FY3464 Quantum Field Theory
Problemset 8

NTNU
Institutt for fysikk

Problem 1
By applying N successive infinitesimal Lorentz transformations, letting $N \to \infty$, and using

$$\frac{\mathrm{i}}{2}\varepsilon_{\nu\lambda}[S^{\nu\lambda},\gamma^{\mu}] = -\varepsilon_{\nu}{}^{\mu}\gamma^{\nu}, \tag{1}$$

show that

$$\Lambda^{\mu}_{\nu}\gamma^{\nu} = U_{\gamma}^{-1}(\Lambda)\gamma^{\mu}U_{\gamma}(\Lambda), \tag{2}$$

with $U_{\gamma}(\Lambda) = \mathrm{e}^{\mathrm{i}\omega_{\mu\nu}S^{\mu\nu}/2}$ and where Λ^{μ}_{ν} now represents a finite Lorentz transformation. How is $\omega_{\mu\nu}$ related to the N and the infinitesimal Lorentz transformation?

Problem 2
In the lectures, we showed that the solution for the Dirac equation for $\boldsymbol{k} \neq 0$ can be obtained via a boost matrix $U_{\gamma}(\Lambda)$. Assuming that the boost-direction was $+\hat{\boldsymbol{z}}$, we found that

$$u(\boldsymbol{k}) = U_{\gamma}(\Lambda)\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix} \tag{3}$$

where ξ is an arbitrary 2×1 spinor and

$$U_{\gamma}(\Lambda) = \mathrm{e}^{-\mathrm{i}\eta S^{03}} = \exp\Big[-\frac{1}{2}\eta\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\Big] \tag{4}$$

Show that the solution

$$u(\boldsymbol{k}) = \begin{pmatrix}\sqrt{k\cdot\sigma}\xi\\ \sqrt{k\cdot\bar{\sigma}}\xi\end{pmatrix} \tag{5}$$

follows from the above equations and that it is valid for any boost direction, in effect any direction of $\boldsymbol{k}$. Above, $\sqrt{A}$ should be understood as a matrix whose eigenvalues are the square root of the eigenvalues of A.

Hint: $U_{\gamma}(\Lambda)$ is a diagonal matrix. Square its blocks to see what it produces. It is also useful to note that $k\cdot\sigma$ is a Hermitian (self-adjoint) matrix. By the way, this problem is tough - don't despair if you can't solve it entirely. A key purpose with this problem is to make you think hard about how you can approach it.