Strictly Stable High Order Difference Methods for the Compressible Euler and Navier-Stokes Equations

Bernhard Müller

Division of Scientific Computing, Department of Information Technology, Uppsala University, Box 337, S-751 05 Uppsala, Sweden Bernhard.Muller@it.uu.se

1 Introduction

High order finite difference methods have been constructed to be strictly stable for linear hyperbolic and parabolic problems. The methods employ high order central approximations in the interior and special boundary stencils to satisfy the summation by parts (SBP) property leading to discrete energy estimates [1][2][3][4][5].

Instead of standard central high order difference approximations, dispersion relation preserving schemes can be used in the interior to yield SBP operators [6]. With entropy splitting, i.e. splitting of the nonlinear Euler and Navier-Stokes equations into a conservative portion and a symmetric non-conservative portion, generalized energy estimates can be derived to yield strict stability [7][8].

Here, the perturbation form of the nonlinear Euler and Navier-Stokes equations has been employed to minimize cancellation errors at low Mach numbers and to reduce to the linearized equations, if the nonlinear terms are neglected [9][10][11]. The SBP operator with sixth order in the interior and third order near the boundaries by B. Strand [2] is employed to approximate the spatial first derivatives in the nonlinear Euler and Navier-Stokes equations. Second derivatives are computed by applying the first order difference operator twice. The explicit fourth order Runge-Kutta method is used for time integration. High wave number oscillations are damped by a sixth order explicit filter. Sound propagation of the Kirchhoff vortex, of rocket-launch noise before lift-off and of point sources in the atmosphere has been simulated by solving the nonlinear and linearized Euler equations [12][13][14][15][11][16][6]. The numerical approach has recently been extended to the Navier-Stokes equations [17]. High accuracy is required for correctly computing not only sound propagation over long distances and times but also sound generation due to viscous and nonlinear effects.

2 Flow Equations in Perturbation Form

The perturbation formulation is used to minimize cancellation errors when discretizing the Euler and Navier-Stokes equations for compressible low Mach number flow [9]. The 2D compressible Navier-Stokes equations in conservative form can be expressed in perturbation form as [10]

$$\mathbf{U}'_t + \mathbf{F}'_x + \mathbf{G}'_y = 0 , \tag{1}$$

where $\mathbf{F}' = \mathbf{F}^{c\prime} - \mathbf{F}^{v\prime}$ and $\mathbf{G}' = \mathbf{G}^{c\prime} - \mathbf{G}^{v\prime}$ denote the differences of the inviscid and viscous flux vectors in the x- and y-directions, respectively, with respect to $\mathbf{F}(\mathbf{U}_0)$ and $\mathbf{G}(\mathbf{U}_0)$, where $\mathbf{U}_0 = (\rho_\infty, 0, 0, (\rho E)_\infty)^T$ and subscript $_\infty$ denotes freestream values. The subscripts in (1) and subsequently denote derivatives. We assume perfect gas. The conservative perturbation variables $\mathbf{U}'$ and the inviscid flux vectors are defined by

$$\mathbf{U}' = \begin{pmatrix} \rho' \\ (\rho u)' \\ (\rho v)' \\ (\rho E)' \end{pmatrix}, \ \mathbf{F}' = \begin{pmatrix} (\rho u)' \\ (\rho u)'u' + p' \\ (\rho v)'u' \\ (\rho_\infty H_\infty + (\rho H)')u' \end{pmatrix}, \ \mathbf{G}' = \begin{pmatrix} (\rho v)' \\ (\rho u)'v' \\ (\rho v)'v' + p' \\ (\rho_\infty H_\infty + (\rho H)')v' \end{pmatrix}$$

The perturbation variables (superscript $'$) are defined by $\rho' = \rho - \rho_\infty, (\rho\mathbf{u})' = \rho\mathbf{u}, (\rho E)' = \rho E - (\rho E)_\infty , (\rho H)' = (\rho E)' + p' , \mathbf{u}' = \frac{(\rho\mathbf{u})'}{\rho_\infty + \rho'}, p' = (\gamma - 1)[(\rho E)' - \frac{1}{2}((\rho\mathbf{u})' \cdot \mathbf{u}')]$. t is time, and x and y are the Cartesian coordinates, respectively. ρ denotes the density, u and v the x- and y-direction velocities, E the specific total energy, p the pressure, H the total enthalpy, $\gamma = 1.4$ the ratio of specific heats for air, $Pr = 0.72$ the Prandtl number. The viscous flux vectors $\mathbf{F}^{v\prime}$ and $\mathbf{G}^{v\prime}$ are the same as for the standard conservative form, except for using the temperature perturbation T' instead of temperature T for the heat flux terms. The perturbation formulation with respect to uniform flow or general base flow is somewhat more elaborate [10]. General geometries are treated by a coordinate transformation $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ [17].

3 High Order Difference Methods

3.1 Approach for Convection Diffusion Equation

The energy method shows us how to construct difference operators such that high order and strict stability including the boundary conditions are guaranteed. As a model equation, we consider the convection diffusion equation with homogeneous boundary conditions

$$u_t + au_x = bu_{xx} , \qquad 0 \le x \le 1 , \tag{2}$$
$$u(x,0) = f(x) , \qquad 0 \le x \le 1 , \tag{3}$$
$$u(0,t) = u(1,t) = 0 , \qquad 0 \le t . \tag{4}$$

where a and b are assumed to be constant and positive.

Let $v_j = v_j(t)$ denote the approximation of the exact solution of the convection diffusion equation at grid point $x_j = jh$ with the grid spacing $h = \frac{1}{N}$. We use the notation $v = [v_0, v_1, ..., v_N]^T$. We define the discrete scalar product and norm

$$(u,v)_h \; = \; hu^T Hv \,, \qquad ||u||_h^2 = (u,u)_h \,, \tag{5}$$

where H is a diagonal and positive definite matrix.

A difference operator Q satisfies the SBP property, if [2]

$$(u,Qv)_h \; = \; u_N v_N - u_0 v_0 \; - \; (Qu,v)_h \,. \tag{6}$$

Thus, the discrete energy estimate for $v_t + aQv \; = \; bQQv$ becomes [17]

$$\frac{d}{dt}||v(t)||_h^2 \; \leq \; av_0^2 + 2b[v_N(Qv)_N - v_0(Qv)_0] \,. \tag{7}$$

The boundary conditions can be weakly imposed by the simultaneous approximation term (SAT) [3] to get strict stability, i.e. continuous dependence on the initial and boundary data and up to $O(\Delta x)$ the same growth rate as the continuous convection diffusion equation [5][18].

3.2 Approach for Euler and Navier-Stokes Equations

The transformed 2D compressible Euler and Navier-Stokes equations in perturbation form are solved on an annulus. ξ and η are the radial and circumferential coordinates, respectively. The ξ-derivatives in the transformed Euler and Navier-Stokes equations and in the metric terms are discretized by the 3-6 SBP operator by B. Strand, which is third order accurate near the boundaries (at the boundary and the 5 adjacent grid points) and corresponds to the standard sixth order central difference operator $Q_\xi^{(6)}$ in the interior [2]. Due to the periodic boundary conditions in the η-direction, the standard sixth order central difference operator $Q_\eta^{(6)}$ is applied in the η-direction. The viscous terms are discretized by first approximating the first ξ and η derivatives of u', v' and T', by B. Strand's 3-6 SBP operator and the standard sixth order central difference operator, respectively. After the flux vectors $\hat{\mathbf{F}}'$ and $\hat{\mathbf{G}}'$ are computed at all grid points, $\hat{\mathbf{F}}'_\xi$ and $\hat{\mathbf{G}}'_\eta$ are approximated by employing the difference operators once more. However, this approximation of the second derivatives leads to a wide stencil and does not damp the high wave number modes $k = \frac{\pi}{\Delta x}$. The high order difference approximations derived for u_{xx} [5] and $(bu_x)_x$ with a variable coefficient $b = b(x,t)$ [18] alleviate those problems, cf. Fig. 1 where ξ denotes the nondimensional wave number.

The classical fourth order explicit Runge-Kutta method is used for time integration. The Euler and Navier-Stokes equations are not only solved in the interior but also at the boundaries. The ingoing characteristic variables and isothermal no-slip wall boundary conditions are imposed at $\xi = 1$ for the

Euler and Navier-Stokes equations, respectively. At the farfield boundary $\xi = \xi_{max}$, the first approximation of the Engquist-Majda nonreflecting boundary conditions is imposed.

Spurious high wave number oscillations are suppressed by a sixth order explicit filter by modifying the numerical solution at the completion of a full time step of the Runge-Kutta method [17]. Although the sixth order explicit filter was found to be less accurate than a characteristic based explicit filter [11], the former has been used here because of its lower complexity.

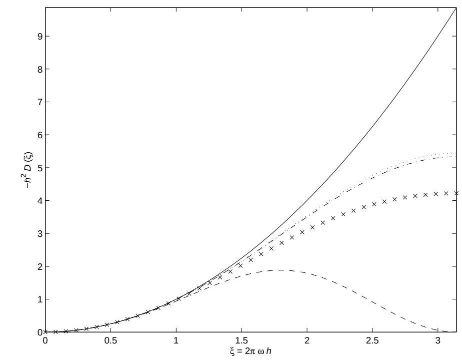

Fig. 1. Damping of modes with nondimensional wave number $\xi = k\Delta x$ of the exact solution (-), the standard central fourth order difference operator for the first derivative D_1 applied twice (- -), the standard central fourth order difference operator for the second derivative D_2 (-·), the fourth order difference operator Q derived for the self-adjoint form (·) and the fourth order operator L based on finite elements with mass lumping (x) [18].

4 Results

To verify the 2D Navier-Stokes solver for computational aeroacoustics, the flow around a circular cylinder generating aeolian tones has been selected. The freestream Mach numbers are $M_\infty = 0.1$ and $M_\infty = 0.2$, respectively. We consider a low Reynolds number ($Re_\infty = 150$ based on the cylinder diameter), where the Kármán vortex street is observed. Each time a vortex is shed from the cylinder, a sound wave is emitted. The frequency of the vortex shedding and the periodic variation of lift and drag are in excellent agreement with a reference solution [17]. At the farfield, the first approximation of the Engquist-Majda absorbing boundary conditions proved to be sufficiently nonreflective., cf. Fig. 2. Whereas the unsteady flow could be correctly predicted even with single precision using the perturbation formulation of the Navier-Stokes equations, double precision was required to correctly simulate sound propagation [17]. With the present approach, the direct numerical simulation of aeolian tones at higher Reynolds numbers can be attacked.

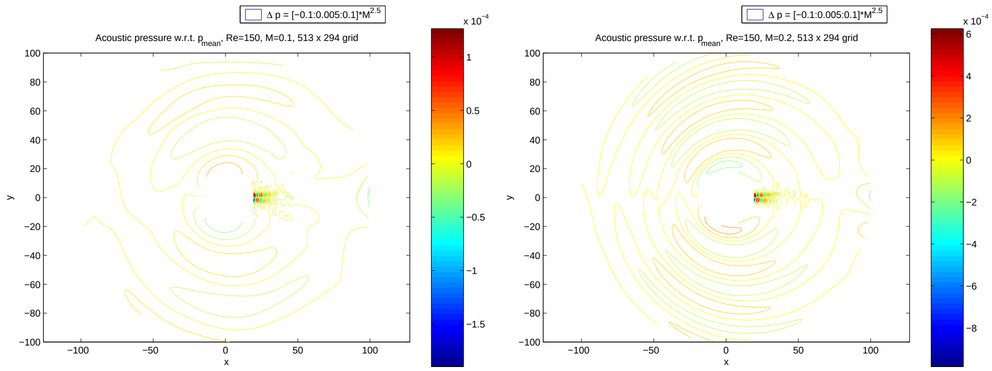

Fig. 2. Nondimensional instantaneous fluctuation pressure $\frac{\tilde{p}'}{\rho_\infty c_\infty^2} = [-0.1 : 0.005 : 0.1]M_\infty^{2.5}$ for $M_\infty = 0.1$ (left) and $M_\infty = 0.2$ (right) at $Re_\infty = 150$.

5 Conclusions

Since strict stability is taken into account in the construction of summation by parts (SBP) difference operators, those strictly stable high order difference methods for the Euler and Navier-Stokes equations have been providing reliable numerical tools for sound generation and sound propagation.

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