in the lectures. Similarly for $\bar{\psi}''(t)$. Since ψ' are just shifted by a constant relative ψ, so that $\mathcal{D}\psi = \mathcal{D}\psi'$, we obtain for Eq. (12) (after renaming ψ' to ψ and $\bar{\psi}''$ to $\bar{\psi}$):

$$S = \int dt \bar{\psi}(t)(i\partial_0 - m + i\varepsilon)\psi(t) - \frac{1}{i}\int\int dt dt' \bar{\eta}(t) S_F(t-t')\eta(t'), \tag{14}$$

where

$$S_F(t-t') = \frac{1}{2\pi}\int d\omega \frac{i}{\omega - m_0 + i\varepsilon} e^{-i\omega(t-t')}. \tag{15}$$

All that remains is to verify that $S_F(t-t')$ satisfies the equation given in the problem text. Using the residue-theorem and that the integrand of Eq. (15) has a pole at $\omega = m - i\varepsilon$, this is straight-forward to verify using the residue-theorem (close the contour in the lower half-plane for $t - t' > 0$ and in the upper half-plane for $t - t' < 0$). Finally, to confirm that $S_F(t-t')$ also satisfies the Eq. (15) for $t = t'$, observe that $S_F \to 1$ for $t -> t'$ from above while $S_F = 0$ for $t -> t'$ from below. Hence, it is a step-function at $t = t'$ and its derivative is a Dirac-delta function, as is consistent with the equation of motion.

Problem 3

Suppose that a massive particle ($m \neq 0$) has spin pointing backward along its line of flight, so that $\langle \boldsymbol{\sigma} \rangle \parallel -\boldsymbol{k}$. It is a left-handed particle. But since it is massive, there exists a Lorentz-boost Λ bringing us to the rest frame of the particle. Boosting even further, beyond this rest frame, brings us to a frame where the particle is moving in the opposite direction of what it did in the original frame. But since its spin is unchanged, the particle is now right-handed.

Instead, if $m = 0$, the particle has no rest frame since it moves with the speed of light c. Thus, we can never change the handedness of the particle. A left-handed particle is always a left-handed particle ina ny reference frame, and is thus Lorentz-invariant.