ROBUST STABILIZATION OF DISCRETE-TIME MULTI-MODEL SYSTEMS USING PIECEWISE AFFINE STATE-FEEDBACK

Olav Slupphaug * **Bjarne A. Foss** *,[1]

* *Norwegian University of Science and Technology*

Department of Engineering Cybernetics

email: Bjarne.Foss@itk.ntnu.no

Abstract: In this paper we develop a BMI based method for nonlinear robust stabilization. Robustness against model uncertainty is handled. The development is based on an uncertain multi-model representation of the plant, and an associated piecewise affine state-feedback structure. Assuming a quadratic Liapunov function, a BMI condition for robust (quadratic) stabilization is found. Control constraints are formulated as BMIs or LMIs. A branch-and-bound algorithm is used for solving the BMI problem, that is, finding the quadratic Liapunov function and the piecewise affine state-feedback. Finally, an example is given.

Keywords: Nonlinear systems, discrete-time, quadratic stabilization, multi-model systems, BMI, constraints

1. INTRODUCTION

Robust controller design is a key factor for implementing controllers. Robust design becomes particularly important, but also challenging, for nonlinear constrained uncertain systems, the outset for this work.

In our approach we have utilized results and ideas from LMI based control (Boyd *et al.*, 1994), (Johansson and Rantzer, 1997), and (Petterson and Lennartson, 1997), and multi-model systems (Murray-Smith and Johansen, 1997).

In recent years much work has been put into the development of nonlinear models which are composed of a set of local models (Murray-Smith and Johansen, 1997). In this work we focus on multi-model systems in which the local models are affine discrete-time state-space models, and we utilize this model structure to describe the model uncertainty class. This structure has at least three important advantages: *(i)* It is possible to utilize the affine structure of the local models for analysis and synthesis; *(ii)* the model class is rich in the sense that it approximates arbitrarily close a very large

[1] Corresponding author

class of nonlinear systems; *(iii)* the model structure is transparent and there exist support tools for model identification (Johansen and Foss, 1997).

In LMI-based control, which also has gained a lot of interest in the last few years, control system analysis- and synthesis problems are formulated as convex optimization problems involving linear matrix inequalities (LMIs). The reason for this interest is the development of very efficient interior-point algorithms for solving such problems (Boyd *et al.*, 1994). Many interesting control problems, in particular robust control problems, can be solved within the LMI framework. There are, however, interesting control problems which are very hard or impossible to formulate within the LMI framework. Some of these problems can be formulated within the more general bilinear matrix inequalities (BMIs) framework (Goh *et al.*, 1994). We utilize a piecewise affine state-feedback structure coupled with the uncertain multi-model to formulate the robust constrained nonlinear stabilization problem as a BMI feasibility problem. BMI problems are much harder than LMI problems since they, in general, are nonconvex. The efficient algorithms developed for the LMI problems do, however, provide a constructive basis on which branch-and-bound algorithms for BMI problems can be developed, see (Tuan *et al.*, 1997; Goh *et al.*, 1994).

The developed method can easily be integrated with other approaches as shown in (Slupphaug and Foss, 1998*a*). In that paper the method developed herein constitutes a basis for developing a robust MPC scheme for constrained uncertain nonlinear systems.

The paper is organized as follows. Firstly, we present the considered multi-model based uncertainty class. Then a BMI is found, which, if it is feasible, guarantees robust constrained stabilization of the origin of the uncertain system. Before the conclusion, the solution of the BMI feasibility problem is discussed, and an example is provided.

Some notation: $I_M := \{1, \ldots, M\}$; $N(x)$ is a neighborhood of x; $P > 0$ ($P \geq 0$) is short for $P = P^T > 0$ ($P = P^T \geq 0$); $||x||_H^2 := x^T H x$ where $H > 0$; let $a, b \in \mathbb{N}$ then $\{a, \ldots, b\} := \emptyset$ and $\{c_l\}_{l=a}^{b} := \emptyset$, when $b < a$.

2. PROBLEM STATEMENT AND UNCERTAINTY MODEL

The problem we investigate is to find a state-feedback which robustly stabilize the origin of a *constrained* uncertain nonlinear plant which is assumed — at any state, for any control input, and for any time — to be described by *some convex combination* of some apriori given *affine discrete-time state-space systems*. There is a finite number of these apriori given affine systems. That is, the uncertain (nonlinear) plant is assumed to be given by

$$x_{k+1} = \sum_{j \in I_{N_m}} \omega_j(x_k, u_k, k)(A_j x_k + B_j u_k + c_j), \tag{1}$$

where $k \geq 0$, x_0 given, $x_k \in X_m \subset \mathbb{R}^n$, $u_k \in U_m \subset \mathbb{R}^m$, the *local models* (A_j, B_j, c_j)'s are triplets which elements have appropriate dimensions, N_m is the number of local models, the uncertain *weights*

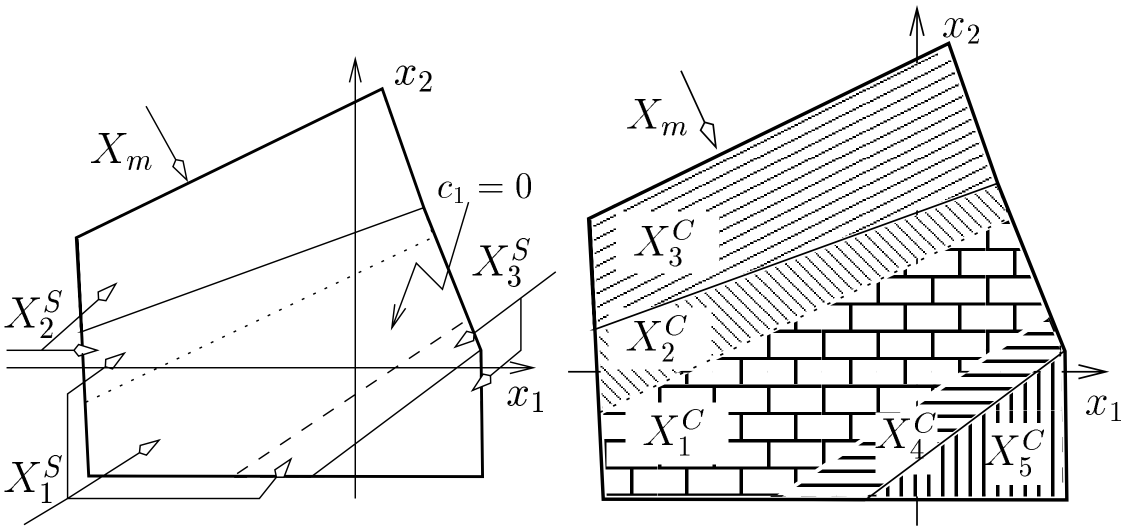

Fig. 1. The state-space supports; X_1^S, X_2^S, and X_3^S for a multi-model system with three local models (left), and the associated 5 clusters (right).

$$\omega_j : X_m \times U_m \times \mathbb{N} \to [0,1],\ \forall j \in I_{N_m},$$

and

$$\sum_{j \in I_{N_m}} \omega_j(x,u,k) = 1,\ \forall (x,u,k) \in X_m \times U_m \times \mathbb{N}.$$

Each of the model validity sets X_m and U_m is assumed to be a connected set containing the origin in its interior. We also — for notational simplicity — let X_m and U_m denote the state- and control constraints, respectively.

Uncertainty is represented by allowing $\omega(\cdot,\cdot,\cdot) := (\omega_1(\cdot,\cdot,\cdot),\ldots,\omega_{N_m}(\cdot,\cdot,\cdot))$ to vary within a predefined set Ω. The uncertainty class $\mathcal{M}$ is defined via this set. Next, Ω is defined.

The uncertainty description, with control synthesis in mind, is based on the assumption that all that is known or utilized about the uncertain weights are their *state-space supports*, X_j^S. That is, knowledge of the sets

$$X_j^S := \bigcup_{(u,k) \in U_m \times \mathbb{N}} \{x \mid \omega_j(x,u,k) > 0\}, \forall j \in I_{N_m}. \tag{2}$$

Notice that the projection on the state-space for all $u \in U_m$ in (2) implies that nonlinearities associated with the control input will be conservatively handled in the sense that one looses the ability to exploit possible knowledge of the nonlinearities associated with the control input. On the other hand, an arbitrary nonlinearity associated with the control input can, in principle, be handled. The easiest remedy for handling nonlinearities associated with the control inputs in a less conservative manner (in the sense above) is to delay the control input one sample and extend the state vector with the one-step delayed control input.

Associated with the state-space supports we define the following sets: for all $j \in I_{N_m}$

$$\Omega_j := \{\tilde{\omega} \mid \tilde{\omega} : X_m \times U_m \times \mathbb{N} \to [0,1] \text{ and } \tilde{\omega}(x,u,k) > 0 \text{ only when } x \in X_j^S\},$$

i.e. the set of all possible weights for local model number j. Now, let

$$\Omega := \{\omega = (\omega_1, \dots, \omega_{N_m}) \in \Omega_1 \times \dots \times \Omega_{N_m} | \sum_{j \in I_{N_m}} \omega_j(x, u, k) = 1, \ \forall (x, u, k) \in X_m \times U_m \times \mathbb{N}\},$$

i.e. the set of all valid convex combinations, and

$$f_\omega(x, u, k) := \sum_{j \in I_{N_m}} \omega_j(x, u, k)(A_j x + B_j u + c_j).$$

Finally

$$\mathcal{M} := \{f_\omega | \ \omega \in \Omega\}.$$

Thus, $\mathcal{M}$ now denotes the assumed multi-model uncertainty class.

Local models with $c_j \neq 0$ are assumed not to have support in some neighborhood of the origin. This amounts to assuming that all the plants $f \in \mathcal{M}$, and in particular the real plant, satisfies $0 = f(0, 0, k)$ for all $k \geq 0$, i.e. the equilibrium state and -control input are assumed to be known.

With the state-space supports, X_j^S, we also associate a *partitioning* of the state space into a set of N_c *clusters*. A *cluster*, X_j^C, is a set on which the same local models have support on the whole set, such that for any extension, at least one of these local models will not have support on the extension and/or at least one other local model has support on the extension. A set Λ_j of local model numbers, is naturally associated with a cluster. In Figure 1 (right part) the 5 clusters associated with the state-space supports given in Figure 1 (left part) are shown, in this case; $\Lambda_1 = \{1\}$, $\Lambda_2 = \{1, 2\}$, $\Lambda_3 = \{2\}$, $\Lambda_4 = \{1, 3\}$ and, $\Lambda_5 = \{3\}$.

3. ROBUST STABILIZATION

The aim of this section is to provide computationally verifiable sufficient conditions for robust stabilizability of the origin of system (1). The outcome of the computation will, when it is successful, be a piecewise affine state-feedback controller, and a quadratic Liapunov function for the origin of the closed-loop.

3.1 *Piecewise Affine State-Feedback*

We finitely parameterize the state-feedback, $u(x)$, as a *piecewise affine state-feedback*. With the cluster containing the origin and the clusters which closure contains the origin, assumed (without loss of generality) to be the first N_c^o clusters, we associate a linear state feedback

$$u(x) = K_l x \text{ when } x \in X_l^C, \ l \in I_{N_c^o}. \tag{3a}$$

With all the other clusters we associate an affine state feedback, i.e. for $l \in \{N_c^o + 1, \dots, N_c\}$

$$u(x) = K_l x + k_l \text{ when } x \in X_l^C. \tag{3b}$$

Remember that the clusters form a partition of X_m, so the above defined piecewise affine state-feedback is indeed well defined.

It should be noted that there is, in principle, no problem associating the piecewise state-feedback with a different partitioning of X_m than the one associated with the clusters. For reasons of clarity, however, we restrict the piecewise affine state-feedback to be associated with the clusters.

3.2 *Set Approximations*

To get BMI conditions for robust (constrained) stabilization it is sensible to approximate the clusters, and state- and control constraints using polytopes or ellipsoids. A short discussion on the outer approximations below is given in (Slupphaug and Foss, 1998*b*). Here, we only note that the given approximations exist for the given sets, and save them for later reference.

Assume that for $l \in I_{N_c^o}$ the polytope

$$\{x \mid E_l x \leq 0\} \supset X_l^C \tag{4}$$

is used as an *outer* approximation of X_l^C. For $l \in \{N_c^o + 1, \dots, N_c^p\}$ assume that the polytope

$$\{x \mid [E_l \; e_l] \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0\} \supset X_l^C \tag{5}$$

is used, and, finally, for $l \in \{N_c^p + 1, \dots, N_c\}$ assume that the ellipsoid

$$\{x \mid \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} E_l & e_l \\ e_l^T & \epsilon_l \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0\} \supset X_l^C \tag{6}$$

is used. N_c^p is the number of clusters outer approximated by polytopes. Furthermore, assume that X_m is *inner* approximated as follows

$$\bigcap_{i \in I_{N_{qx}}} \{x \mid ||x - x_{i,c}||^2_{H_{i,x}} \leq 1\} \subset X_m,$$

i.e. by an intersection of ellipsoids where $x_{i,c}$ denotes the centers of the ellipsoids, and N_{qx} denotes the number of ellipsoids. Similarly, we assume

$$\bigcap_{i \in I_{N_{qu}}} \{u| \; ||u - u_{i,c}||^2_{H_{i,u}} \leq 1\} \subset U_m. \tag{7}$$

3.3 *BMI for Robust Stabilization*

In this subsection, we investigate *quadratic stability* of the origin of the closed-loop using the piecewise affine state-feedback defined by (3). We will let $U_m = \mathbb{R}^m$, i.e. it assumed that no input constraints are present. The constrained case is deferred to Section 5.

Firstly, we precisely define quadratic stability in the present context. Based on (Corless, 1994) the following definition is adopted.

Definition 1

Given an uncertain system

$$x_{k+1} = f(x_k, k) \tag{9a}$$

$$f \in \tilde{\mathcal{M}} \tag{9b}$$

where $k \geq 0$, $x_k \in \mathbb{R}^n$, x_0 given, and all $f \in \tilde{\mathcal{M}}$ satisfies: $f : \tilde{X}_m \times \mathbb{N} \to \mathbb{R}^n$ and $f(0,k) = 0$ for all $k \geq 0$. We say that the origin is a *quadratically stable equilibrium for system* (9) if there $\exists M,\ P > 0,\ N(0)$ such that $N(0) \subset \tilde{X}_m$ and $\forall (a,i) \in N(0) \times \mathbb{N}$

$$f(a,i)^T P f(a,i) - a^T P a \leq -a^T M a.$$

If, in addition, there exists $\alpha \in (0,\infty)$ such that for a given set $\tilde{R}_A$

$$\tilde{R}_A \subset \{x \mid x^T P x \leq \alpha\} \subset N(0)$$

then the origin is said to be a *quadratically stable equilibrium for system* (9) *with a region of attraction associated with* $\tilde{R}_A$ *of at least* $\{x \mid x^T P x \leq \alpha\}$. $\triangle$

Next, the main result is presented.

THEOREM 1

If, restricting the W_l*'s to be symmetric and have nonnegative elements, there* $\exists M > 0, P = P^T, S = S^T, \{K_l\}_{l=1}^{N_c}, \{k_l\}_{l=N_c^o+1}^{N_c}, \{W_l\}_{l=1}^{N_c^p}$,
$\{\tau_l \in \mathbb{R}\}_{l=N_c^p+1}^{N_c}$ *such that*
$\forall l \in I_{N_c^o},\ j \in \Lambda_l$,

$$\begin{bmatrix} S & A_j + B_j K_l \\ \star & P - M - E_l^T W_l E_l \end{bmatrix} \geq 0, \tag{10a}$$

$\forall l \in \{N_c^o + 1, \ldots, N_c^p\},\ j \in \Lambda_l$,

$$\begin{bmatrix} S & A_j + B_j K_l & B_j k_l + c_j \\ \star & P - M - E_l^T W_l E_l & -E_l^T W_l e_l \\ \star & \star & -e_l^T W_l e_l \end{bmatrix} \geq 0, \tag{10b}$$

$\forall l \in \{N_c^p + 1, \ldots, N_c\},\ j \in \Lambda_l$,

$$\begin{bmatrix} S & A_j + B_j K_l & B_j k_l + c_j \\ \star & P - M + \tau_l E_l & \tau_l e_l \\ \star & \star & \tau_l \epsilon_l \end{bmatrix} \geq 0, \tag{10c}$$

and

$$SP + PS \leq 2I, \tag{11}$$

then the origin is a quadratically stable equilibrium for the closed-loop.
If, in addition, there exist reals α *and* β *such that*

$$\begin{bmatrix} P - \beta R_A & 0 \\ 0 & \beta - \alpha \end{bmatrix} \leq 0, \tag{12a}$$

and reals $\{\lambda_i\}_{i \in I_{N_{qx}}}$ *such that* $\forall i \in I_{N_{qx}}$

$$\begin{bmatrix} \lambda_i H_{i,x} - P & -\lambda_i H_{i,x} x_{i,c}, \\ \star & \lambda_i (x_{i,c}^T H_{i,x} x_{i,c} - 1) + \alpha \end{bmatrix} \leq 0, \tag{12b}$$

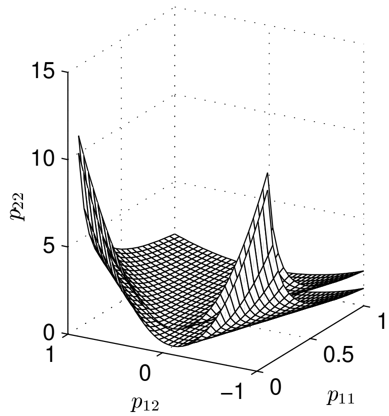

Fig. 2. Satisfaction of $P^{-1} \geq P$ and $P > 0$ vs. $P > 0$ only.

then the origin is a quadratically stable equilibrium for the closed-loop with a region of attraction associated with $\{x \mid ||x||_{R_A}^2 \leq 1\}$ *of at least* $\{x \mid x^T P x \leq \alpha\}$.

$\triangle$

The $\star$ elements are induced by symmetry of the associated matrices. The given set $\{x \mid ||x||_{R_A}^2 \leq 1\}$ would typically denote *the smallest acceptable region of attraction*.

The proof (Slupphaug and Foss, 1998*b*) proceeds by using the so-called $\mathcal{S}$-procedure and Schur complements (Boyd *et al.*, 1994), and some other results involving matrix inequalities.

The LMIs (10) are conditions for the decrease, in the different clusters, of the Liapunov function $x \mapsto x^T P x$ along all possible closed-loop trajectories which can be generated by plants in $\mathcal{M}$ under the state-feedback (3). The LMIs (12) are conditions for the Liapunov level set $\{x \mid x^T P x \leq \alpha\}$ to contain $\{x \mid ||x||_{R_A}^2 \leq 1\}$ (12a), while simultaneously being contained in X_m (12b). The BMI (11) origins from the inequality $P^{-1} \geq S$, which emanates from using Schur complements to get the LMIs (10).

It seems very hard to get an equivalent LMI condition. It should be noted that there exists an LMI which feasibility implies feasibility of the BMI in Theorem 1. However, this LMI will, in general, be very conservative. The existence of such an LMI can be seen by forcing $S = P$ and replacing the BMI (11) by the LMI

$$\begin{bmatrix} 2(I+P) & P+I \\ \star & I \end{bmatrix} \geq 0,$$

which stems from $P^{-1} \geq P$. To illustrate why this may be very conservative take $P \in \mathbb{R}^{2\times 2}$, then, for $P^{-1} \geq P$ and $P > 0$ to be satisfied, the elements of P must lie between the two surfaces in Figure 2, whereas $P > 0$ for any point above the lowest surface.

4. BRANCH-AND-BOUND FOR THE BMI FEASIBILITY PROBLEM

With the BMI feasibility problem associated with Theorem 1 we associate the following eigenvalue optimization problem denoted $\mathcal{P}_{\mathrm{EV}}$:

$$\min \vartheta$$

subject to

$$\begin{aligned} M &> -\vartheta I, \\ \mathcal{A}(P, S, M, \{K_l\}_{l=1}^{N_c}, \{k_l\}_{l=N_c^\circ+1}^{N_c}, \dots) &\geq -\vartheta I, \\ \mathcal{B}(P, S) &\geq -\vartheta I. \end{aligned}$$

The minimization is over all matrix and scalar variables in the matrix inequalities. The affine symmetric matrix valued mapping $\mathcal{A}(\cdot, \dots, \cdot)$ is given by (10), (12), and the nonnegativity condition on the elements of the W_ls. The biaffine symmetric matrix valued mapping $\mathcal{B}(\cdot, \cdot)$ is given by (11).

It is clear that if $\vartheta^* < 0$, where ϑ^* is the value of ϑ at the optimum, then the BMI and LMIs in Theorem 1 are satisfiable. $\mathcal{P}_{\mathrm{EV}}$ is a biconvex non-smooth optimization problem (Goh *et al.*, 1994).

We use branch-and-bound algorithm 3 in (Tuan *et al.*, 1997) for solving $\mathcal{P}_{\mathrm{EV}}$ (of course we stop when a feasible $\vartheta < 0$ is found). In algorithm 3 the branching is done on a set of lower dimension, in our case much lower, than the total problem dimension, as opposed to (Goh *et al.*, 1994) where the branching is done on a set with dimension equal to the total problem size. The number of so-called *complicating variables* gives the dimension of this lower dimensional set. The number of complicating variables is the smallest number of variables that need to be fixed to make the BMI an LMI. In our case the BMI structure arises due to the single BMI (11) (when control constraints are represented as LMIs). Since $P = P^T \in \mathbb{R}^{n \times n}$, this gives $(n^2 + n)/2$ complicating variables (the number of independent elements in P) which is much lower than the total problem size which might be ten times the number of complicating variables.

5. CONTROL INPUT CONSTRAINTS AS BMIS OR LMIS

To satisfy the control constraints, U_m, on all possible closed-loop trajectories starting within $\{x \mid ||x||_{R_A}^2 \leq 1\}$ it is sufficient to satisfy them on the *positively invariant* embracing set $\{x \mid x^T P x \leq \alpha\} \subset X_m$. For this to be the case it is sufficient that for all $(i, l) \in I_{N_{qu}} \times I_{N_c}$ and for all $x \in X_l^C \cap \{x \mid x^T P x \leq \alpha\}$

$$||K_l x + k_l - u_{i,c}||_{H_{i,u}}^2 \leq 1. \tag{14}$$

When outer approximating the clusters X_l^C in connection with formulating BMI or LMI conditions for satisfying control constraints one can use anyone of the approximations (4), (5), or (6). One is not restricted to use (4) when $l \in I_{N_c^\circ}$. We present the results when using (6), similar results for the other two cases can be derived. In this case we get, by invoking the $\mathcal{S}$-procedure followed by using Schur complements, that (14) can be transformed into the following sufficient BMI condition

$$\begin{bmatrix} \tau_{l,i}^P P + \tau_{l,i}^E E_l & \tau_{l,i}^E e_l & K_l^T \\ \star & 1 - \tau_{l,i}^P \alpha + \tau_{l,i}^E \epsilon_l & k_l - u_{i,c}^T \\ \star & \star & H_{i,u}^{-1} \end{bmatrix} \geq 0.$$

where all the (scalar) τs must be nonnegative, and $k_l := 0$ when $l \in I_{N_c^o}$. Notice that this results in only one more complicating variable (α). Forcing the τ^Ps to zero, which is equivalent to dropping the intersection with the Liapunov level set in (14), this BMI condition reduce to a, more conservative though, LMI condition.

6. EXAMPLE

In this example we show that the proposed design procedure provides stabilizing controllers for multi-model systems where there are uncontrollable local models, as well as local models which are affine and not merely linear. In particular, having uncontrollable local models means that no controller can simultaneously stabilize all the local models on the whole state-space.

The uncertain multi-model consists of three affine local models, and has three clusters. The local models are as follows

$$A_1 = \begin{bmatrix} -1 & 2 \\ -1.2 & 1 \end{bmatrix} B_1 = 100 \begin{bmatrix} .1 & 0 \\ .1 & 1 \end{bmatrix} c_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.1 & 0 \\ .3 & -1.1 \end{bmatrix} B_2 = 100 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} c_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1.5 & 0 \\ .4 & -.7 \end{bmatrix} B_3 = 100 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} c_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Notice that all local models are open-loop unstable. Furthermore, is local model number 3 not controllable. In this example we let $U_m = [-1, 1] \times [-1, 1]$, and $X_m = \{x \mid x_1^2 + x_2^2 - 100 \leq 0\}$. The smallest acceptable region of attraction is taken to be a ball with radius 6.

X_m, being itself an ellipsoid, is equal to its inner approximation. U_m is inner approximated using two ellipsoids with centers at the origin and with longest axes ten times the length of the shortest.

The clusters and sets of associated local models are

$$X_1^C = X_m \setminus (X_2^C \cup X_3^C),$$
$$X_2^C = \{x \mid (x_1 - 3)^2 + (x_2 - 3)^2 - 1 \leq 0\},$$
$$X_3^C = \{x \mid (x_1 + 3)^2 + (x_2 + 3)^2 - 1 \leq 0\},$$
$$\Lambda_1 = \{1\},\ \Lambda_2 = \{1, 2\},\ \Lambda_3 = \{1, 3\}.$$

X_1^C is outer approximated by X_m, while X_2^C and X_3^C equal their outer approximations, and an LMI, as described above, is used to represent the control constraints.

Applying the developed BMI based robust design strategy to this example system, we get the following piecewise affine state feedback controller

$$K_1 = \begin{bmatrix} 0.0637 & -0.0746 \\ 0.0061 & -0.0043 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.0117 & 0.0010 \\ 0.0121 & -0.0129 \end{bmatrix} \; k_2 = \begin{bmatrix} -0.0094 \\ 0.0011 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 0.1459 & -0.1554 \\ -0.0051 & 0.0060 \end{bmatrix} \; k_3 = \begin{bmatrix} 0.0666 \\ -0.0087 \end{bmatrix},$$

and Liapunov matrix

$$P = \begin{bmatrix} 0.8760 & 0.2634 \\ 0.2634 & 1.8861 \end{bmatrix}$$

accompanied by $\alpha = 73.8766$.

A phase-space simulation plot showing two closed-loop motions ('o' and 'x') is given in the left part of Figure 3 ('o' starts at $x \approx (-9, 4.5)^T$ and 'x' starts at $x \approx (3.5, 3.5)^T$). We observe that the closed loop trajectories are confined to the set $\{x \mid x^T P x \leq \alpha\}$ when starting inside it, and that the prescribed relationship $\{x \mid ||x||_{R_A} \leq 1\} \subset \{x \mid x^T P x \leq \alpha\} \subset X_m$ is satisfied (the borders of each of these sets are shown). In the right part of Figure 3 we observe that the Liapunov function is indeed decreasing, and that the control input constraints are satisfied (these plots show the 'o' simulation). In the simulations the following weights defined the real system

$$\omega_1 = 1 \text{ on } X_1^C, \; 0.2 \text{ on } X_2^C, \text{ and } 0.6 \text{ on } X_3^C,$$

$$\omega_2 = 0 \text{ on } X_1^C, \; 0.8 \text{ on } X_2^C, \text{ and } 0 \text{ on } X_3^C,$$

$$\omega_3 = 0 \text{ on } X_1^C, \; 0 \text{ on } X_2^C, \text{ and } 0.4 \text{ on } X_3^C.$$

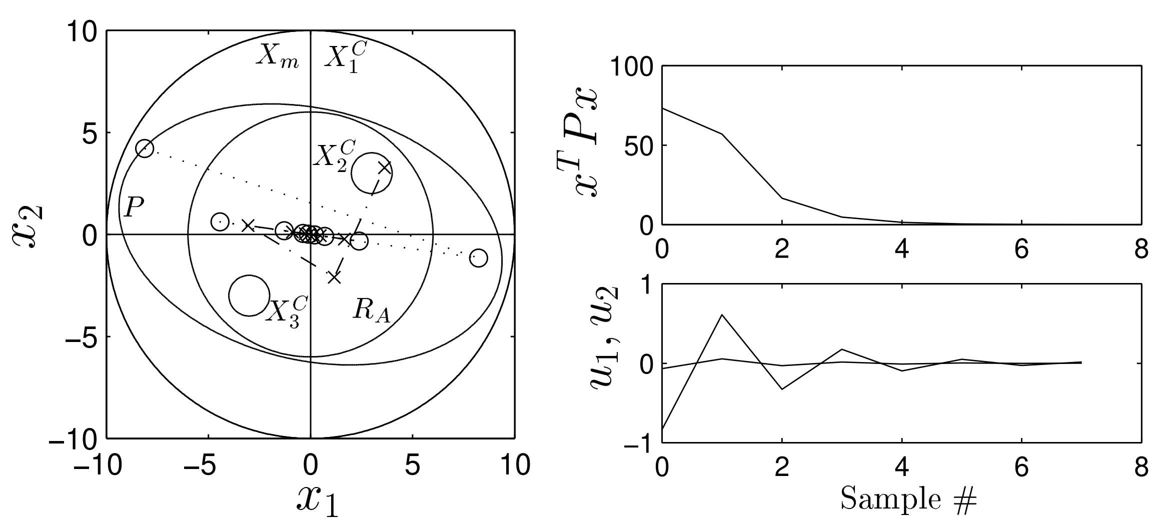

Fig. 3. Simulation results.

7. DISCUSSION AND CONCLUSION

The major drawback with the proposed method is that a BMI feasibility problem, which number of complicating variables grows quadratically with the number of states, has to be solved. Thus, presently only problems with a low number of states $(3-4)$ can be attempted solved using a *global* approach. A natural next step, probably very hard though, would be to establish the necessity of a BMI formulation, and, if it turns out to be unnecessary, find an equivalent LMI. Anyway, it should be possible to formulate an equivalent BMI which does not, or at least less dramatically, suffer from this growth in complexity. Another sensible parallel track to follow would, perhaps, be to check the performance of different *local* approaches on problems with a high number of states.

Also, we note that important problems such as output feedback and disturbance rejection cannot, at the moment, be addressed by the given design procedure. However, the work (Dussy and El Ghaoui, 1997) provides hope for such extensions.

In conclusion, a computationally solvable robust constrained controller synthesis method, is developed. Robustness against model uncertainty is investigated, and the problem solved is a stabilization problem. Although much work remains to be done before the approach may serve as a general tool for handling robust constrained nonlinear control problems, it may provide an interesting basis on which to continue.

In (Slupphaug, 1998) a more thorough investigation of the topics addressed in this paper is given.

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