FY3464 Quantum Field Theory
Problemset 1

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SUGGESTED SOLUTION

Problem 1
We require that

$$(x')^{\mu}(x')_{\mu} = x^{\alpha}x_{\alpha}. \tag{1}$$

Starting with the lhs, we get:

$$(x')^{\mu}(x')_{\mu} = \eta_{\mu\nu}(x')^{\mu}(x')^{\nu} = \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}x^{\alpha}x^{\beta}. \tag{2}$$

Consider now the rhs, which equals

$$x^{\alpha}x_{\alpha} = \eta_{\alpha\beta}x^{\alpha}x^{\beta}. \tag{3}$$

Comparing the two equations, it is clear that

$$\eta_{\alpha\beta} = \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}. \tag{4}$$

As for the inverse, we note that Eq. (4) can be written as

$$\eta_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\Lambda_{\mu\beta}. \tag{5}$$

Multiply this by $\eta^{\beta\gamma}$ (and sum over β, as indicated by the repeated index):

$$\delta^{\gamma}_{\alpha} = \Lambda^{\mu}{}_{\alpha}\Lambda_{\mu}{}^{\gamma}. \tag{6}$$

Comparing this with the definition of the inverse given in the problem text, we see that

$$\Lambda_{\mu}{}^{\gamma} = (\Lambda^{-1})^{\gamma}{}_{\mu}. \tag{7}$$

which is what we set out to prove (after renaming indices).
Problem 2
We get:

$$\begin{aligned}
H|\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots\rangle &= \int \frac{d^3k}{(2\pi)^3}\omega(\boldsymbol{k})a^{\dagger}_{\boldsymbol{k}}a_{\boldsymbol{k}}\Big(\prod_{i=1}^{N}\sqrt{2\omega(\boldsymbol{k}_i)}a^{\dagger}_{\boldsymbol{k}_i}\Big)|0\rangle \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\omega(\boldsymbol{k})\sqrt{2\omega(\boldsymbol{k}_1)}\sqrt{2\omega(\boldsymbol{k}_2)}\ldots\times a^{\dagger}_{\boldsymbol{k}}[a^{\dagger}_{\boldsymbol{k}_1}a_{\boldsymbol{k}} + (2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}_1)]a^{\dagger}_{\boldsymbol{k}_2}\ldots|0\rangle \\
&= \omega(\boldsymbol{k}_1)|\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots\rangle + \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\omega(\boldsymbol{k})a^{\dagger}_{\boldsymbol{k}}\sqrt{2\omega(\boldsymbol{k}_1)}\sqrt{2\omega(\boldsymbol{k}_2)}\ldots\times a^{\dagger}_{\boldsymbol{k}_1}[a^{\dagger}_{\boldsymbol{k}_2}a_{\boldsymbol{k}} + (2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}_2)]a^{\dagger}_{\boldsymbol{k}_3}\ldots|0\rangle \\
&= \omega(\boldsymbol{k}_1)|\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots\rangle + \omega(\boldsymbol{k}_2)|\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots\rangle + \ldots \\
&= \sum_{i=1}^{N}\omega(\boldsymbol{k}_i)|\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots\rangle
\end{aligned} \tag{8}$$

Problem 3

Writing out the Poisson brackets explicitly, we have

$$\dot{\phi}(\boldsymbol{x}) = \int d\boldsymbol{x}' \Big(\frac{\delta\phi(\boldsymbol{x})}{\delta\phi(\boldsymbol{x}')} \frac{\delta H}{\delta\Pi(\boldsymbol{x}')} - \frac{\delta\phi(\boldsymbol{x})}{\delta\Pi(\boldsymbol{x}')} \frac{\delta H}{\delta\phi(\boldsymbol{x}')} \Big),$$
$$\dot{\Pi}(\boldsymbol{x}) = \int d\boldsymbol{x}' \Big(\frac{\delta\Pi(\boldsymbol{x})}{\delta\phi(\boldsymbol{x}')} \frac{\delta H}{\delta\Pi(\boldsymbol{x}')} - \frac{\delta\Pi(\boldsymbol{x})}{\delta\Pi(\boldsymbol{x}')} \frac{\delta H}{\delta\phi(\boldsymbol{x}')} \Big). \quad (9)$$

To compute these expressions, we need $\mathcal{H}$:

$$\mathcal{H} = \Pi\dot{\phi} - \mathcal{L} = \frac{1}{2}[\Pi^2 + (\partial_i\phi)^2 + 2\zeta\phi^2], \quad (10)$$

where we used that $\Pi = \partial_0\phi = \partial^0\phi$. Since $H = \int d\boldsymbol{x}\mathcal{H}(\boldsymbol{x})$, we obtain:

$$\frac{\delta H}{\delta\Pi(\boldsymbol{x}')} = \Pi(\boldsymbol{x}'),$$
$$\frac{\delta H}{\delta\phi(\boldsymbol{x}')} = \frac{\partial\mathcal{H}}{\partial\phi(\boldsymbol{x}')} - \nabla\cdot\frac{\partial\mathcal{H}}{\partial[\nabla\phi(\boldsymbol{x}')]} = 2\zeta\phi(\boldsymbol{x}') + \partial_i\partial^i\phi(\boldsymbol{x}'). \quad (11)$$

Note that ∇ has components $\partial_i = \frac{\partial}{\partial x^i}$, in effect it is a covariant derivative which thus differentiates with respect to contravariant variables. We also used that $\partial_i = -\partial^i$. The Poisson brackets in Eq. (9) then take the form:

$$\dot{\phi}(\boldsymbol{x}) = \pi(\boldsymbol{x}),$$
$$\dot{\Pi}(\boldsymbol{x}) = -2\zeta\phi(\boldsymbol{x}) - \partial_i\partial^i\phi(\boldsymbol{x}). \quad (12)$$

Eliminating Π from these equations, we arrive at the equation of motion:

$$\ddot{\phi} + \partial_i\partial^i\phi + 2\zeta\phi = 0 \quad (13)$$

or alternatively

$$\partial_\mu\partial^\mu\phi + 2\zeta\phi = 0. \quad (14)$$