NGSSC CFD Course Part 2
Compressible Fluid Flow

18 August 2003

II. Governing Equations (B. Müller)

Exercise 2

Objective

To get familiar with the Euler equations, learn to determine characteristics and Riemann invariants and to use them as boundary conditions.

a) Show that the 2D compressible Euler equations in differential conservative form for homentropic flow, i.e. entropy $s = constant$ throughout the flow, and by neglecting the y-derivatives and source term simplify to:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} + u\frac{\partial u}{\partial x} = 0 \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = 0 \tag{3}$$

Hint: The following consequence of Gibbs relation for perfect gas may be used without proof: $ds = 0 \Longleftrightarrow dp = c^2 d\rho$.

b) Derive the characteristics $C_\pm : \frac{dx}{dt} = u \pm c$ and $C_0 : \frac{dx}{dt} = u$.

Hint: Express equations (1) - (3) as system $\mathbf{V}_t + \mathbf{B}\mathbf{V}_x = 0$ and determine the eigenvalues of $\mathbf{B}$.

c) Determine the characteristic form of the hyperbolic system defined by equations (1) - (3).

Hint: Determine the eigenvectors of $\mathbf{B}$, which define the right eigenvector matrix $\mathbf{T}$ and multiply the system $\mathbf{V}_t + \mathbf{B}\mathbf{V}_x = 0$ with the left eigenvector matrix $\mathbf{T}^{-1}$ from the left.

d) Show that v is a Riemann invariant on C_0 and that $R_{\pm} = u \pm \frac{2}{\gamma-1}c$ are Riemann invariants on $C_{\pm}$.

Hint: Compute $\partial \mathbf{W} = \mathbf{T}^{-1}\partial \mathbf{V}$ and integrate $\frac{dW_l}{dt} = 0$ on characteristics to determine Riemann invariant W_l, $l = 1, 2, 3$.
$c = \sqrt{\gamma R T}$ is the speed of sound. For $s = s_0$, $\frac{T}{T_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$. Thus, $c = c(\rho)$ for homentropic flow may be used for the integration in

$$R_{\pm}(\underline{U}_1) - R_{\pm}(\underline{U}_0) = \int_{u_0}^{u_1} du \pm \int_{\rho_0}^{\rho_1} \frac{c}{\rho} d\rho \; . \tag{4}$$

e) Discuss the application of the Riemann invariants as boundary conditions at a subsonic inlet and outlet of a channel.