FY3464 Quantum Field Theory
Problemset 10

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SUGGESTED SOLUTION

Problem 1
The Dirac Lagrangian is

$$\mathcal{L} = \mathrm{i}\bar{\psi}(x)\partial\!\!\!/\psi(x) - m\bar{\psi}(x)\psi(x), \tag{1}$$

with $\partial\!\!\!/ = \partial_\mu\gamma^\mu$. Performing the transformation given in the problem text gives

$$\mathcal{L} = \mathrm{i}\bar{\psi}'(x')\partial\!\!\!/'\psi'(x') - m\bar{\psi}'(x')\psi'(x'), \tag{2}$$

where $\partial\!\!\!/' = \partial'_\mu\gamma^\mu$. We know that

$$\psi'(x') = U_\gamma\psi(x). \tag{3}$$

Based on this, it follows that

$$\bar{\psi}'(x') = \bar{\psi}(x)U_\gamma^{-1}. \tag{4}$$

by using that $U_\gamma = \mathrm{e}^{\mathrm{i}\omega_{\mu\nu}S^{\mu\nu}/2}$ and $(S^{\mu\nu})^\dagger = \gamma^0 S^{\mu\nu}\gamma^0$. The latter can be verified from its definition. Taken in combination, this gives

$$\gamma^0 U_\gamma^\dagger \gamma^0 = U_\gamma^{-1}. \tag{5}$$

Based on Eqs. (3) and (4), it is clear that the mass-term in $\mathcal{L}$ is invariant:

$$\begin{aligned} m\bar{\psi}'(x')\psi'(x') &= m^2[\bar{\psi}(x)U_\gamma^{-1}]U_\gamma\psi(x) \\ &= m\bar{\psi}(x)\psi(x). \end{aligned} \tag{6}$$

Now for the kinetic term, which requires a bit more care. We have

$$\bar{\psi}'(x')\gamma^\mu\partial'_\mu\psi'(x')\bar{\psi}'(x') = \bar{\psi}(x)U_\gamma^{-1}\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu U_\gamma\psi(x). \tag{7}$$

Now recall that we previously derived the following identity:

$$\Lambda^\mu{}_\nu\gamma^\nu = U_\gamma^{-1}\gamma^\mu U_\gamma. \tag{8}$$

Since $\delta_\mu^\sigma = \Lambda^\rho{}_\mu\Lambda_\rho{}^\sigma$, multiplying Eq. (8) with $\Lambda_\mu{}^\lambda$ gives

$$\gamma^\lambda = U_\gamma^{-1}\Lambda_\mu{}^\lambda\gamma^\mu U_\gamma. \tag{9}$$

Using this in Eq. (7), we get

$$\bar{\psi}(x)U_\gamma^{-1}\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu U_\gamma\psi(x) = \bar{\psi}(x)\gamma^\nu\partial_\nu\psi(x), \tag{10}$$

which proves in the invariance of $\mathcal{L}$.

Problem 2

You were given a head-start on this problem in the lectures, but let's start from the beginning any way.

We recall first that when the Lagrangian transforms as

$$\mathcal{L} \to \mathcal{L} + \varepsilon\partial_\mu J^\mu \tag{11}$$

under a field transformation $\phi(x) \to \phi(x) + \varepsilon f[\phi(x)]$, then the conserved current is

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} f(\phi) - J^\mu. \tag{12}$$

Now we consider the specific transformation that we're interested in:

$$\psi(x) \to \psi'(x) = \psi(x) + \delta\psi(x). \tag{13}$$

We need to find out what $\delta\psi(x)$ is for an infinitesimal transformation. We have previously derived that

$$\psi'(x') = U_\gamma(\Lambda)\psi(x),\ U_\gamma(\Lambda) = \mathrm{e}^{\mathrm{i}\varepsilon_{\mu\nu}S^{\mu\nu}/2}, \tag{14}$$

for an infinitesimal transformation $\varepsilon_{\mu\nu}$. Moreover, we know that

$$x' = U^{-1}(\Lambda)x,\ U(\Lambda) = \mathrm{e}^{\mathrm{i}\varepsilon_{\mu\nu}J^{\mu\nu}/2} \tag{15}$$

where $J^{\mu\nu} = \mathrm{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)$. From now on, drop the argument (Λ) in the U and U_γ operators for brevity of notation, but keep the important distinction between U_γ and U. Taylor-expanding, it is seen that

$$\psi(Ux) = U\psi(x) \tag{16}$$

for an infinitesimal transformation $U = 1 - \varepsilon_{\mu\nu}J^{\mu\nu}/2$ and using the definition of $J^{\mu\nu}$. Therefore, we have that

$$\psi'(x) = \psi(x) + \frac{\mathrm{i}}{2}\varepsilon_{\mu\nu}(S^{\mu\nu} + J^{\mu\nu})\psi(x), \tag{17}$$

allowing us to identify:

$$\delta\psi(x) = \frac{\mathrm{i}}{2}[S^{\mu\nu} + \mathrm{i}(x^\mu\partial^\nu - x^\nu\partial^\mu). \tag{18}$$

This expression gives us part of the conserved current [the f-term in Eq. (12)]. Next, we need to identify $\delta\mathcal{L}$ to see if there is a J-term as well. The transformed Lagrangian is

$$\begin{aligned}\mathcal{L}'(x) = \mathcal{L}(x) + \delta\mathcal{L}(x) &= \bar{\psi}'(x)(\mathrm{i}\partial\!\!\!/ - m)\psi'(x) \\ &= \psi^\dagger(Ux)U_\gamma^\dagger\gamma^0\mathrm{i}\gamma^\mu\partial_\mu U_\gamma\psi(Ux) \\ &- \psi^\dagger(Ux)U_\gamma^\dagger\gamma^0 U_\gamma\psi(Ux)m. \end{aligned}\tag{19}$$

For an infinitesimal transformation $U = 1 - \frac{1}{2}\varepsilon_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)$, we saw previously that $\psi(Ux) = U\psi(x)$. Using also that the U_γ matrices are not unitary, since $U_\gamma^\dagger = \gamma^0 U_\gamma^{-1}\gamma^0$, we obtain

$$\mathcal{L}'(x) = \mathrm{i}U\bar{\psi}(x)\Lambda^\mu{}_\nu\gamma^\nu\partial_\mu U\psi(x) - U\bar{\psi}(x)U\psi(x)m. \tag{20}$$

To obtain this equation, we made use of $\Lambda^\mu{}_\nu\gamma^\nu = U_\gamma^{-1}\gamma^\mu U_\gamma$ and that

$$\psi^\dagger(Ux) = [\psi(Ux)]^\dagger = [U\psi(x)]^\dagger = U\psi^\dagger(x) \tag{21}$$

since the † means the complex conjugate and matrix transpose operation here, leaving the sign of the ∂ operators in U unaffected by the † operation (unlike the † denoting an adjoint operator, defined from the inner product involving an operator, in which case $\partial_\mu^\dagger = -\partial_\mu$). In passing, we make note of the following property which will be useful later:

$$(\partial^\nu x^\mu - \partial^\mu x^\nu)\mathcal{L} = (x^\mu\partial^\nu - x^\nu\partial^\mu)\mathcal{L}. \tag{22}$$

Now insert into Eq. (20)

$$U = 1 - \frac{1}{2}\varepsilon_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu) \tag{23}$$

and also $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \varepsilon_\nu{}^\mu$ for our infinitesimal Lorentz-transformation. We then obtain [dropping from now on the argument (x) on all $\bar{\psi}$ and ψ for brevity of notation]:

$$\begin{aligned}\mathcal{L}'(x) &= \mathrm{i}\bar{\psi}\partial\!\!\!/\psi - \bar{\psi}m\psi \\ &\quad + [-\frac{\mathrm{i}}{2}\varepsilon_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\bar{\psi}\delta^\lambda_\sigma\gamma^\sigma\partial_\lambda\psi + \mathrm{i}\bar{\psi}\varepsilon_\nu{}^\mu\gamma^\nu\partial_\mu\psi \\ &\quad + \mathrm{i}\bar{\psi}\delta^\lambda_\sigma\gamma^\sigma\partial_\lambda(-\frac{1}{2})\varepsilon_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi + \frac{m}{2}\varepsilon_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\bar{\psi}\cdot\psi + \frac{m}{2}\bar{\psi}\varepsilon_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi].\end{aligned} \tag{24}$$

After some simplifications, we can write this as

$$\begin{aligned}\mathcal{L}'(x) &= \mathcal{L} + \frac{1}{2}\varepsilon_{\mu\nu}\partial_\lambda[m(\eta^{\lambda\nu}x^\mu - \eta^{\lambda\mu}x^\nu)(\bar{\psi}\psi)] \\ &\quad + \frac{\varepsilon_{\mu\nu}}{2}[2\mathrm{i}\bar{\psi}\eta^{\nu\lambda}\gamma^\mu\partial_\lambda\psi - \mathrm{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\bar{\psi}\cdot\partial\!\!\!/\psi - \mathrm{i}\bar{\psi}\partial\!\!\!/(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi],\end{aligned} \tag{25}$$

where we, among other things, used that $\varepsilon_\nu{}^\mu = \eta^{\sigma\mu}\varepsilon_{\nu\sigma}$ and afterwards renamed the indices for the corresponding term.

We're almost there now, hang in there a little while longer. First, we want to move the x^μ and x^ν in the last term to the left of the $\partial\!\!\!/$. This is achieved by using the commutator

$$[\partial_\lambda, x^\mu] = \delta^\mu_\lambda \tag{26}$$

and by using that the infinitesimal displacement tensor $\varepsilon_{\mu\nu}$ is antisymmetric. Moving the x^μ and x^ν terms past the $\partial\!\!\!/$ in this way then *cancels* the first term on the second line: $2\mathrm{i}\bar{\psi}\gamma^\mu\partial^\nu\psi$. We are then left with

$$\begin{aligned}\mathcal{L}' &= \mathcal{L} + \frac{\varepsilon_{\mu\nu}}{2}\partial_\lambda[-(\eta^{\nu\lambda}x^\mu - \eta^{\mu\lambda}x^\nu)\mathcal{L}_m] \\ &\quad + \frac{\varepsilon_{\mu\nu}}{2}[-\mathrm{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\bar{\psi}\cdot\partial\!\!\!/\psi - \mathrm{i}\bar{\psi}(x^\mu\partial^\nu - x^\nu\partial^\mu)\partial\!\!\!/\psi],\end{aligned} \tag{27}$$

where $\mathcal{L}_m$ is the m-dependent part of $\mathcal{L}$. Making use of Eq. (22) on the second line, we get

$$\begin{aligned}\mathcal{L}' = \mathcal{L} + \frac{\varepsilon_{\mu\nu}}{2}\partial_\lambda[-(\eta^{\nu\lambda}x^\mu - \eta^{\lambda\mu}x^\nu)\mathcal{L}_m] \\ + \frac{\varepsilon_{\mu\nu}}{2}\partial_\lambda[-\mathrm{i}(\eta^{\nu\lambda}x^\mu - \eta^{\lambda\mu}x^\nu)\mathcal{L}_{\text{kin}}]\end{aligned} \tag{28}$$

where $\mathcal{L}_{\text{kin}}$ is the kinetic energy part of the Dirac Lagrangian. Since $\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_m$, we have therefore arrived at

$$\mathcal{L}' = \mathcal{L} + \frac{\varepsilon_{\mu\nu}}{2}\partial_\lambda[-(\eta^{\nu\lambda}x^\mu - \eta^{\lambda\mu}x^\nu)\mathcal{L}]. \tag{29}$$

Here, ∂_λ acts on the $\psi(x), \bar{\psi}(x)$ terms in $\mathcal{L} = \mathcal{L}(x)$. By instead considering $\mathcal{L}$ as a function of $\psi, \bar{\psi}$, and x, i.e. $\mathcal{L} = \mathcal{L}(\psi, \bar{\psi}, x)$, we can replace ∂_λ with a total derivative d_λ and thus identify

$$J^{\lambda\mu\nu} = -(\eta^{\nu\lambda}x^\mu - \eta^{\lambda\mu}x^\nu)\mathcal{L}. \tag{30}$$

At long last, the conserved Noether current satisfying $d_\lambda j^{\lambda\mu\nu} = 0$ is now established:

$$j^{\lambda\mu\nu} = \frac{\partial\mathcal{L}}{\partial[\partial_\lambda\psi(x)]}(i)[S^{\mu\nu} + \mathrm{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)]\psi(x) + (\eta^{\nu\lambda}x^\mu - \eta^{\mu\lambda}x^\nu)\mathcal{L}. \tag{31}$$

If you managed to derive this current without looking at this solution, I am impressed.