Multimodality in the Kalman Filter and Ensemble Kalman Filter

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Kalman Model

Process model’s assumptions:

1. Gaussian initial distribution \( f(x_1) \)
2. Single site dependence and conditional independence
3. Gauss-linear forward and likelihood model:

\[
f(x_{t+1}|x_t) = \mathcal{N}(x_{t+1}, B x_t, \Sigma_{x|x}) \\
f(d_t|x_t) = \mathcal{N}(d_t, H x_t, \Sigma_{d|x})
\]
Kalman Model

Properties:
1. Analytically tractable, conjugate prior
2. Models linear unimodal processes
Selection Gaussian distribution

Let \( A \subset \mathbb{R}^q \), and

\[
\begin{bmatrix}
  x_0 \\
  \nu
\end{bmatrix}
\sim \mathcal{N}_{p+q} \left( \begin{bmatrix}
  x_0 \\
  \nu
\end{bmatrix} ; \mu = \begin{bmatrix}
  \mu_x_0 \\
  \mu_\nu
\end{bmatrix}, \Sigma = \begin{bmatrix}
  \Sigma_{11} & \Sigma_{12} \\
  \Sigma_{21} & \Sigma_{22}
\end{bmatrix} \right)
\]

then \( x_{0,A} = [x_0 | \nu \in A] \) is Selection Gauss.

**Flexibility**

1. Skewness
2. Multimodality
3. Conjugate prior to a Gauss-linear likelihood

**Bimodality**
Selection Gaussian Kalman Model

Model’s assumptions:

1. Selection Gaussian initial distribution \( f(x_1) \)
2. Single site response and conditional independence
3. Gauss-linear forward and likelihood model:

\[
\begin{align*}
    f(x_{t+1} | x_t) &= \mathcal{N}(x_{t+1}; Bx_t, \Sigma_x | x) \\
    f(d_t | x_t) &= \mathcal{N}(d_t; Hx_t, \Sigma_d | x)
\end{align*}
\]
Selection Gaussian Kalman Model

Properties

1. Analytically tractable
2. Models multimodality
3. Easy to implement

Marginal smoothing distribution

![Smoothing distribution graph with temperature in Celsius on the x-axis and density on the y-axis. The distribution shows two peaks, indicating multimodality.]
We start with $\begin{bmatrix} x_1 \\ \nu \end{bmatrix}$ that is Gaussian

We increment (update) to $\begin{bmatrix} x_1 \\ \nu \\ d_1 \end{bmatrix}$ that is still Gaussian

We increment (forward) to $\begin{bmatrix} x_2 \\ x_1 \\ \nu \\ d_1 \end{bmatrix}$

etc ...
Implementation

This allows for:

1. Access to filtering and smoothing
2. Fast computation
3. Keep a Gaussian structure
Example: Backtracking the 2D Heat equation

The heat equation:

\[ \frac{\partial T}{\partial t} - \nabla^2 T = 0 \]
\[ \nabla T \cdot n = 0 \]

Modelled using finite differences on \([0, 1] \times [0, 1]\), it gives the following Gauss-linear forward model:

\[ f(T_{t+1}|T_t) = \varphi_p(T_{t+1}, BT_t, \Sigma_T|T) \] (1)

Data is collected at 5 different locations using the following Gauss-linear likelihood model:

\[ f(d_t|T_t) = \varphi_p(d_t, HT_t, \Sigma_d|T) \] (2)
Example: Backtracking the 2D Heat equation

Initial Heat map

Initial Marginal distribution

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Example: Backtracking the 2D Heat equation

Data collected (outer point) Vs True process

Data collection
1. Data collected at every time step \( (dt = 3s) \)
2. Small variance \( \sum_{d|x} = 0.5I \)
Example: Backtracking the 2D Heat equation

MMAP of the smoothing distribution $[x_1|d_1, \ldots, d_T]$
Example: Backtracking the 2D Heat equation

Now only considering $[x_1|d_1, \ldots, d_T]$ for the Sel-Gauss initial distribution:
Example: Backtracking the 2D Heat equation

Same data collection points, new heat map:

Sel-Gauss initial distribution

Gaussian initial distribution
Ongoing work: EnKF with Sel-Gauss initial distribution

1. Working algorithm.
2. Numerical convergence to the Selection Gaussian Kalman Filter when $n_e \to \infty$.
3. Test it on the transport equation for an oil slick.