II. Governing Equations

by Bernhard Müller

Outline

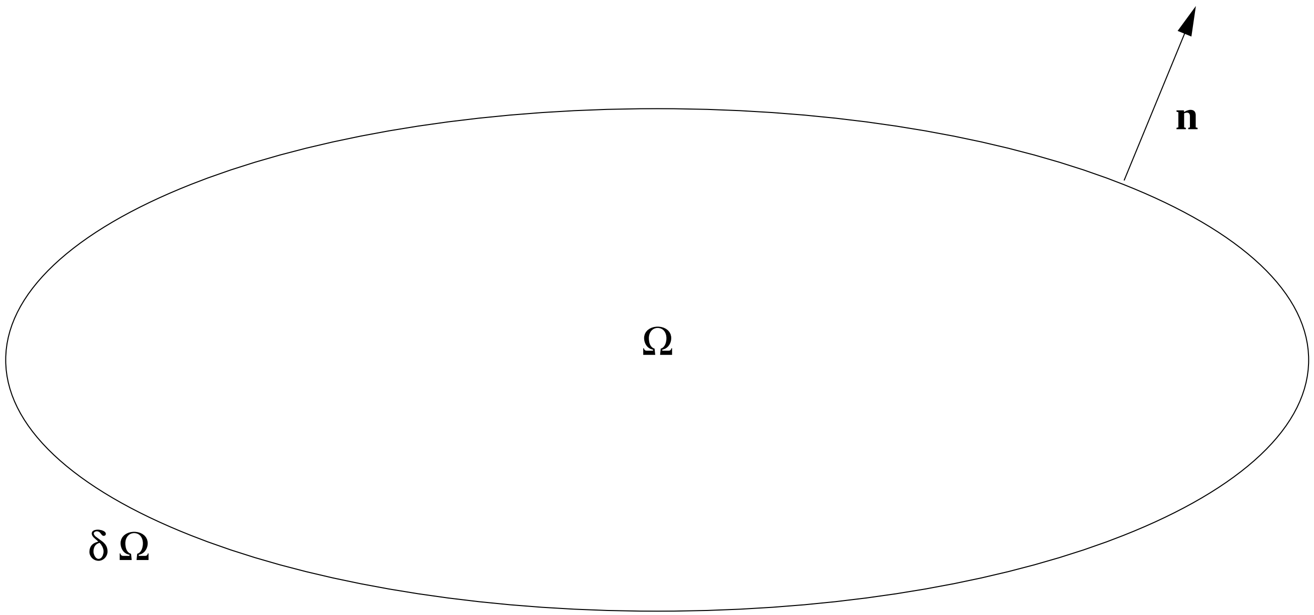
The conservation laws for mass, momentum and energy of a compressible fluid flow are derived. In CFD, this mixed hyperbolic-parabolic system is called the Navier-Stokes equations. We consider perfect gas and discuss different forms of the equations.

For inviscid flow, the Navier-Stokes equations simpify to the hyperbolic Euler equations. The characteristic relations and Riemann invariants normal to a boundary allow the formulation of boundary conditions at artificial boundaries and will be used in module V. to derive Riemann solvers. Simplified forms of the Euler equations like the potential and wave equations are derived.

1. Navier-Stokes Equations

We derive the Navier-Stokes Equations, i.e. the conservation laws for mass, momentum and energy.

In a fluid flow, consider a time dependent control volume $\Omega(t)$ moving with the fluid velocity $\mathbf{u}$. Denote the boundary by $\partial\Omega$ and the outer unit normal by $\mathbf{n}$.



1.1. Continuity Equation

Conservation of mass:
total rate of mass change in $\Omega(t)$ is zero, i.e.

$$\frac{d\mathcal{M}}{dt} = \frac{d}{dt}\int_{\Omega(t)} \rho \, dV = 0 \, ,$$

where $\mathcal{M}$ is the mass in $\Omega(t)$, t time and ρ density.

Mathematically identical and valid also for a *stationary* control volume Ω:
rate of mass change in Ω + mass flow over $\partial\Omega = 0$, i.e.

$$\int_{\Omega} \frac{\partial \rho}{\partial t} dV + \int_{\partial\Omega} \rho \mathbf{u} \cdot \mathbf{n} \, dA = 0 \, . \tag{2}$$

1.2. Momentum Equation

Newton's second law of motion:
total rate of momentum change in $\Omega(t)$ is equal to sum of acting forces $\mathcal{K}$, i.e.

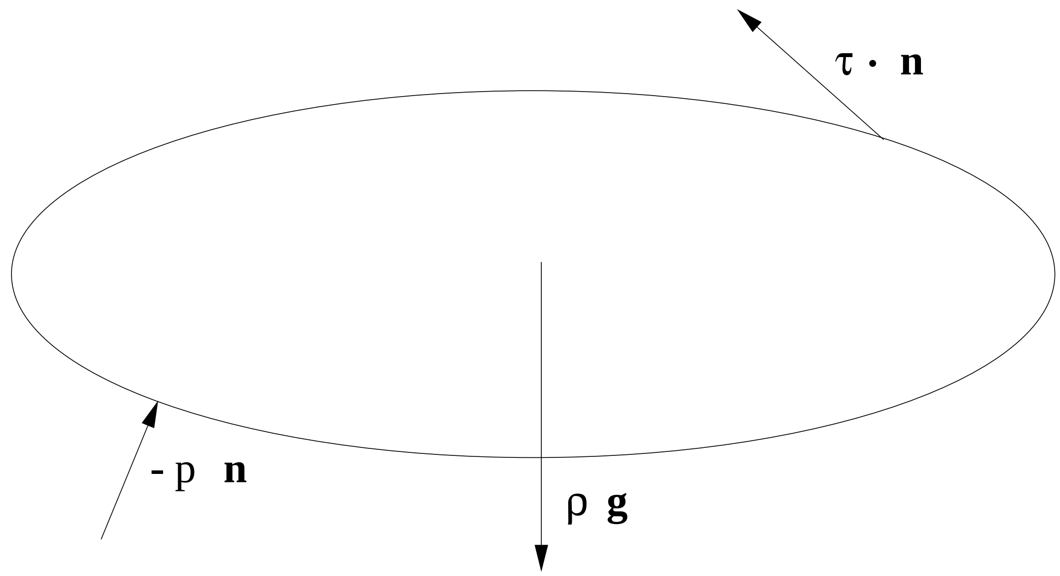
$$\frac{d\mathbf{m}}{dt} = \frac{d}{dt}\int_{\Omega(t)} \rho\mathbf{u}\, dV = \mathcal{K} ,$$

where $\mathbf{m}$ is the momentum in $\Omega(t)$.

Thus, for a *stationary* control volume Ω considering pressure, viscous and volume forces:
rate of momentum change in Ω + momentum flow over $\partial\Omega$ =
pressure and viscous forces on $\partial\Omega$ + volume force on Ω , i.e.

$$\int_{\Omega} \frac{\partial \rho \mathbf{u}}{\partial t} dV + \int_{\partial\Omega} \rho \mathbf{u}\, \mathbf{u} \cdot \mathbf{n}\, dA \;=\; -\int_{\partial\Omega} p\, \mathbf{n}\, dA + \int_{\partial\Omega} \boldsymbol{\tau} \cdot \mathbf{n}\, dA + \int_{\Omega} \rho \mathbf{f} dV \;, \tag{4}$$

where p pressure, $\boldsymbol{\tau}$ shear stress tensor and $\rho\mathbf{f}$ external force density, e.g. $\rho\mathbf{g}$ for gravity.



Surface and volume forces

Shear Stress Tensor of a Newtonian Fluid

$$\boldsymbol{\tau} \;=\; \mu[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \;-\; \frac{2}{3}\mu \nabla \cdot \mathbf{u}\, \mathbf{I} \tag{5}$$

with the (dynamic) viscosity μ and assuming Stokes' hypothesis $\mu' = -\frac{2}{3}\mu$ for the second viscosity coefficient μ'.

Sutherland's formula yields μ in SI units:

$$\mu = C_1 \frac{T^{3/2}}{T + C_2}\;, \tag{6}$$

where $C_1 = 1.458 \times 10^{-6} \frac{kg}{ms\sqrt{K}}$ and $C_2 = 110.4K$ for air at moderate temperatures T.

$\mathbf{I} = (\delta_{\mathbf{ij}})$ is the unit tensor defined by the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

In Cartesian coordinates $(x_i) = (x, y, z)^T$:

$$\nabla \mathbf{u} = (\frac{\partial u_j}{\partial x_i})$$

and

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \; ,$$

where $\mathbf{u} = (u_i) = (u, v, w)^T$ and $\nabla = (\frac{\partial}{\partial x_i}) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^T$.

1.3. Energy Equation

First law of thermodynamics:
total rate of total energy change in $\Omega(t)$ is equal to the rate of work $\mathcal{L}$ done on the fluid by the acting forces $\mathcal{K}$ plus the rate of heat added $\mathcal{W}$, i.e.

$$\frac{d\mathcal{E}}{dt} \;=\; \frac{d}{dt}\int_{\Omega(t)} \rho E \, dV \;=\; \mathcal{L} + \mathcal{W}\,,$$

where $\mathcal{E}$ is the total energy in $\Omega(t)$ and ρE the total energy per unit volume. Thus, for a *stationary* control volume Ω:

rate of total energy change in Ω $+$ total energy flow over $\partial\Omega$ $=$ rate of work of pressure and viscous forces on $\partial\Omega$ $+$ rate of work of volume force on Ω $+$ the rate of heat added over $\partial\Omega$, i.e.

$$\int_\Omega \frac{\partial \rho E}{\partial t} dV \; + \; \int_{\partial\Omega} \rho E \, \mathbf{u} \cdot \mathbf{n} \, dA \; = $$
$$- \int_{\partial\Omega} p \, \mathbf{u} \cdot \mathbf{n} \, dA \; + \; \int_{\partial\Omega} (\boldsymbol{\tau} \cdot \mathbf{u}) \cdot \mathbf{n} \, dA \; + \; \int_\Omega \rho \mathbf{f} \cdot \mathbf{u} \, dV \; - \; \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, dA \quad (8)$$

The specific total energy E is the sum of internal energy e and kinetic energy:

$$E \; = \; e \; + \; \frac{1}{2} |\mathbf{u}|^2 \, .$$

Fourier's heat conduction law yields the heat flux $\mathbf{q}$ as

$$\mathbf{q} \; = \; -k \boldsymbol{\nabla} T \, ,$$

where k is the thermal conductivity. For constant Prandtl number $Pr \; = \; \frac{c_p \mu}{k}$, where c_p is the specific heat at constant pressure, e.g. $Pr \; = \; 0.72$ for air at standard conditions, k is obtained from

$$k \; = \; \frac{c_p}{Pr} \, \mu \, .$$

1.4 Equations of State for Perfect Gas

For perfect gas, pressure p, density ρ, temperature T and internal energy e are related by the equations of state

$$p = \rho R T \tag{9}$$

$$e = c_v T \,. \tag{10}$$

$R = c_p - c_v$ is the specific gas constant, c_p and c_v the (constant) specific heats at constant pressure and volume, respectively.

$\gamma = \frac{c_p}{c_v}$ is the ratio of specific heats.

$\gamma = 1.4$ and $R = 287 \frac{m^2}{s^2 K}$ for air at standard conditions.

Thus, $c_v = \frac{R}{\gamma - 1} = 717.5 \frac{m^2}{s^2 K}$ and $c_p = \frac{\gamma R}{\gamma - 1} = 1004.5 \frac{m^2}{s^2 K}$.

With the equations of state and the definition of E, we get

$$p = (\gamma - 1)\,\rho e = (\gamma - 1)\,[\rho E - \frac{1}{2}\rho|\mathbf{u}|^2]\,. \tag{11}$$

Using $\mathbf{u} = \frac{\rho\mathbf{u}}{\rho}$ in (11), p is a function of the conservative variables

$$\mathbf{U} = (\rho, \rho\mathbf{u}, \rho E)^T\,. \tag{12}$$

With similar arguments, $T = T(\mathbf{U})$.

Thus, all flow variables in the conservation laws (2), (4) and (8), the Navier-Stokes equations, can be expressed as functions of the conservative variables $\mathbf{U}$.

Note that $\int_\Omega \mathbf{U}\,dV$ is the vector of mass, momentum and energy in control volume Ω.

1.5 Forms of Navier-Stokes Equations
1.5.1. Integral Form

Defining the inviscid (i.e. convective) and viscous normal flux vectors by

$$\mathbf{F}^c \cdot \mathbf{n} = \begin{pmatrix} \rho \mathbf{u} \cdot \mathbf{n} \\ \rho \mathbf{u}\, \mathbf{u} \cdot \mathbf{n} + p \mathbf{I} \cdot \mathbf{n} \\ (\rho E + p)\, \mathbf{u} \cdot \mathbf{n} \end{pmatrix}, \mathbf{F}^v \cdot \mathbf{n} = \begin{pmatrix} 0 \\ \boldsymbol{\tau} \cdot \mathbf{n} \\ (\boldsymbol{\tau} \cdot \mathbf{u}) \cdot \mathbf{n} + k(\boldsymbol{\nabla} T) \cdot \mathbf{n} \end{pmatrix},$$

the flux tensor by $\mathbf{F} = \mathbf{F}^c - \mathbf{F}^v$ and the (external) source strength vector by $\mathbf{F}_e = (0, \mathbf{f}, \mathbf{f} \cdot \mathbf{u})^T$, the Navier-Stokes equations in integral form read:

$$\int_\Omega \frac{\partial \mathbf{U}}{\partial t} dV + \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, dA = \int_\Omega \rho \mathbf{F_e}\, dV. \tag{13}$$

1.5.2. Conservative and Non-Conservative Forms

Provided the flux tensor $\mathbf{F}$ is differentiable, we obtain from (13) with the Gauss theorem

$$\int_\Omega \left(\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{F} - \rho \mathbf{F}_e\right) dV = 0 .$$

Since this integral equation is valid for arbitrary control volumes Ω, we obtain the differential conservative form of the Navier-Stokes equations

$$\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{F} = \rho \mathbf{F}_e . \tag{14}$$

In Cartesian coordinates $(x_i) = (x, y, z)^T$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial(\mathbf{F}_1^c - \mathbf{F}_1^v)}{\partial x_1} + \frac{\partial(\mathbf{F}_2^c - \mathbf{F}_2^v)}{\partial x_2} + \frac{\partial(\mathbf{F}_3^c - \mathbf{F}_3^v)}{\partial x_3} = \rho \mathbf{F}_e \,, \qquad (15)$$

where

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E \end{pmatrix}, \; \mathbf{F}_j^c = \begin{pmatrix} \rho u_j \\ \rho u_1 u_j + p\delta_{1j} \\ \rho u_2 u_j + p\delta_{2j} \\ \rho u_3 u_j + p\delta_{3j} \\ \rho H u_j \end{pmatrix}, \; \mathbf{F}_e = \begin{pmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \\ \mathbf{f} \cdot \mathbf{u} \end{pmatrix},$$

where $H = E + \frac{p}{\rho}$ is the total enthalpy, $(u_i) = (u, v, w)^T$,

$$\mathbf{F}_j^v = \begin{pmatrix} 0 \\ \tau_{1j} \\ \tau_{2j} \\ \tau_{3j} \\ \tau_{j1}u_1 + \tau_{j2}u_2 + \tau_{j3}u_3 + k\frac{\partial T}{\partial x_j} \end{pmatrix},$$

with $\tau_{ij} = \mu(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}) - \frac{2}{3}\mu(\sum_{l=1}^{3} \frac{\partial u_l}{\partial x_l})\delta_{ij}$.

With the product rule, the continuity and momentum equations and the equations of state, we can derive from (14) the following non-conservative form of the Navier-Stokes equations:

$$
\begin{aligned}
\frac{D\rho}{Dt} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{u} &= 0\,, \\
\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\boldsymbol{\nabla}p &= \frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \mathbf{f}\,, \\
\frac{Dp}{Dt} + \gamma p\boldsymbol{\nabla}\cdot\mathbf{u} &= (\gamma-1)[(\boldsymbol{\tau}\cdot\boldsymbol{\nabla})\cdot\mathbf{u} + \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}T)]\,,
\end{aligned}
\tag{16}
$$

where

$$
\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\rho
$$

denotes the substantial derivative of ρ, etc.

1.6. Boundary Conditions

1.6.1. Solid Wall

If a solid wall is moving with the velocity $\mathbf{v}$, the no-slip boundary condition states that the fluid velocity at the wall coincides with the wall velocity, i.e. $\mathbf{u}_w = \mathbf{v}$. Thus, for a stationary solid wall

$$\mathbf{u}_w \;=\; 0\,,$$

Usually, either the wall temperature or the normal wall heat flux are given.

Example isothermal wall with constant temperature *const*:

$$T_w = const\,.$$

Example adiabatic wall, i.e. $(\mathbf{q} \cdot \mathbf{n})_w = 0$:

$$\frac{\partial T_w}{\partial n} = 0 \, .$$

A boundary condition for the wall pressure is not needed, because the wall pressure can be determined by means of the normal momentum equation, e.g. for a stationary solid wall:

$$\frac{\partial p_w}{\partial n} \; = \; [(\boldsymbol{\nabla} \cdot \boldsymbol{\tau}) \cdot \mathbf{n} \, + \, \rho \mathbf{f} \cdot \mathbf{n}]_w$$

Without external forces, i.e. $\mathbf{f} = 0$, and for high Reynolds numbers, the wall normal momentum equation can be simplified to the boundary layer approximation $\frac{\partial p_w}{\partial n} \; = \; 0$, except for the neighbourhood of separation and reattachment points.

1.6.2. Inflow and Outflow

At inflow and outflow boundaries, the energy method yields well-posed boundary conditions for the Navier-Stokes equations, cf.
B. Gustafsson, A. Sundström, "Incompletely Parabolic Problems in Fluid Dynamics", SIAM J. Appl. Math, Vol. 35, No. 2, Sept. 1978, pp. 343-357.

However, since these rigorous boundary conditions require information often not available, the artificial boundaries are usually placed in regions where simplified boundary conditions, e.g. for the Euler or boundary layer equations, can be employed.

1.7. Nondimensional Form

The nondimensional form of the Navier-Stokes equations

- yields conditions for similar flow patterns, e.g. for airplane and windtunnel model,
- allows the derivation of simplified equations.

Choose reference conditions, e.g. u_∞, ρ_∞, T_∞, μ_∞, k_∞, and a characteristic length scale L. For external and internal flow, freestream and stagnation conditions, respectively, are often chosen. Then, define the nondimensional flow varaiables. For example:

$$\mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad t^* = \frac{t}{L/u_\infty},$$

$$\mathbf{u}^* = \frac{\mathbf{u}}{u_\infty}, \quad \rho^* = \frac{\rho}{\rho_\infty}, \quad p^* = \frac{p}{\rho_\infty u_\infty^2}, \quad T^* = \frac{T}{T_\infty}, \quad e^* = \frac{e}{u_\infty^2}, \quad E^* = \frac{E}{u_\infty^2}.$$

If we then define (for zero source term $\mathbf{f} = 0$)

$$\mu^* = \frac{1}{Re_{\infty L}} \frac{\mu}{\mu_\infty}, \quad k^* = \frac{1}{(\gamma - 1) M_\infty^2 Pr_\infty Re_{\infty L}} \frac{k}{k_\infty},$$

the nondimensional equations take the same form as the corresponding dimensional equations, e.g. (15) with nondimensional variables ρ^* etc. instead of the dimensional variables ρ etc.

The reference Reynolds, Mach and Prandtl numbers are defined by

$$Re_{\infty L} = \frac{\rho_\infty u_\infty L}{\mu_\infty}, \quad M_\infty = \frac{u_\infty}{c_\infty}, \quad Pr_\infty = \frac{c_p \mu_\infty}{k_\infty}$$

with $c_\infty = \sqrt{\gamma \frac{p_\infty}{\rho_\infty}} = \sqrt{\gamma R T_\infty}$ the reference speed of sound.

2. Euler Equations

If viscous stresses and heat conduction are neglected, i.e. $\mu \equiv 0$ and $k \equiv 0$, the Navier-Stokes equations degenerate to the Euler equations.

2.1. Integral and Differential Forms

Integral form of the Euler equations (cf. (13)):

$$\int_{\Omega} \frac{\partial \mathbf{U}}{\partial t} dV + \int_{\partial\Omega} \mathbf{F}^c \cdot \mathbf{n}\, dA = \int_{\Omega} \rho \mathbf{F_e}\, dV, \tag{17}$$

where the conservative variables $\mathbf{U}$ and the inviscid flux tensor $\mathbf{F}^c$ are

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\mathbf{u} \\ \rho E \end{pmatrix}, \mathbf{F}^c(\mathbf{U}) = \begin{pmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\,\mathbf{u} + p\mathbf{I} \\ \rho H\,\mathbf{u} \end{pmatrix}.$$

Differential conservative form of the Euler equations (cf. (14)):

$$\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{F}^c \;=\; \rho \mathbf{F}_e \,. \tag{18}$$

In Cartesian coordinates $(x_i) = (x, y, z)^T$ (cf. (15)):

$$\frac{\partial \mathbf{U}}{\partial t} \;+\; \frac{\partial \mathbf{F}_1^c}{\partial x_1} \;+\; \frac{\partial \mathbf{F}_2^c}{\partial x_2} \;+\; \frac{\partial \mathbf{F}_3^c}{\partial x_3} \;=\; \rho \mathbf{F}_e \,,$$

$$\mathbf{F}_1^c = \begin{pmatrix} \rho u \\ \rho u u + p \\ \rho v u \\ \rho w u \\ \rho H u \end{pmatrix}, \; \mathbf{F}_2^c = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v v + p \\ \rho w v \\ \rho H v \end{pmatrix}, \; \mathbf{F}_3^c = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w w + p \\ \rho H w \end{pmatrix}.$$

Non-conservative form of the Euler equations (cf. (16)):

$$
\begin{aligned}
\frac{D\rho}{Dt} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{u} &= 0\,, \\
\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\boldsymbol{\nabla} p &= \mathbf{f}\,, \\
\frac{Dp}{Dt} + \gamma p \boldsymbol{\nabla}\cdot\mathbf{u} &= 0\,.
\end{aligned}
\tag{19}
$$

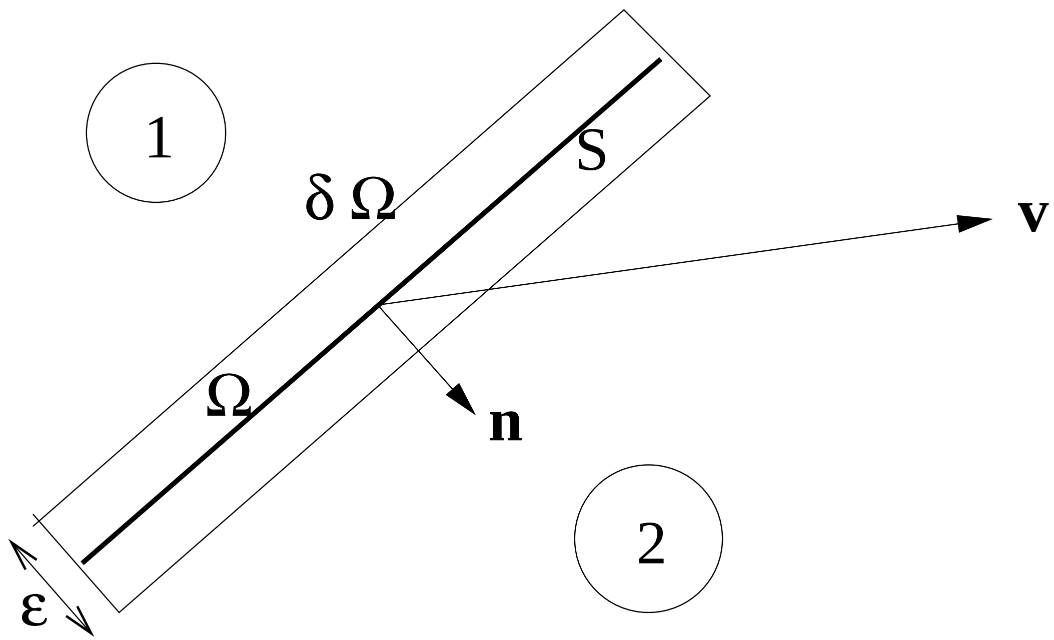
2.2. Rankine-Hugoniot Relations

Opposed to the Navier-Stokes equations, the Euler equations allow for discontinuities, namely shocks and contact discontinuities.

Suppose a flow discontinuity is propagating with the constant velocity $\mathbf{v}$. Consider a control volume Ω containing the discontinuity and moving with that velocity $\mathbf{v}$. The Euler equations (17) become:

$$\int_\Omega \frac{\partial \mathbf{U}}{\partial t} dV + \int_{\partial\Omega} (\mathbf{U}\,(\mathbf{u} - \mathbf{v}) + \mathbf{P}) \cdot \mathbf{n}\, dA = \int_\Omega \rho \mathbf{F_e}\, dV\,, \tag{20}$$

where $\mathbf{P} = p \begin{pmatrix} 0 \\ \mathbf{I} \\ \mathbf{u} \end{pmatrix}$. Note that $\mathbf{F}^c = \mathbf{U}\,\mathbf{u} + \mathbf{P}$.



Control volume Ω moving with velocity $\mathbf{v}$ of flow discontinuity S.

Let the front faces shrink to zero, i.e. $\epsilon \longrightarrow 0$. Then, the volume integrals in (20) become zero, and the boundary $\partial\Omega$ becomes the upstream (state (1)) and downstream (state (2)) sides of the discontinuity S.

Since $\mathbf{n}_1 = -\mathbf{n}_2$, the Euler equations (20) reduce to

$$\int_S [\mathbf{U}\,(\mathbf{u} - \mathbf{v}) \;+\; \mathbf{P}] \cdot \mathbf{n}\, dA \;=\; 0\,,$$

where $[\mathbf{u}] = \mathbf{u}_2 - \mathbf{u}_1$ denotes the jump of $\mathbf{u}$ across the discontinuity, etc. As that relation holds for any surface S along the discontinuity, we obtain the Rankine-Hugoniot relations:

$$[\mathbf{U}\,\mathbf{u} \;+\; \mathbf{P}] \cdot \mathbf{n} \;=\; [\mathbf{U}]\,\mathbf{v} \cdot \mathbf{n}\,. \tag{21}$$

For a stationary discontinuity, $\mathbf{v} = 0$ and the Rankine-Hugoniot relations (21) simplify to

$$[\mathbf{U}\,\mathbf{u} \;+\; \mathbf{P}] \cdot \mathbf{n} \;=\; 0\,. \tag{22}$$

If the discontinuity is moving with the velocity $\mathbf{v}$ in the inertial frame of reference, the discontinuity is stationary in the frame of reference moving with $\mathbf{v}$. The velocities in the moving and inertial frames are related by $\mathbf{u}_{movingframe} = \mathbf{u}_{inertialframe} - \mathbf{v}$.

The Rankine-Hugoniot relations for a stationary discontinuity (22) can be expressed as

$$[\rho\,\mathbf{u}\cdot\mathbf{n}] = 0\,,$$

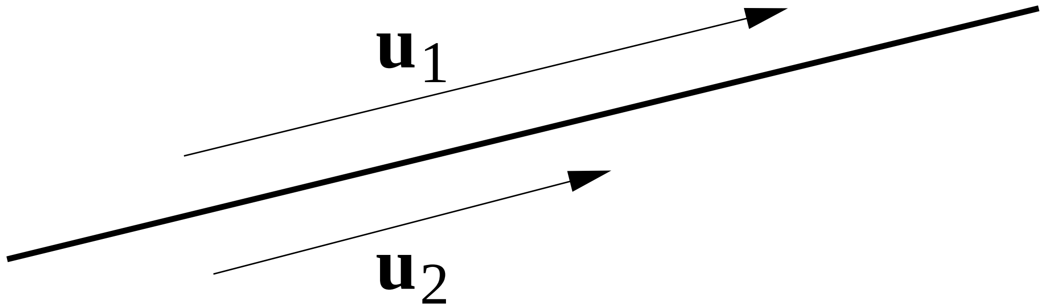
$$[\mathbf{u}]\rho\,\mathbf{u}\cdot\mathbf{n}\, + \,[p]\mathbf{n} = 0\,,$$

$$[H]\rho\,\mathbf{u}\cdot\mathbf{n} = 0\,.$$

If there is no mass flow through the discontinuity, i.e. $\rho\,\mathbf{u}\cdot\mathbf{n} = 0$, it is a contact discontinuity:

$$[u_n] = 0, \quad [p] = 0, \quad [s] \neq 0, \qquad \text{but in general } [\rho] \neq 0, \quad [u_t] \neq 0, \quad [H] \neq 0\,,$$

$u_n = \mathbf{u}\cdot\mathbf{n}$ and $u_t = \mathbf{u} - u_n\mathbf{n}$ normal and tangential velocities, resp.

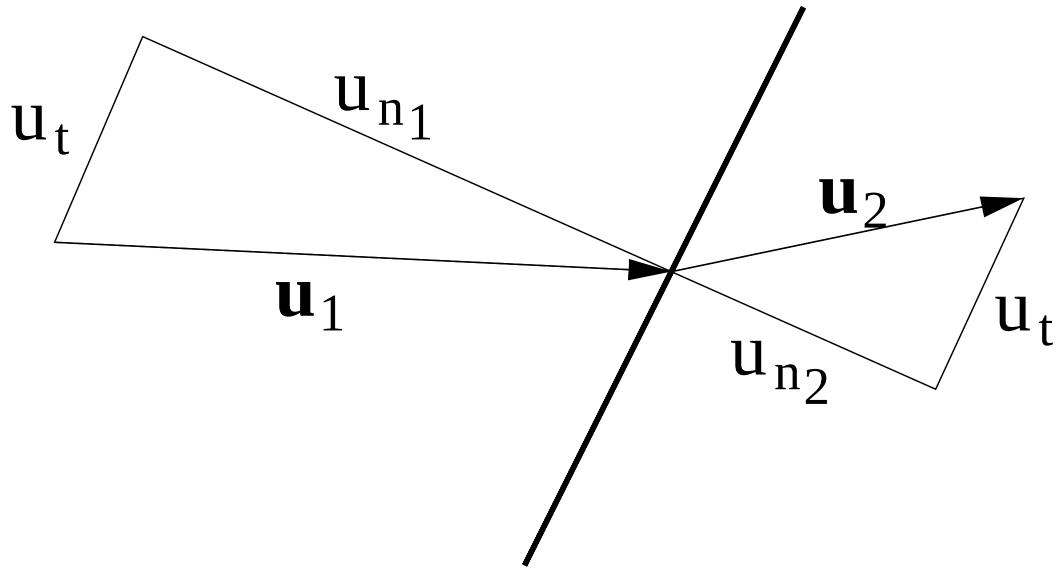


Contact discontinuity.

For mass flow through the discontinuity, i.e. $\rho\,\mathbf{u}\cdot\mathbf{n} \neq 0$, it is a shock:

$$[u_t] = 0, \quad [H] = 0, \quad [u_n] < 0, \quad [\rho] > 0, \quad [p] > 0, \quad [s] > 0\,,$$

where s is entropy.



Shock.

2.3. Hyperbolic System

We consider the non-conservative form of the Euler equations (19) in Cartesian coordinates $(x_i) = (x, y, z)^T$ ($\mathbf{V}$ primitive variables):

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{B}_1 \frac{\partial \mathbf{V}}{\partial x_1} + \mathbf{B}_2 \frac{\partial \mathbf{V}}{\partial x_2} + \mathbf{B}_3 \frac{\partial \mathbf{V}}{\partial x_3} = \mathbf{G}_e \tag{23}$$

where $\mathbf{V} = (\rho, u_1, u_2, u_3, p)^T$, $\mathbf{G}_e = (0, f_1, f_2, f_3, 0)^T$,

$$k_1\mathbf{B}_1 + k_2\mathbf{B}_2 + k_3\mathbf{B}_3 = \begin{pmatrix} \mathbf{k}\cdot\mathbf{u} & k_1\rho & k_2\rho & k_3\rho & 0 \\ 0 & \mathbf{k}\cdot\mathbf{u} & 0 & 0 & \frac{k_1}{\rho} \\ 0 & 0 & \mathbf{k}\cdot\mathbf{u} & 0 & \frac{k_2}{\rho} \\ 0 & 0 & 0 & \mathbf{k}\cdot\mathbf{u} & \frac{k_3}{\rho} \\ 0 & k_1\gamma p & k_2\gamma p & k_3\gamma p & \mathbf{k}\cdot\mathbf{u} \end{pmatrix},$$

$\mathbf{k} = (k_1, k_2, k_3)^T$. Other choices of $\mathbf{V}$: e.g. T or s instead of ρ.

The Euler equations (23) are a hyperbolic system, because:

1. For all directions $\mathbf{k} \in \mathbf{R}^3$, the eigenvalues λ_i of

$$\mathbf{P}(\mathbf{k}) \;=\; k_1\mathbf{B}_1 + k_2\mathbf{B}_2 + k_3\mathbf{B}_3$$

   are real, namely

$$\begin{aligned}
\lambda_1(\mathbf{k}) &= \mathbf{k}\cdot\mathbf{u} - c|\mathbf{k}|, \\
\lambda_2(\mathbf{k}) &= \lambda_3(\mathbf{k}) = \lambda_4(\mathbf{k}) = \mathbf{k}\cdot\mathbf{u} \qquad (24) \\
\lambda_5(\mathbf{k}) &= \mathbf{k}\cdot\mathbf{u} + c|\mathbf{k}|\,.
\end{aligned}$$

   These eigenvalues signify the wave speeds in the **k**-direction.

2. For all directions $\mathbf{k} \in \mathbf{R}^3$, $|\mathbf{k}| = 1$, there is a uniformly bounded, non-singular, real transformation matrix $\mathbf{T}(\mathbf{k})$, i.e.

$\exists K \; \forall |\mathbf{k}| = 1 \; \exists \mathbf{T}(\mathbf{k}) \quad \|\mathbf{T}(\mathbf{k})\| + \|\mathbf{T}^{-1}(\mathbf{k})\| \leq K$, such that the transformed matrix is diagonal, i.e.

$$\mathbf{T}^{-1}(\mathbf{k})\mathbf{P}(\mathbf{k})\mathbf{T}(\mathbf{k}) = \mathbf{\Lambda}(\mathbf{k}) = \begin{pmatrix} \lambda_1(\mathbf{k}) & & & & 0 \\ & \lambda_2(\mathbf{k}) & & & \\ & & \lambda_3(\mathbf{k}) & & \\ & & & \lambda_4(\mathbf{k}) & \\ 0 & & & & \lambda_5(\mathbf{k}) \end{pmatrix} \tag{25}$$

For $\mathbf{k} \in \mathbf{R}^3$, $|\mathbf{k}| = 1$, choose $\mathbf{l}, \mathbf{m} \in \mathbf{R}^3$ such that $\mathbf{k}, \mathbf{l}, \mathbf{m}$ are orthogonal and $|\mathbf{l}| = |\mathbf{m}| = 1$. Choose e.g.

$$\mathbf{T}(\mathbf{k}) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -k_1\frac{c}{\rho} & 0 & \frac{l_1}{\rho} & \frac{m_1}{\rho} & k_1\frac{c}{\rho} \\ -k_2\frac{c}{\rho} & 0 & \frac{l_2}{\rho} & \frac{m_2}{\rho} & k_2\frac{c}{\rho} \\ -k_3\frac{c}{\rho} & 0 & \frac{l_3}{\rho} & \frac{m_3}{\rho} & k_3\frac{c}{\rho} \\ c^2 & 0 & 0 & 0 & c^2 \end{pmatrix},$$

$$\mathbf{T}^{-1}(\mathbf{k}) = \begin{pmatrix} 0 & -k_1\frac{\rho}{2c} & -k_2\frac{\rho}{2c} & -k_3\frac{\rho}{2c} & \frac{1}{2c^2} \\ 1 & 0 & 0 & 0 & -\frac{1}{c^2} \\ 0 & l_1\rho & l_2\rho & l_3\rho & 0 \\ 0 & m_1\rho & m_2\rho & m_3\rho & 0 \\ 0 & k_1\frac{\rho}{2c} & k_2\frac{\rho}{2c} & k_3\frac{\rho}{2c} & \frac{1}{2c^2} \end{pmatrix}.$$

In 2D: $\mathbf{k} \in \mathbf{R}^2$, $|\mathbf{k}| = 1$, $\mathbf{l} = (-k_2, k_1)^T$, and skip 4th columns and 4th rows of matrices. In 1D: $k_1 = 1$, and skip 3rd and 4th columns and 3rd and 4th rows of matrices.

2.4. Characteristic Formulation

Multiplying the non-conservative Euler equations (23) from the left by $\mathbf{T}^{-1}(\mathbf{k})$, we obtain the characteristic form of the Euler equations:

$$\begin{aligned}\frac{\partial \mathbf{W}(\mathbf{k})}{\partial t} &+ \mathbf{T}^{-1}(\mathbf{k})\mathbf{B}_1\mathbf{T}(\mathbf{k})\frac{\partial \mathbf{W}(\mathbf{k})}{\partial x_1} + \mathbf{T}^{-1}(\mathbf{k})\mathbf{B}_2\mathbf{T}(\mathbf{k})\frac{\partial \mathbf{W}(\mathbf{k})}{\partial x_2} \\ &+ \mathbf{T}^{-1}(\mathbf{k})\mathbf{B}_3\mathbf{T}(\mathbf{k})\frac{\partial \mathbf{W}(\mathbf{k})}{\partial x_3} = \mathbf{T}^{-1}(\mathbf{k})\mathbf{G}_e\,, \end{aligned} \tag{26}$$

where the characteristic variables $\mathbf{W}(\mathbf{k})$ are defined by
$\partial \mathbf{W}(\mathbf{k}) = \mathbf{T}^{-1}(\mathbf{k})\partial \mathbf{V}$.

If we choose $\mathbf{k} = \mathbf{e}_1 = (1,0,0)^T$, the first coefficient matrix in (26) will be diagonal because of (25) $\mathbf{T}^{-1}(\mathbf{e}_1)\mathbf{B}_1\mathbf{T}(\mathbf{e}_1) = \mathbf{\Lambda}(\mathbf{e}_1)$. Thus:

$$
\begin{aligned}
\frac{\partial \mathbf{W}(\mathbf{e}_1)}{\partial t} &+ \boldsymbol{\Lambda}(\mathbf{e}_1)\frac{\partial \mathbf{W}(\mathbf{e}_1)}{\partial x_1} + \mathbf{T}^{-1}(\mathbf{e}_1)\mathbf{B}_2\mathbf{T}(\mathbf{e}_1)\frac{\partial \mathbf{W}(\mathbf{e}_1)}{\partial x_2} \\
&+ \mathbf{T}^{-1}(\mathbf{e}_1)\mathbf{B}_3\mathbf{T}(\mathbf{e}_1)\frac{\partial \mathbf{W}(\mathbf{e}_1)}{\partial x_3} = \mathbf{T}^{-1}(\mathbf{e}_1)\mathbf{G}_e\,. \qquad (27)
\end{aligned}
$$

The jth component of this equation is called the compatibility equation for the eigenvalue $\lambda_j(\mathbf{e}_1)$. We shall study (27) for the simplified equation, where the y- and z-derivatives and the source term are neglected:

$$
\frac{\partial \mathbf{W}(\mathbf{e}_1)}{\partial t} + \boldsymbol{\Lambda}(\mathbf{e}_1)\frac{\partial \mathbf{W}(\mathbf{e}_1)}{\partial x_1} = 0\,. \qquad (28)
$$

2.5. Characteristics, Riemann Invariants

Equation (28) corresponds to the 5 decoupled equations ($\mathbf{l} = \mathbf{e}_2, \mathbf{m} = \mathbf{e}_3$ used in $\mathbf{T}^{-1}(\mathbf{e}_1)$ to get $\partial \mathbf{W} = \mathbf{T}^{-1}(\mathbf{e}_1)\partial \mathbf{V}$):

$$\frac{\partial W_j}{\partial t} + \lambda_j \frac{\partial W_j}{\partial x} \; = \; 0 \, , \quad j = 1, ..., 5 \, , \tag{29}$$

where
$$\begin{aligned}
\partial W_1 &= -\tfrac{\rho}{2c}\partial u + \tfrac{1}{2c^2}\partial p \quad , & \lambda_1 &= u - c \, , \\
\partial W_2 &= \partial \rho - \tfrac{1}{c^2}\partial p \quad , & \lambda_2 &= u \, , \\
\partial W_3 &= \rho \partial v \quad , & \lambda_3 &= u \, , \\
\partial W_4 &= \rho \partial w \quad , & \lambda_4 &= u \, , \\
\partial W_5 &= \tfrac{\rho}{2c}\partial u + \tfrac{1}{2c^2}\partial p \quad , & \lambda_5 &= u + c \, .
\end{aligned}$$

Note that we may multiply each equation (29) by an arbitrary value, as we can multiply each left eigenvector of $\mathbf{B}_1$, i.e. each row of

$\mathbf{T}^{-1}(\mathbf{e}_1)$, by an arbitrary value.

On the characteristic C_j, i.e. the curve $x = x(t)$ defined by $\frac{dx}{dt} = \lambda_j$, each scalar PDE (29) simplifies to the ODE

$$\frac{dW_j}{dt} \;=\; 0\,, \quad j = 1, ..., 5\,, \tag{30}$$

because $\frac{dW_j}{dt} = \frac{\partial W_j}{\partial t} + \frac{\partial W_j}{\partial x}\frac{dx}{dt} = \frac{\partial W_j}{\partial t} + \frac{\partial W_j}{\partial x}\lambda_j = 0$.

Thus, the characteristic variable W_j is constant on the characteristic C_j. Such a quantity that is invariant along a characteristic is called Riemann invariant.

The Euler equations have 3 different eigenvalues and therefore 3 different characteristics.

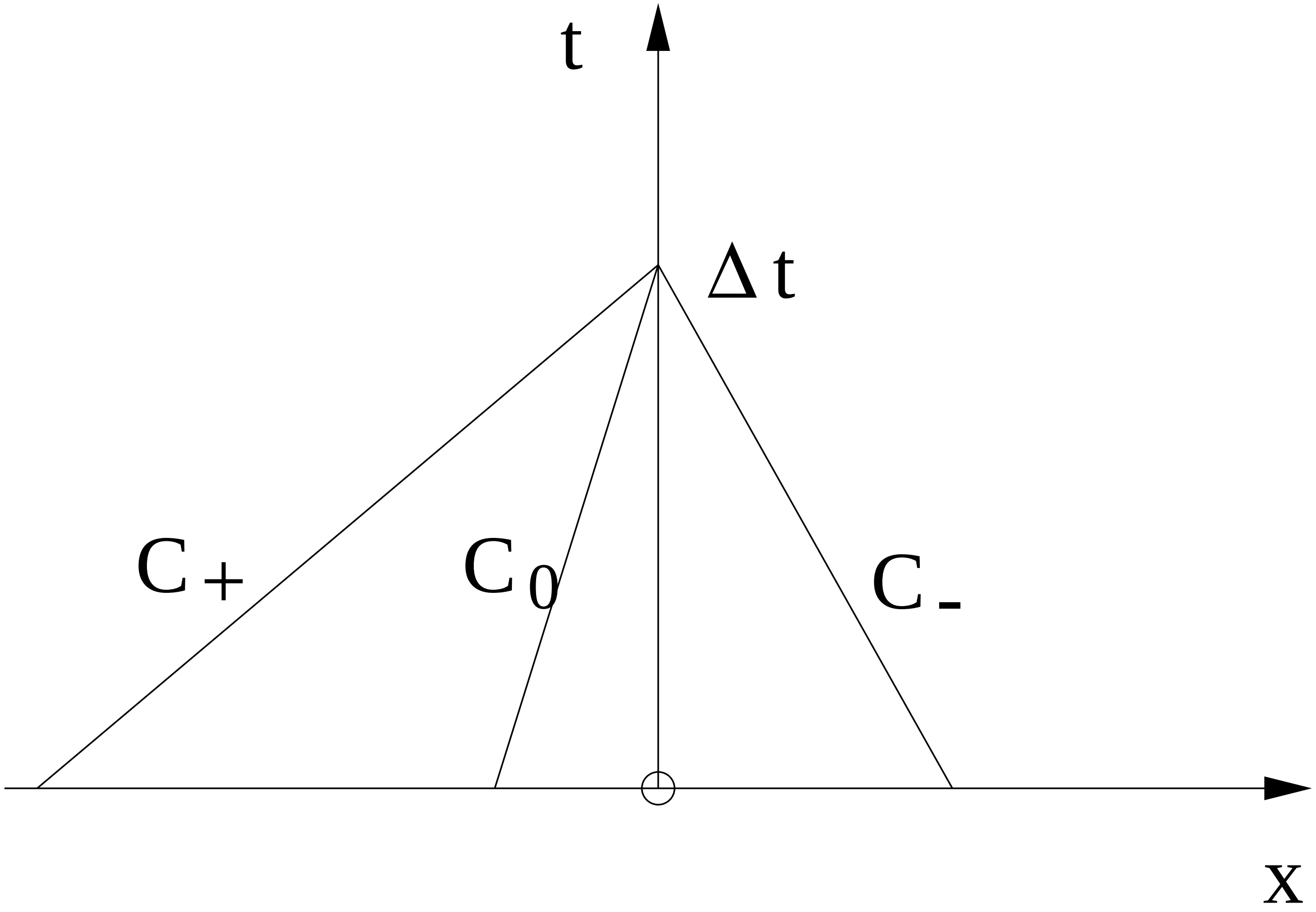
Riemann Invariants

On the path line $C_0 := C_2 = C_3 = C_4$, i.e. $\frac{dx}{dt} = u$, entropy s and the tangential velocity components v and w are Riemann invariants. To see that $\frac{ds}{dt} = 0$ on the path line, multiply $\frac{dW_2}{dt} = 0$ by $-c_p/\rho$ and use the equations of state for perfect gas and the Gibbs relation

$$T\,ds \;=\; de + p\,d\left(\frac{1}{\rho}\right) . \tag{31}$$

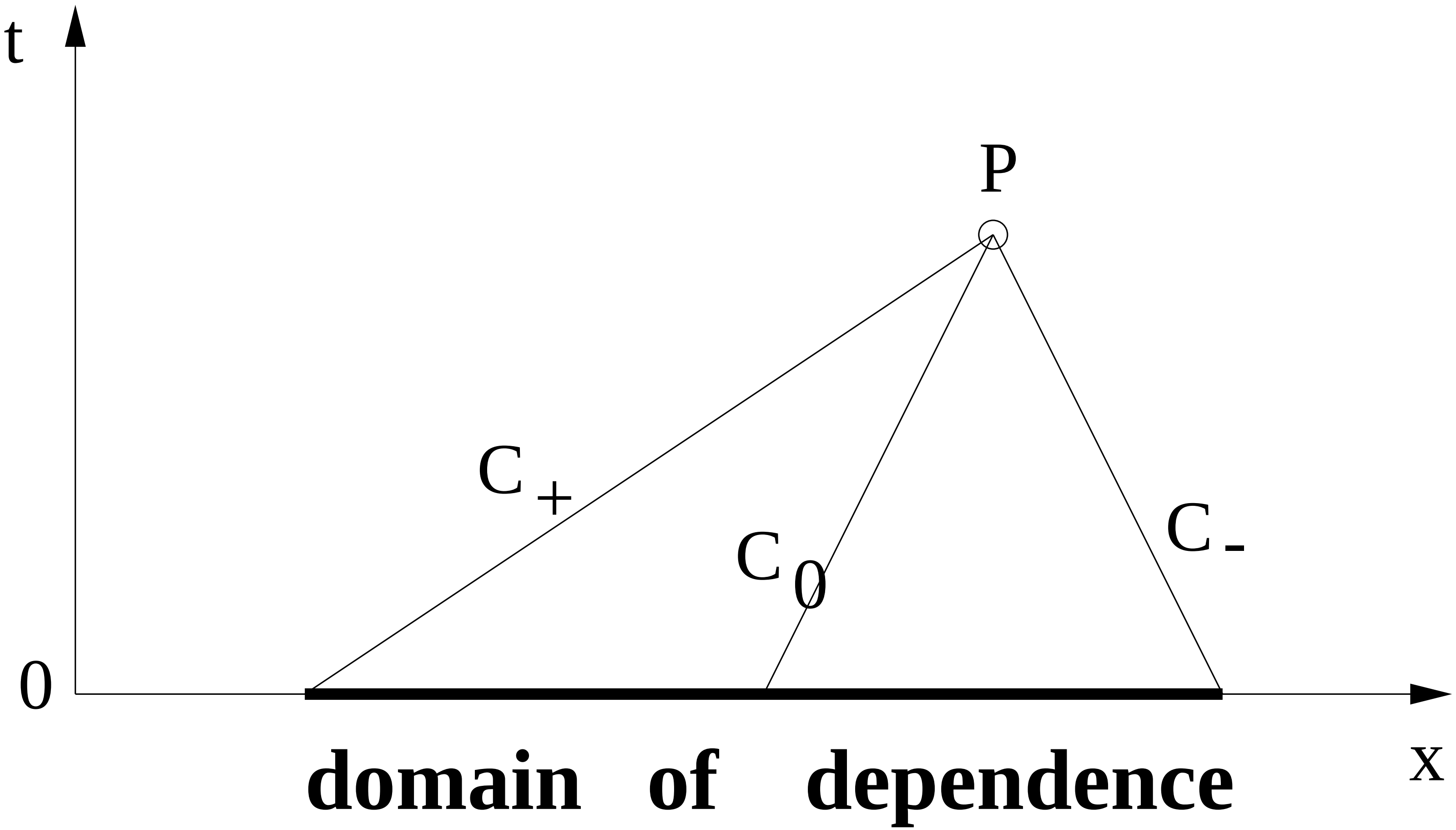
On the Mach lines $C_- := C_1$, i.e. $\frac{dx}{dt} = u - c$, and $C_+ := C_5$, i.e. $\frac{dx}{dt} = u + c$, we have $\frac{du}{dt} \mp \frac{1}{\rho c}\frac{dp}{dt} = 0$. For homentropic flow, i.e. $s = \mathit{constant}$ throughout the flow, that relation can be integrated and yields that $u - \frac{2}{\gamma-1}c$ and $u + \frac{2}{\gamma-1}c$ are Riemann invariants on C_- and C_+ , respectively.

Characteristics



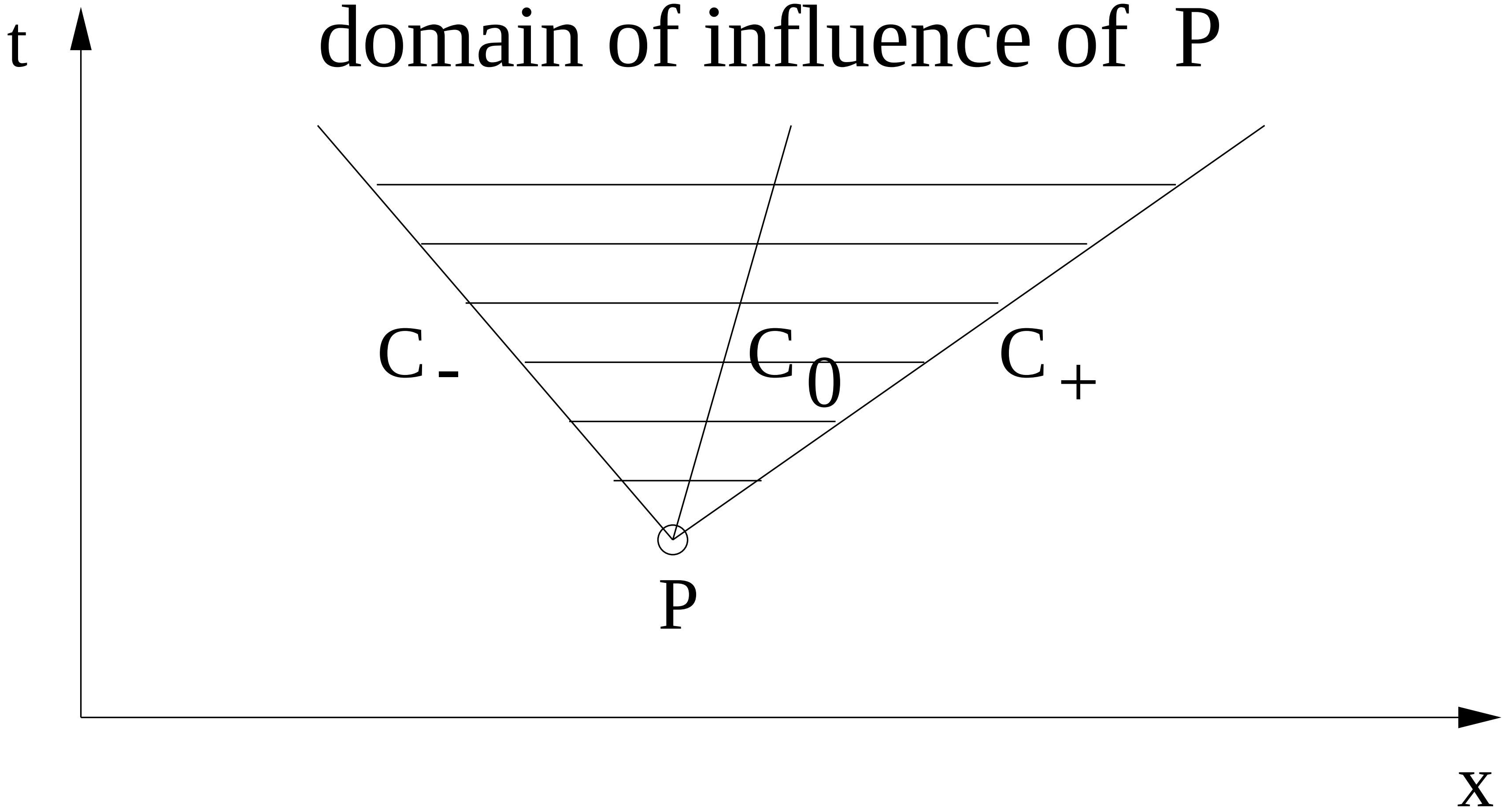
Characteristics in subsonic flow.

Domain of Dependence



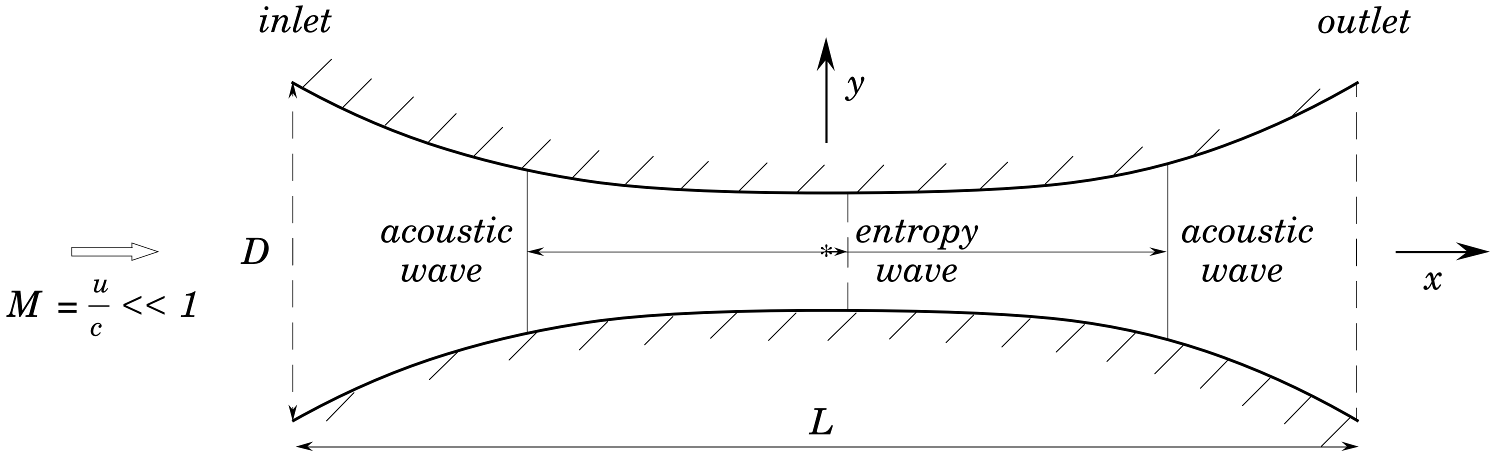
The solution at P depends on the domain of dependence of P.

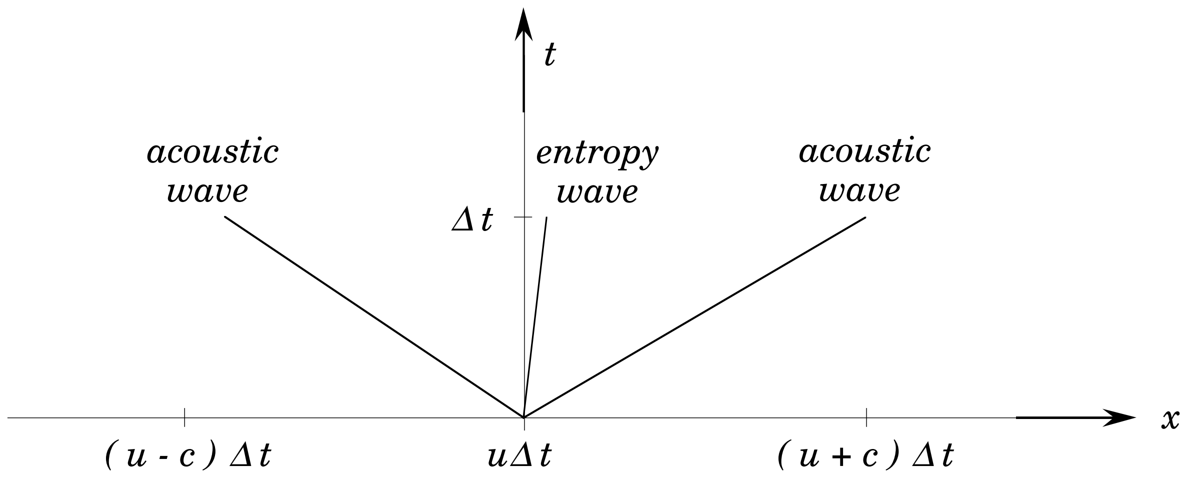
Domain of Influence



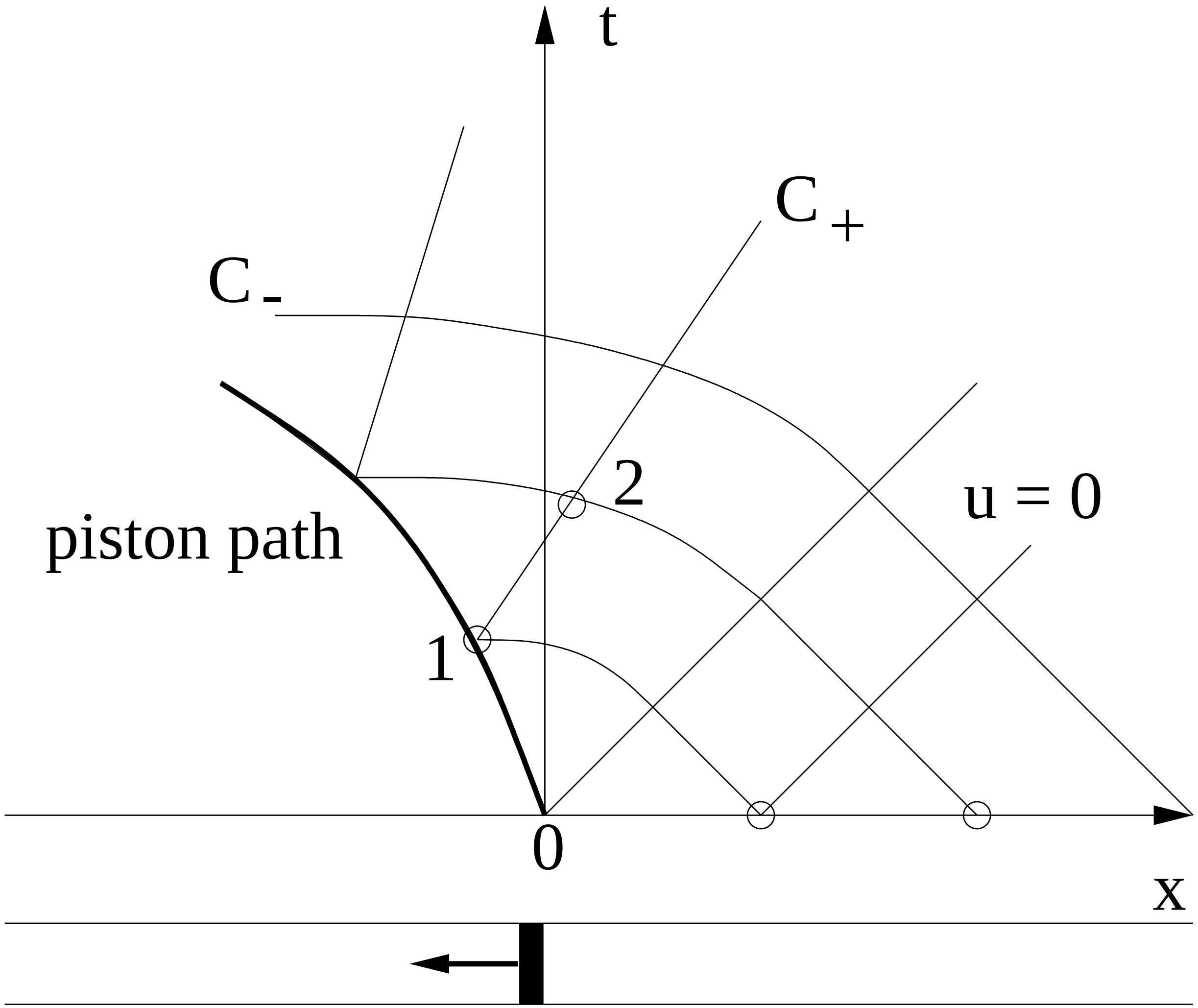
The values at P influence any point in the domain of influence of P.

Acoustic and Entropy Waves





Method of Characteristics



The method of characteristics is illustrated for flow expansion behind a piston in a cylinder.

Suppose in a gas at rest (i.e. $u = 0$, $c = c_0$), a piston is moved with velocity $v < 0$ to the left. Assuming 1D homentropic flow, the flow expansion can be computed by the method of characteristics as follows. On the right of the characteristics C_+ from the origin, the flow is at rest.

$u_1 = v(t_1)$, because of the boundary condition.
$u_1 - \frac{2}{\gamma-1}c_1 = -\frac{2}{\gamma-1}c_0$, because it is a Riemann invariant on C_- through point 1. Since c_0, v and γ are known, u_1 and c_1 can be determined.

$u_2 + \frac{2}{\gamma-1}c_2 = u_1 + \frac{2}{\gamma-1}c_1$, because it is a Riemann invariant on C_+ through point 1.
$u_2 - \frac{2}{\gamma-1}c_2 = -\frac{2}{\gamma-1}c_0$, because it is a Riemann invariant on C_- through point 2. Since u_1 and c_1 have already been determined, u_2 and c_2 can be computed.

In this particular case, $u_2 - \frac{2}{\gamma-1}c_2 = u_1 - \frac{2}{\gamma-1}c_1$, because both C_--characteristics originate from the same flow state. Thus, we get $u_2 = u_1$ and $c_2 = c_1$. The same argument holds for any point on the C_+-characteristics through point 1. Since $\frac{dx}{dt} = u + c$ on C_+ and 1 is an arbitrary point on the piston path, the C_+-characteristics are straight lines.

This is an example of a simple wave solution: all Riemann invariants except for one are constant.

$ds = c_v(\frac{dT}{T} - (\gamma - 1)\frac{d\rho}{\rho}) = c_v(\frac{dp}{p} - \gamma\frac{d\rho}{\rho})$ from the Gibbs relation (31). Thus, if the entropy is constant between two states 1 and 2, integration yields $\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$ and $\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$. If we know flow state 1 and either p_2, ρ_2, T_2 or $c_2(= \sqrt{\gamma R T_2})$, all thermodynamic variables at state 2 are determined for $s_2 = s_1$.

2.6. Boundary Conditions

2.6.1. Solid Wall

If a solid wall is moving with the velocity $\mathbf{v}$, the impermeability boundary condition states: $\mathbf{u}_w \cdot \mathbf{n}_w = \mathbf{v} \cdot \mathbf{n}_w$. Thus, for a stationary solid wall

$$\mathbf{u}_w \cdot \mathbf{n}_w \;=\; 0\,.$$

If a stationary solid wall is plane, it is a symmetry boundary, e.g. $v_w = \frac{\partial u_w}{\partial y} = \frac{\partial w_w}{\partial y} = \frac{\partial \rho_w}{\partial y} = \frac{\partial p_w}{\partial y}$ for $\mathbf{n}_w = \mathbf{e}_2$. For a curved stationary solid wall with radius of curvature R_w, the wall normal momentum equation yields (without source terms)

$$\frac{\partial p_w}{\partial r} = \frac{\rho_w |\mathbf{u}_w|^2}{R_w}\;,$$

where r is the radial coordinate in 2D polar or 3D spherical coordinates. Since $R_w \longrightarrow \infty$, if a curved wall becomes plane, $\frac{\partial p_w}{\partial n} = 0$

is correctly recovered.

For steady flow without source terms, total enthalpy H and except for shocks also entropy s are constant along streamlines, i.e. $(\mathbf{u} \cdot \boldsymbol{\nabla})H = 0$ and except for shocks also $(\mathbf{u} \cdot \boldsymbol{\nabla})s = 0$. Since stationary solid walls are streamlines (as impermeability implies wall tangential flow), we have conditions for H_w and s_w .

2.6.2. Inflow and Outflow

We look at the characteristics normal to an artificial boundary, where we may have inflow or outflow. If we choose $\mathbf{k} = \mathbf{n}$ in (26), where $\mathbf{n}$ is the outer normal unit vector, and neglect the tangential derivatives and the source term, we obtain instead of (28)

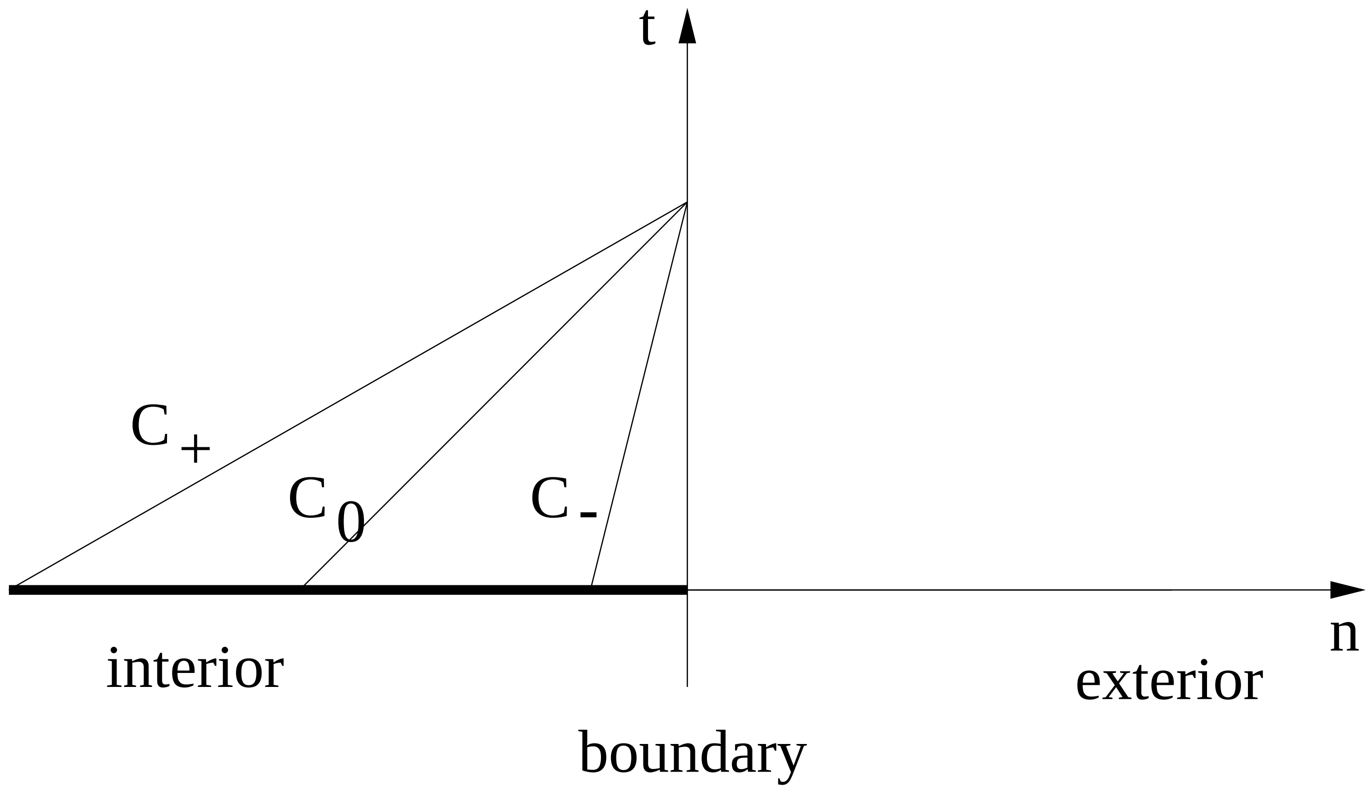
$$\frac{\partial \mathbf{W}(\mathbf{n})}{\partial t} + \boldsymbol{\Lambda}(\mathbf{n}) \frac{\partial \mathbf{W}(\mathbf{n})}{\partial n} \; = \; 0 \; . \tag{32}$$

where $\frac{\partial}{\partial n} = \mathbf{n} \cdot \boldsymbol{\nabla}$. We obtain relations similar to (29) and (30) with u replaced by the normal velocity component $\mathbf{u} \cdot \mathbf{n}$ and v and w replaced by the velocity components tangential to the boundary.

At the boundary, the characteristics tell us, where the waves come from. If a characteristics comes from the interior, the corresponding Riemann invariant is known from previous time and a boundary condition must not be prescibed. If a characteristics comes from the exterior, the corresponding Riemann invariant is unknown from previous time and a boundary condition must be prescibed.

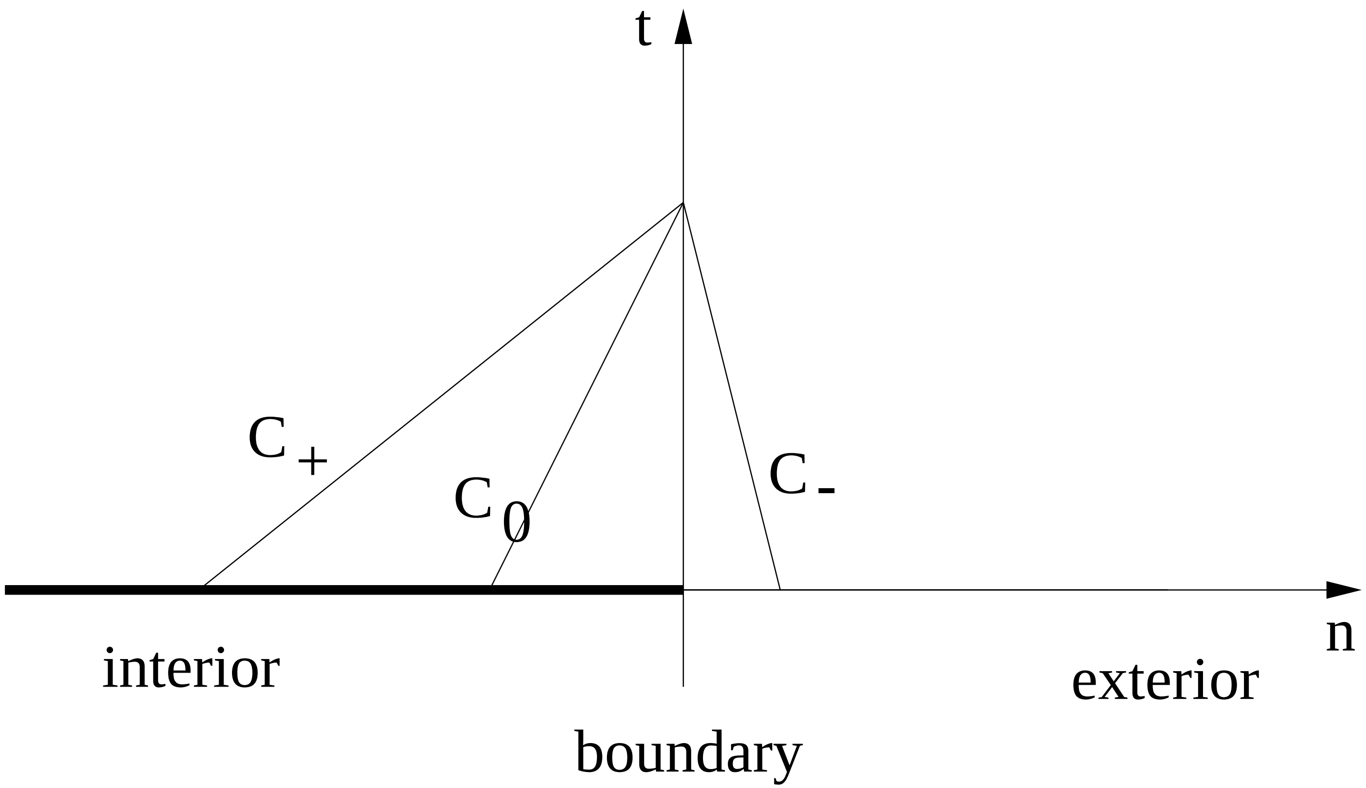
We distinguish 4 cases:

(i) Supersonic Outflow $\quad c < \mathbf{u} \cdot \mathbf{n}$



All waves come from the interior. No boundary condition must be given. The Euler equations can be used at the boundary.

(ii) Subsonic Outflow $\quad 0 < \mathbf{u} \cdot \mathbf{n} < c$



All waves except for the one on C_- come from the interior. For the outgoing waves, no boundary condition must be given. Instead, the compatibility equations for the eigenvalues $\lambda_j(\mathbf{n})$, $j = 2, 3, 4, 5$ (cf. (27)) may be used. For the incoming wave on C_- , a boundary condition must be provided. Often, the ambient pressure p_a is prescribed, i.e.

$$p = p_a\,.$$

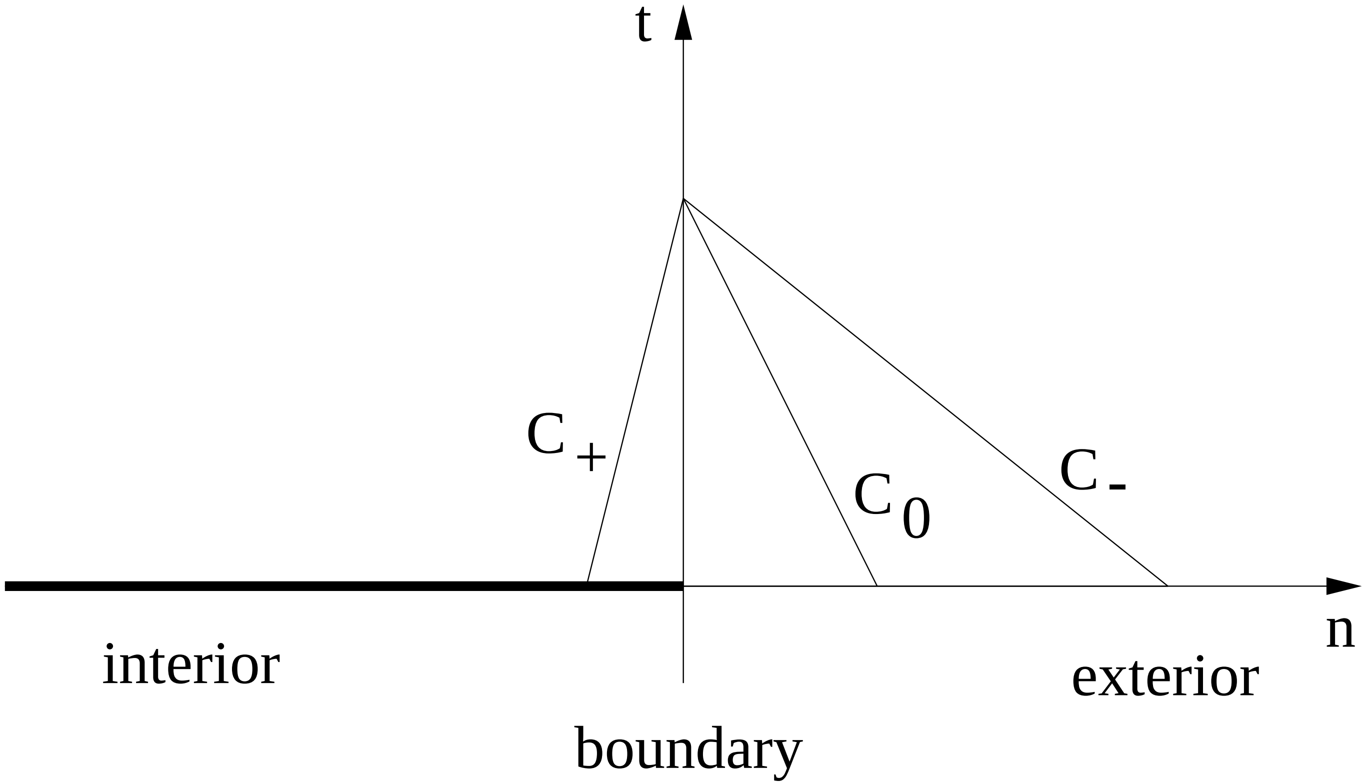
In the farfield, the Riemann invariant for homentropic flow

$$\mathbf{u}\cdot\mathbf{n} - \frac{2}{\gamma-1}c = \mathbf{u}_\infty\cdot\mathbf{n} - \frac{2}{\gamma-1}c_\infty$$

may be presribed, where subscript ∞ denotes uniform farfield flow. Non-reflecting 1D boundary conditions require $\frac{\partial W_1}{\partial t} = 0$ or

$$\frac{\partial p}{\partial t} - \rho c\frac{\partial \mathbf{u}\cdot\mathbf{n}}{\partial t} = 0\,.$$

(iii) Subsonic Inflow $\quad -c < \mathbf{u} \cdot \mathbf{n} < 0$



All waves except for the one on C_+ come from the exterior. For the outgoing wave on C_+, no boundary condition must be given. Instead, the compatibility equation for the eigenvalue $\lambda_5(\mathbf{n})$ (cf.

(27)) may be used. For each incoming wave, a boundary condition must be provided. For internal flow, often total enthalpy $H = \frac{\gamma p}{(\gamma-1|\rho} + \frac{1}{2}|\mathbf{u}|^2$, entropy s and either the tangential velocity components or the flow angles are prescribed, e.g. for a boundary with $x = constant$:

$$H \;=\; H_0\,,\;\frac{p}{\rho^\gamma} \;=\; \frac{p_0}{\rho_0^\gamma}\,,\;\frac{v}{|\mathbf{u}|} \;=\; \alpha_y\,,\;\frac{w}{|\mathbf{u}|} \;=\; \alpha_z,$$

where the subscript 0 denotes stagnation conditions. In the farfield of external flow, one often uses the Riemann invariants

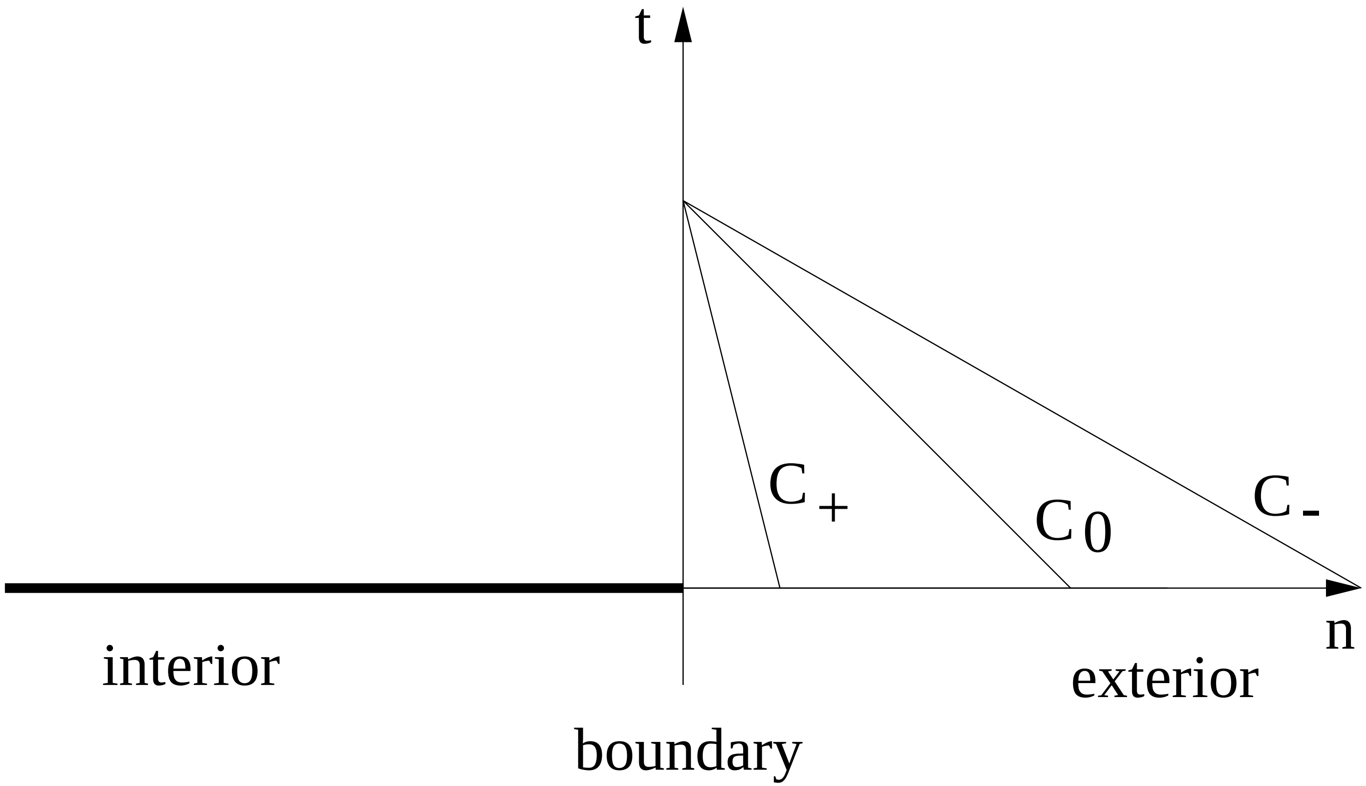
$$\begin{aligned}\mathbf{u}\cdot\mathbf{n} - \tfrac{2}{\gamma-1}c \;&=\; \mathbf{u}_\infty\cdot\mathbf{n} - \tfrac{2}{\gamma-1}c_\infty\;,\\ \tfrac{p}{\rho^\gamma} \;&=\; \tfrac{p_\infty}{\rho_\infty^\gamma}\,,\\ \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \;&=\; \mathbf{u}_\infty - (\mathbf{u}_\infty\cdot\mathbf{n})\mathbf{n}\;.\end{aligned}$$

Non-reflecting 1D boundary conditions require $\frac{\partial W_j}{\partial t} = 0\,,$

$j = 1, 2, 3, 4\,,$ or

$$\frac{\partial p}{\partial t} - \rho c \frac{\partial \mathbf{u} \cdot \mathbf{n}}{\partial t} = 0\ , \qquad c^2 \frac{\partial \rho}{\partial t} - \frac{\partial p}{\partial t} = 0\ , \qquad \frac{\partial \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}}{\partial t} = 0\,.$$

(iv) Supersonic Inflow $\mathbf{u} \cdot \mathbf{n} < -c$



All waves come from the exterior. Thus, for each wave, a boundary condition must be provided, e.g. in the farfield

$$\mathbf{V} = \mathbf{V}_\infty$$

or equivalently

$$\mathbf{U} = \mathbf{U}_\infty .$$

2.6.3. Well-Posedness

With these boundary conditions, the Euler equations are expected to be well-posed, i.e.

1. a unique solution exists, and
2. the solution depends continuously on the data.

The latter condition says that small perturbations of the source term, the initial or boundary conditions lead to small perturbations of the solution. Cf. Kreiss, Lorenz (1989) and Gustafsson, Kreiss, Oliger (1995) for details, e.g. proof of well-posedness for the linearized Euler equations with characteristic boundary conditions.

2.7 Simplified Forms

2.7.1. Potential Equation

Assumptions:

1. steady flow, i.e. $\frac{\partial}{\partial t} = 0$,
2. irrotational flow, i.e. $\omega = 0$, where $\omega = \mathbf{\nabla} \times \mathbf{u}$ is called vorticity,
3. homentropic flow, i.e. $s = constant$,
4. no external force, i.e. $\mathbf{f} = 0$.

Since $\mathbf{\nabla} \times \mathbf{u} = 0$, there exists a scalar function ϕ such that

$$\mathbf{u} \; = \; \mathbf{\nabla}\,\phi\,. \tag{33}$$

ϕ is called the velocity potential. By definition, it satisfies

$\boldsymbol{\nabla} \times \mathbf{u} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = 0$. In Cartesian coordinates, we have

$$u = \frac{\partial \phi}{\partial x}\,, \qquad v = \frac{\partial \phi}{\partial y}\,, \qquad w = \frac{\partial \phi}{\partial z}\,.$$

In Cartesian coordinates, the continuity equation, i.e. the 1st component of (18), reads for steady flow:

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho \frac{\partial \phi}{\partial z}\right) = 0\,. \qquad (34)$$

Using the identity $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}(\frac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\omega} \times \mathbf{u}$, the relation $dp = c^2\, d\rho$ for homentropic flow and the assumptions above, the inviscid momentum equation, i.e. 2nd, 3rd and 4th components of (23), simplifies to

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}|\mathbf{u}|^2\right) = -\frac{c^2}{\rho}\frac{\partial}{\partial x_i}\rho\,, \quad i = 1, 2, 3\,.$$

Using (33) and the above relations to replace the density derivatives in (34), we obtain

$$\left(1 - \frac{\phi_x^2}{c^2}\right)\phi_{xx} \quad + \quad \left(1 - \frac{\phi_y^2}{c^2}\right)\phi_{yy} + \left(1 - \frac{\phi_z^2}{c^2}\right)\phi_{zz} -$$

$$2\frac{\phi_x\phi_y}{c^2}\phi_{xy} \quad - \quad 2\frac{\phi_x\phi_z}{c^2}\phi_{xz} - 2\frac{\phi_y\phi_z}{c^2}\phi_{yz} \;=\; 0\,, \tag{35}$$

where $\phi_x = \frac{\partial\phi}{\partial x}$, $\phi_{xy} = \frac{\partial^2\phi}{\partial x\partial y}$, etc.

Using the compressible Bernoulli equation $H = H_\infty$ or

$$\frac{c^2}{\gamma - 1} + \frac{1}{2}|\mathbf{u}|^2 \;=\; \frac{c_\infty^2}{\gamma - 1} + \frac{1}{2}|\mathbf{u}_\infty|^2\,,$$

we can express c^2 as a function of $|\mathbf{u}|^2 = \phi_x^2 + \phi_y^2 + \phi_z^2$ and the known states c_∞ and $|\mathbf{u}_\infty|$. Using the isentropic relation between ρ and c, cf. derivation of Riemann invariants for $s = \mathit{constant}$, ρ in (34) can be expressed as function of $|\boldsymbol{\nabla}\phi|^2$. In numerical applications, the conservative form (34) is usually preferred.

Both the conservative and non-conservative forms of the potential equation (34) and (35) reduce to Laplace's equation $\Delta\phi = 0$ for incompressible flow because of $\rho \rightarrow constant$ and $c \rightarrow \infty$, respectively.

2.7.2. TSP and Prandtl-Glauert Equations

Thin airfoils cause only small disturbances of uniform flow. Denoting the velocity disturbance in 2D by $(u', v')^T$, the velocity components can be expressed as

$$u = u_\infty + u' , \qquad v = v'$$

or with the perturbation potential

$$u = \phi_x = u_\infty + \phi'_x , \qquad v = \phi_y = \phi'_y .$$

Using the compressible Bernoulli equation, c^2 can be expressed as a function of the perturbation potential and the reference state. Inserting the ansatz for u , v and c^2 into the potential equation (35) and neglecting small perturbation terms like $(\phi'_x)^2/c_\infty^2$ and $(\phi'_y)^2/c_\infty^2$ (and using the order of magnitude argument: $\frac{v'}{u_\infty} = O(\epsilon)$, $\frac{\partial}{\partial x} = O(1)$ and irrotationality $\frac{\partial v'/u_\infty}{\partial x} = \frac{\partial u'/u_\infty}{\partial y}$ imply $\frac{u'}{u_\infty} = O(\epsilon^{2/3})$, $\frac{\partial}{\partial y} = O(\epsilon^{1/3})$) , we obtain the transonic small-perturbation (TSP) equation

$$[1 - M_\infty^2 \; - \; (\gamma + 1) M_\infty^2 \frac{\phi'_x}{u_\infty}] \phi'_{xx} \; + \; \phi'_{yy} \; = \; 0 \qquad (36)$$

where $M_\infty = \frac{u_\infty}{c_\infty}$ is the Mach number of the uniform flow. The nonlinear term is only important for transonic flow $M \approx 1$.

Thus, for pure subsonic or supersonic flow, the TSP equation (36) reduces to the linear Prandtl-Glauert equation

$$(1 - M_\infty^2) \phi'_{xx} \; + \; \phi'_{yy} \; = \; 0 \qquad (37)$$

The equation is either elliptic, parabolic or hyperbolic, depending on whether the flow is subsonic, sonic (where it is not valid) or supersonic.

Using $H = H_\infty$ (cf. 2.7.1), $\frac{p}{p_\infty} = \left(\frac{\rho}{\rho_\infty}\right)^\gamma$ and the TSP ansatz, the pressure coefficient can be determined by

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right) \approx -2\frac{\phi_x'}{u_\infty}\,.$$

2.7.3. Linearized Euler Equations

Sound propagation can usually be described by the linearized Euler equations. These equations for the perturbations $(\rho', \mathbf{u}', p')^T$ are obtained from the Euler equations (23) without source term linearized around the mean state $(\rho_0, \mathbf{u}_0, p_0)^T = constant$:

$$\frac{\partial \rho'}{\partial t} + (\mathbf{u}_0 \cdot \boldsymbol{\nabla})\rho' + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{u}' = 0 \tag{38}$$

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}_0 \cdot \boldsymbol{\nabla})\mathbf{u}' + \frac{1}{\rho_0} \boldsymbol{\nabla} p' = 0 \tag{39}$$

$$\frac{\partial p'}{\partial t} + (\mathbf{u}_0 \cdot \boldsymbol{\nabla})p' + \gamma p_0 \boldsymbol{\nabla} \cdot \mathbf{u}' = 0 \tag{40}$$

Thus,

$$\rho = \rho_0 + \rho', \quad \mathbf{u} = \mathbf{u}_0 + \mathbf{u}', \quad p = p_0 + p' \tag{41}$$

2.7.4. Wave Equation

For stagnant mean flow $\mathbf{u}_0 = 0$, we obtain from (38 to 40) the acoustic approximation of the Euler equations, from which the wave equation for the velocity potential φ can be derived for the irrotational acoustic flow:

$$\frac{1}{c_0^2}\frac{\partial^2 \varphi}{\partial t^2} - \triangle \varphi = 0\,. \tag{42}$$

Velocity, pressure and density perturbations are obtained from

$$\mathbf{u} = \boldsymbol{\nabla}\varphi, \quad p' = -\rho_0 \frac{\partial \varphi}{\partial t}, \quad \rho' = \frac{1}{c_0^2} p'\,. \tag{43}$$