Output Feedback Boundary Control of a Ginzburg-Landau Model of Vortex Shedding*

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Abstract

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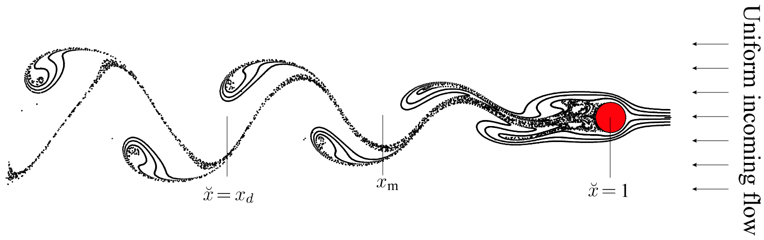

Figure 1: Vortex shedding from a cylinder.

1 Introduction

$$\frac{\partial A(\breve{x},t)}{\partial t} = a_1 \frac{\partial^2 A(\breve{x},t)}{\partial \breve{x}^2} + a_2(\breve{x}) \frac{\partial A(\breve{x},t)}{\partial \breve{x}} + a_3(\breve{x}) A(\breve{x},t) + a_4 |A(\breve{x},t)|^2 A(\breve{x},t) \quad (1)$$

We consider the linearized Ginzburg-Landau equation given by

$$\frac{\partial A(\breve{x},t)}{\partial t} = a_1 \frac{\partial^2 A(\breve{x},t)}{\partial \breve{x}^2} + a_2(\breve{x}) \frac{\partial A(\breve{x},t)}{\partial \breve{x}} + a_3(\breve{x}) A(\breve{x},t) \quad (2)$$

for $\breve{x} \in (-\infty, 1)$, with boundary conditions

$$A(1,t) = u(t), \quad (3)$$
$$A(-\infty,t) = 0, \quad (4)$$

and where $A : (-\infty, 1] \times \mathbb{R}_+ \to \mathbb{C}$, $a_2 \in C^2((-\infty,1];\mathbb{C})$, $a_3 \in C^1((-\infty,1];\mathbb{C})$, $a_1 \in \mathbb{C}$, and $u : \mathbb{R}_+ \to \mathbb{C}$ is the control input. a_1 is assumed to have strictly positive real part.

2 Problem Statement

We now truncate the semi-infinte domain $(-\infty, 1]$ at $\breve{x} = x_d \in (-\infty, 1)$, and rewrite the equation to obtain two coupled partial differential equations in real variables and coefficients by defining

$$\rho(x,t) = \Re(B(x,t)) = \frac{1}{2}\left(B(x,t) + \bar{B}(x,t)\right), \quad (5)$$
$$\iota(x,t) = \Im(B(x,t)) = \frac{1}{2i}\left(B(x,t) - \bar{B}(x,t)\right), \quad (6)$$

where

$$x = \frac{\breve{x} - x_d}{1 - x_d}, \text{ and } B(x,t) = A(\breve{x},t) \exp\left(\frac{1}{2a_1}\int_{x_d}^{\breve{x}} a_2(\tau)\, d\tau\right). \quad (7)$$

i denotes the imaginary unit, and $\bar{}$ denotes complex conjugation. Equation (2) becomes

$$\rho_t = a_R \rho_{xx} + b_R(x)\rho - a_I \iota_{xx} - b_I(x)\iota, \tag{8}$$

$$\iota_t = a_I \rho_{xx} + b_I(x)\rho + a_R \iota_{xx} + b_R(x)\iota, \tag{9}$$

for $x \in (0,1)$, with boundary conditions

$$\rho(0,t) = 0,\ \iota(0,t) = 0, \tag{10}$$

$$\rho(1,t) = u_R(t),\ \iota(1,t) = u_I(t), \tag{11}$$

and where

$$a_R \triangleq \frac{1}{(1-x_d)^2}\Re(a_1),\ a_I \triangleq \frac{1}{(1-x_d)^2}\Im(a_1), \tag{12}$$

$$b_R(x) \triangleq \Re\left(a_3(\breve{x}) - \frac{1}{2}a_2'(\breve{x}) - \frac{1}{4a_1}a_2^2(\breve{x})\right), \tag{13}$$

$$b_I(x) \triangleq \Im\left(a_3(\breve{x}) - \frac{1}{2}a_2'(\breve{x}) - \frac{1}{4a_1}a_2^2(\breve{x})\right). \tag{14}$$

3 Stabilization by State Feedback

In [1], extending the results in [3, 4], the state feedback stabilization problem was solved by searching for a coordinate transformation on the form

$$\tilde{\rho}(x,t) = \rho(x,t) - \int_0^x [k(x,y)\rho(y,t) + k_c(x,y)\iota(y,t)]\,dy, \tag{15}$$

$$\tilde{\iota}(x,t) = \iota(x,t) - \int_0^x [-k_c(x,y)\rho(y,t) + k(x,y)\iota(y,t)]\,dy, \tag{16}$$

transforming system (8)–(11) into

$$\tilde{\rho}_t = a_R\tilde{\rho}_{xx} + f_R(x)\tilde{\rho} - a_I\tilde{\iota}_{xx} - f_I(x)\tilde{\iota}, \tag{17}$$

$$\tilde{\iota}_t = a_I\tilde{\rho}_{xx} + f_I(x)\tilde{\rho} + a_R\tilde{\iota}_{xx} + f_R(x)\tilde{\iota}, \tag{18}$$

for $x \in (0,1)$, with boundary conditions

$$\tilde{\rho}(0,t) = 0,\ \tilde{\iota}(0,t) = 0,\ \tilde{\rho}(1,t) = 0,\ \tilde{\iota}(1,t) = 0. \tag{19}$$

By the choice of f_R and f_I, system (17)–(19) can be given any desired level of stability. The corresponding stable behaviour for the original system is ensured by the control input

$$u_R(t) = \int_0^1 \left[k_1(y)\,\rho(y,t) + k_{c,1}(y)\,\iota(y,t)\right] dy, \tag{20}$$

$$u_I(t) = \int_0^1 \left[-k_{c,1}(y)\,\rho(y,t) + k_1(y)\,\iota(y,t)\right] dy, \tag{21}$$

where

$$k_1(y) = k(1,y), \tag{22}$$

$$k_{c,1}(y) = k_c(1,y). \tag{23}$$

The skew-symmetric form of (20)–(21), is postulated from the skew-symmetric form of (8)–(9). The following result was proven in [1].

Theorem 1

i. *The pair of kernels, $k(x,y)$ and $k_c(x,y)$, satisfy the partial differential equation*

$$k_{xx} = k_{yy} + \beta(x,y)k + \beta_c(x,y)k_c, \tag{24}$$

$$k_{c,xx} = k_{c,yy} - \beta_c(x,y)k + \beta(x,y)k_c, \tag{25}$$

for $(x,y) \in T = \{x,y : 0 < y < x < 1\}$, with boundary conditions

$$k(x,x) = -\frac{1}{2}\int_0^x \beta(\gamma,\gamma)d\gamma, \tag{26}$$

$$k_c(x,x) = \frac{1}{2}\int_0^x \beta_c(\gamma,\gamma)d\gamma, \tag{27}$$

$$k(x,0) = 0, \tag{28}$$

$$k_c(x,0) = 0, \tag{29}$$

where

$$\beta(x,y) = \left[a_R\left(b_R(y) - f_R(x)\right) + a_I\left(b_I(y) - f_I(x)\right)\right] / \left(a_R^2 + a_I^2\right), \tag{30}$$

$$\beta_c(x,y) = \left[a_R\left(b_I(y) - f_I(x)\right) - a_I\left(b_R(y) - f_R(x)\right)\right] / \left(a_R^2 + a_I^2\right). \tag{31}$$

The equation (24)–(25) with boundary conditions (26)–(29) has a unique $C^2(\mathcal{T})$ solution, given by

$$
\begin{aligned}
k(x,y) &= \sum_{n=0}^{\infty} G_n(x+y, x-y), & (32)\\
k_c(x,y) &= \sum_{n=0}^{\infty} G_{c,n}(x+y, x-y), & (33)
\end{aligned}
$$

where

$$
\begin{aligned}
G_0(\xi,\eta) &= -\frac{1}{4}\int_{\eta}^{\xi} b(\tau,0)\,d\tau, & (34)\\
G_{c,0}(\xi,\eta) &= \frac{1}{4}\int_{\eta}^{\xi} b_c(\tau,0)\,d\tau, & (35)\\
G_{n+1}(\xi,\eta) &= \frac{1}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} b(\tau,s)\,G_n(\tau,s)\,dsd\tau + \frac{1}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} b_c(\tau,s)\,G_{c,n}(\tau,s)\,dsd\tau, & (36)\\
G_{c,n+1}(\xi,\eta) &= -\frac{1}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} b_c(\tau,s)\,G_n(\tau,s)\,dsd\tau + \frac{1}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} b(\tau,s)\,G_{c,n}(\tau,s)\,dsd\tau, & (37)
\end{aligned}
$$

and

$$
b(\xi,\eta) = \beta\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right), b_c(\xi,\eta) = \beta_c\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right). \quad (38)
$$

ii. *The inverse of (15)–(16) exists and is in the form*

$$
\begin{aligned}
\rho(x,t) &= \tilde{\rho}(x,t) - \int_0^x \left[l(x,y)\,\tilde{\rho}(y,t) + l_c(x,y)\,\tilde{\iota}(y,t)\right]dy, & (39)\\
\iota(x,t) &= \tilde{\iota}(x,t) - \int_0^x \left[-l_c(x,y)\,\tilde{\rho}(y,t) + l(x,y)\,\tilde{\iota}(y,t)\right]dy, & (40)
\end{aligned}
$$

where l and l_c are $C^2(\mathcal{T})$ functions.

iii. *If f_R and f_I are chosen such that*

$$
\sup_{x\in[0,1]}\left(f_R(x) + \frac{1}{2}\left|f_I'(x)\right|\right) \le -c, \quad (41)
$$

where $c \ge 0$, then for any intial data $(\rho_0, \iota_0) \in H_1(0,1)$, the system (8)–(11) in closed loop with the control law (20)–(23) has a unique classical solution $(\rho,\iota) \in C^{2,1}((0,1)\times(0,\infty))$ and is exponentially stable at the origin in the $L_2(0,1)$ and $H_1(0,1)$ norms.

It was also shown in [1], that for the numerical coefficients given in [6], the target system (17)–(19) can be chosen such that the kernels have compact support and are independent of the choice of x_d. In this case, we have that $k_1(y) = 0$ and $k_{c,1}(y) = 0$ when $y < x_s$ for some $x_s \in (0,1)$, and stability of the zero solution is ensured for the system evolving on the semi-infinite domain ($x_d \to -\infty$). It follows that even though the system evolves on a semi-infinite domain, we need to design an observer that provides an estimate of the state in $(x_s, 1)$, only. In the anti-collocated case, that is when the measurement is downstream of the cylinder, we can design the observer independent of the choice of x_d, guaranteeing output feedback stabilization on the semi-infinite domain. In the collocated case, that is when sensing and actuation are both at the location of the cylinder, stability is guaranteed when the system is truncated to a finite domain. In the latter case, the semi-infinite case can be approximated with arbitrary accuracy by increasing the domain on which the estimate is computed. It is interesting to notice that also in this case, the output injection gains are independent of the size of the domain.

In the next two sections, we design observers for the anti-collocated and collocated cases for system (8)–(11) by modifying the results in [5] to deal with two coupled pde's and the semi-infinite domain.

4 Anti-Collocated Output Feedback Design

Suppose that $\rho(x_m,t)$, $\iota(x_m,t)$, $\rho_x(x_m,t)$, and $\iota_x(x_m,t)$ are available for measurement for some $x_m \le x_s$. Without loss of generality, we set $x_m = 0$ from now on.[1] Consider the following observer

$$\hat{\rho}_t = a_R \hat{\rho}_{xx} + b_R(x)\hat{\rho} - a_I \hat{\iota}_{xx} - b_I(x)\hat{\iota} + p_1(x)\left(\rho(0,t) - \hat{\rho}(0,t)\right) + p_{c,1}(x)\left(\iota(0,t) - \hat{\iota}(0,t)\right), \quad (42)$$

$$\hat{\iota}_t = a_I \hat{\rho}_{xx} + b_I(x)\hat{\rho} + a_R \hat{\iota}_{xx} + b_R(x)\hat{\iota} - p_{c,1}(x)\left(\rho(0,t) - \hat{\rho}(0,t)\right) + p_1(x)\left(\iota(0,t) - \hat{\iota}(0,t)\right), \quad (43)$$

[1] We can always rescale the semi-infinite interval $(-\infty, 1]$ such that the point $x_m \in (-\infty, 1)$ is moved to the origin in the new coordinates.

for $x \in (0,1)$, with boundary conditions

$$\hat{\rho}_x(0,t) = \rho_x(0,t) + p_0(\rho(0,t) - \hat{\rho}(0,t)) + p_{c,0}(\iota(0,t) - \hat{\iota}(0,t)), \tag{44}$$

$$\hat{\iota}_x(0,t) = \iota_x(0,t) - p_{c,0}(\rho(0,t) - \hat{\rho}(0,t)) + p_0(\iota(0,t) - \hat{\iota}(0,t)), \tag{45}$$

$$\hat{\rho}(1,t) = u_R(t), \ \hat{\iota}(1,t) = u_I(t). \tag{46}$$

In (42)–(46), $p_1(x)$, $p_{c,1}(x)$, p_0, and $p_{c,0}$ are output injection gains to be designed. The skew-symmetric form of the output injections is postulated from the skew-symmetric form of system (8)–(9). Notice that both $\rho(0,t)$, $\iota(0,t)$ and their first spatial derivate are used in the observer. Defining the observer error $\tilde{\rho}(x,t) = \rho(x,t) - \hat{\rho}(x,t)$, the error dynamics is given by

$$\tilde{\rho}_t = a_R\tilde{\rho}_{xx} + b_R(x)\tilde{\rho} - a_I\tilde{\iota}_{xx} - b_I(x)\tilde{\iota} - p_1(x)\tilde{\rho}(x_s,t) - p_{c,1}(x)\tilde{\iota}(x_s,t), \tag{47}$$

$$\tilde{\iota}_t = a_I\tilde{\rho}_{xx} + b_I(x)\tilde{\rho} + a_R\tilde{\iota}_{xx} + b_R(x)\tilde{\iota} + p_{c,1}(x)\tilde{\rho}(x_s,t) - p_1(x)\tilde{\iota}(x_s,t), \tag{48}$$

for $x \in (0,1)$, with boundary conditions

$$\tilde{\rho}_x(0,t) = -p_0\tilde{\rho}(0,t) - p_{c,0}\tilde{\iota}(0,t), \tag{49}$$

$$\tilde{\iota}_x(0,t) = p_{c,0}\tilde{\rho}(0,t) - p_0\tilde{\iota}(0,t), \tag{50}$$

$$\tilde{\rho}(1,t) = 0, \ \tilde{\iota}(1,t) = 0. \tag{51}$$

The observer gains $p_1(x)$, $p_{c,1}(x)$, p_0, and $p_{c,0}$ should be chosen to stabilize the system (47)–(51). Towards that end, we look for a transformation

$$\tilde{\rho}(x,t) = \tilde{\sigma}(x,t) - \int_{x_s}^{x} [p(x,y)\tilde{\sigma}(y,t) - p_c(x,y)\tilde{\kappa}(y,t)]\,dy, \tag{52}$$

$$\tilde{\iota}(x,t) = \tilde{\kappa}(x,t) - \int_{x_s}^{x} [p_c(x,y)\tilde{\sigma}(y,t) + p(x,y)\tilde{\kappa}(y,t)]\,dy, \tag{53}$$

transforming system (47)–(51) into the exponentially stable system

$$\tilde{\sigma}_t = a_R\tilde{\sigma}_{xx} + f_R(x)\tilde{\sigma} - a_I\tilde{\kappa}_{xx} - f_I(x)\tilde{\kappa}, \tag{54}$$

$$\tilde{\kappa}_t = a_I\tilde{\sigma}_{xx} + f_I(x)\tilde{\sigma} + a_R\tilde{\kappa}_{xx} + f_R(x)\tilde{\kappa}, \tag{55}$$

for $x \in (0,1)$, with boundary conditions

$$\begin{aligned} \tilde{\sigma}_x(0,t) &= 0,\ \tilde{\kappa}_x(0,t)=0, && (56)\\ \tilde{\sigma}(1,t) &= 0,\ \tilde{\kappa}(1,t)=0. && (57)\end{aligned}$$

Once the coordinate transformation (52)–(53) is found, the output injection terms are given by

$$\begin{aligned} p_1(x) &= a_R p_y(x,0) + a_I p_{c,y}(x,0), && (58)\\ p_{c,1}(x) &= -a_I p_y(x,0) + a_R p_{c,y}(x,0), && (59)\\ p_0 &= p(0,0),\ \text{and } p_{c,0} = p_c(0,0). && (60)\end{aligned}$$

By subtracting (47)–(51) from (54)–(57) and using (52)–(53) it can be shown that the kernels p and p_c must satisfy

$$\begin{aligned} p_{xx} &= p_{yy} - \beta(y,x)\,p - \beta_c(y,x)\,p_c, && (61)\\ p_{c,xx} &= p_{c,yy} + \beta_c(y,x)\,p - \beta(y,x)\,p_c, && (62)\end{aligned}$$

with boundary conditions

$$\begin{aligned} \frac{dp(x,x)}{dx} &= \frac{1}{2}\beta(x,x), && (63)\\ \frac{dp_{c,x}(x,x)}{dx} &= -\frac{1}{2}\beta_c(x,x), && (64)\\ p(1,y) &= p_c(1,y) = 0. && (65)\end{aligned}$$

Changing coordinates according to

$$\breve{x} = 1-y,\ \breve{y} = 1-x, \qquad (66)$$

and defining

$$\begin{aligned} \breve{\beta}(\breve{x},\breve{y}) &\triangleq \beta(y,x),\ \breve{\beta}_c(\breve{x},\breve{y}) \triangleq \beta_c(y,x), && (67)\\ \breve{p}(\breve{x},\breve{y}) &\triangleq p(x,y),\ \breve{p}_c(\breve{x},\breve{y}) \triangleq p_c(x,y), && (68)\end{aligned}$$

we obtain

$$\begin{aligned} \breve{p}_{\breve{y}\breve{y}} &= \breve{p}_{\breve{x}\breve{x}} - \breve{\beta}(\breve{x},\breve{y})\,\breve{p} - \breve{\beta}_c(\breve{x},\breve{y})\,\breve{p}_c, && (69)\\ \breve{p}_{c,\breve{y}\breve{y}} &= \breve{p}_{c,\breve{x}\breve{x}} + \breve{\beta}_c(\breve{x},\breve{y})\,\breve{p} - \breve{\beta}(\breve{x},\breve{y})\,\breve{p}_c, && (70)\end{aligned}$$

with boundary conditions

$$\breve{p}(\breve{x},\breve{x}) = -\frac{1}{2}\int_0^{\breve{x}} \breve{\beta}(\gamma,\gamma)\,d\gamma, \tag{71}$$

$$\breve{p}_c(\breve{x},\breve{x}) = \frac{1}{2}\int_0^{\breve{x}} \breve{\beta}_c(\gamma,\gamma)\,d\gamma, \tag{72}$$

$$\breve{p}(\breve{x},0) = \breve{p}_c(\breve{x},0) = 0. \tag{73}$$

Equation (69)–(73) is the same as equation (24)–(29). Thus, we get the following result directly from Theorem 1.

Theorem 2 *Suppose that f_R and f_I satisfy (41), and that p, p_c is a solution of (61)–(65). Then for any intial data $(\tilde{\rho}_0, \tilde{\imath}_0) \in H_1(0,1)$, the system (47)–(51) with output injection gains given by (58)–(60) has a unique classical solution $(\tilde{\rho}, \tilde{\imath}) \in C^{2,1}((0,1)\times(0,\infty))$ and is exponentially stable at the origin in the $L_2(0,1)$ and $H_1(0,1)$ norms.*

Having found a convergent observer and a stabilizing state feedback control law, it follows from standard results that the closed loop consisting of replacing the state with its estimate in the state feedback control law is exponentially stable at the origin [2]. We now formulate the solution to the output-feedback problem.

Theorem 3 *Suppose that f_R and f_I satisfy (41), and let k, k_c be the solution of (24)–(29) and p, p_c the solution of (61)–(65). Then for any intial data $(\rho_0, \iota_0), (\hat{\rho}_0, \hat{\imath}_0) \in H_1(0,1)$, the system (8)–(10) with the controller*

$$\rho(1,t) = \int_0^1 \left[k_1(y)\,\hat{\rho}(y,t) + k_{c,1}(y)\,\hat{\imath}(y,t)\right]dy, \tag{74}$$

$$\iota(1,t) = \int_0^1 \left[-k_{c,1}(y)\,\hat{\rho}(y,t) + k_1(y)\,\hat{\imath}(y,t)\right]dy, \tag{75}$$

and the observer

$$\hat{\rho}_t = a_R\hat{\rho}_{xx} + b_R(x)\,\hat{\rho} - a_I\hat{\imath}_{xx} - b_I(x)\,\hat{\imath} + p_1(x)\left(\rho(0,t) - \hat{\rho}(0,t)\right) + p_{c,1}(x)\left(\iota(0,t) - \hat{\imath}(0,t)\right), \tag{76}$$

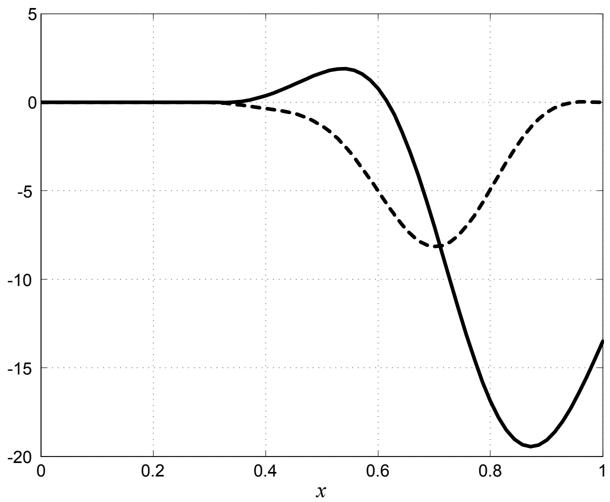

Figure 2: State feedback gain kernels.

$$\hat{\iota}_t = a_I \hat{\rho}_{xx} + b_I(x)\hat{\rho} + a_R \hat{\iota}_{xx} + b_R(x)\hat{\iota} \\ - p_{c,1}(x)(\rho(0,t) - \hat{\rho}(0,t)) + p_1(x)(\iota(0,t) - \hat{\iota}(0,t)), \quad (77)$$

$$\begin{aligned} \hat{\rho}_x(0,t) &= \rho_x(0,t) + p_0(\rho(0,t) - \hat{\rho}(0,t)) + p_{c,0}(\iota(0,t) - \hat{\iota}(0,t)), \\ \hat{\iota}_x(0,t) &= \iota_x(0,t) - p_{c,0}(\rho(0,t) - \hat{\rho}(0,t)) + p_0(\iota(0,t) - \hat{\iota}(0,t)), \\ \hat{\rho}(1,t) &= \int_0^1 [k_1(y)\hat{\rho}(y,t) + k_{c,1}(y)\hat{\iota}(y,t)]\,dy, \\ \hat{\iota}(1,t) &= \int_0^1 [-k_{c,1}(y)\hat{\rho}(y,t) + k_1(y)\hat{\iota}(y,t)]\,dy, \end{aligned}$$

have unique classical solutions $(\rho, \iota), (\hat{\rho}, \hat{\iota}) \in C^{2,1}((0,1) \times (0,\infty))$ *and are exponentially stable at the origin in the* $L_2(0,1)$ *and* $H_1(0,1)$ *norms.*

Figure 2: State feedback gain kernels, $k_1(y)$ and $k_{c,1}(y)$.

Figure 3: Open-loop plant response.

Figure 4: Output injection gains, $p_1(x)$ and $p_{c,1}(x)$.

Figure 5: Open-loop observer error.

Figure 6: Closed-loop plant response.

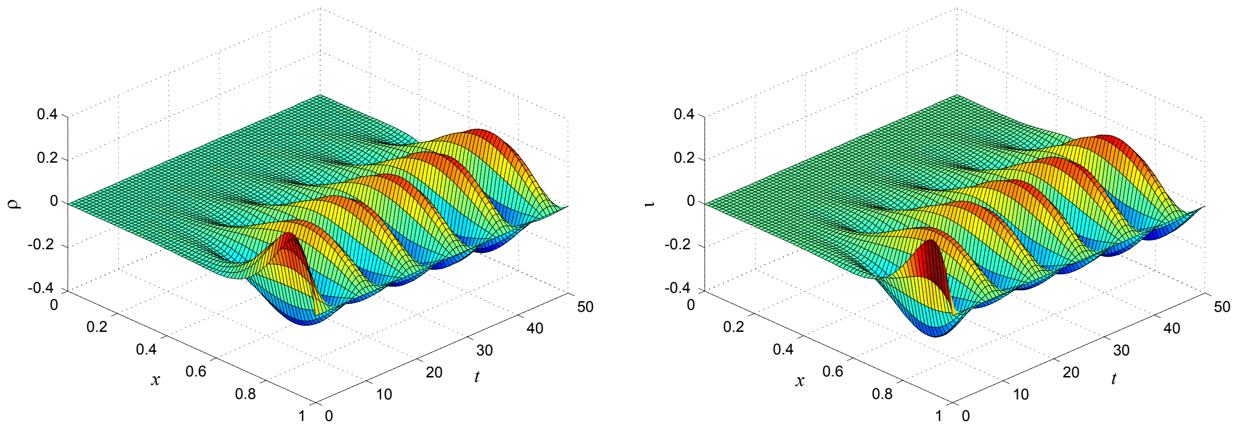

Figure 3: Open loop plant response.

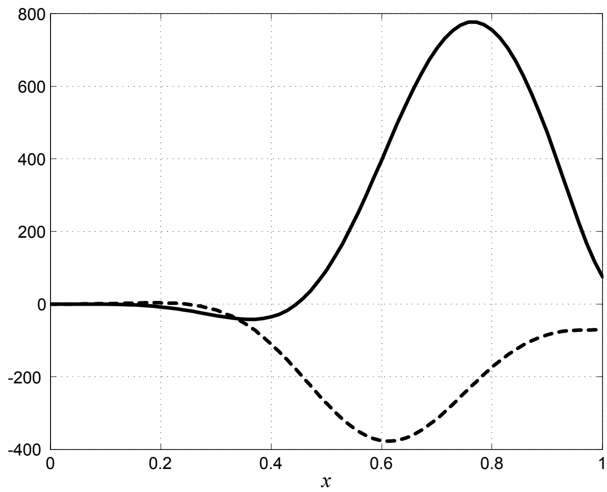

Figure 4: Observer gains.

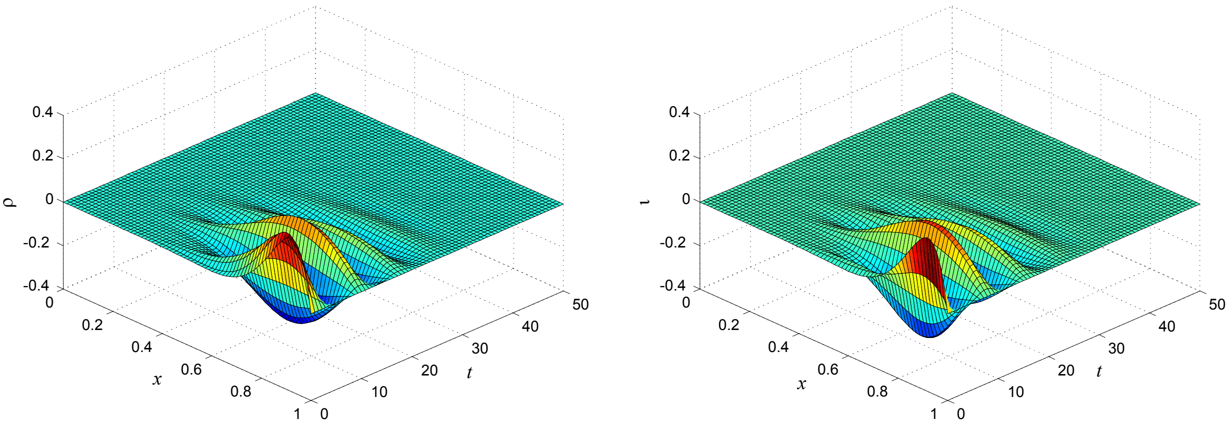

Figure 5: Observer error.

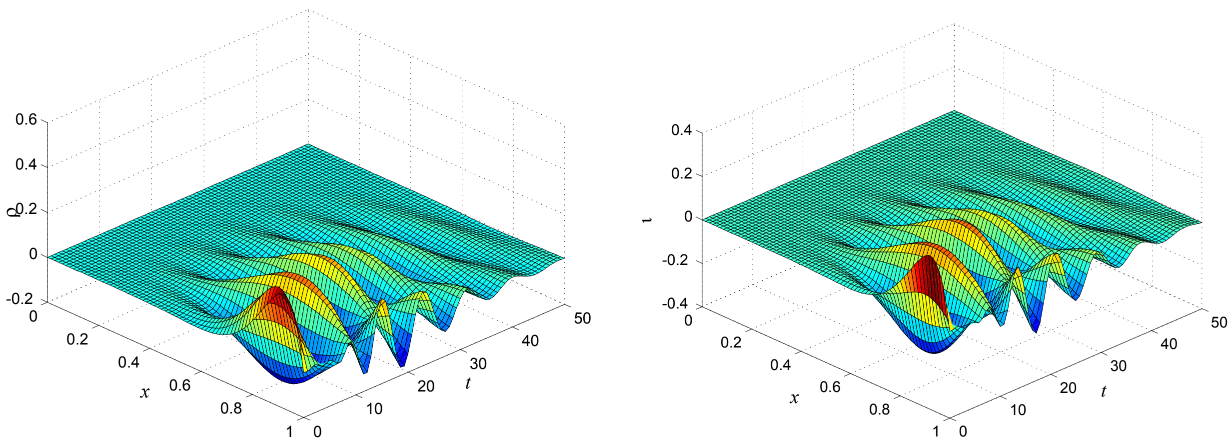

Figure 6: Plant response.

5 Collocated Output Feedback Design

In the collocated case, measurements are taken at the same location as the control input, that is on the cylinder surface. The measurements are $\rho(1,t)$ and $\iota(1,t)$, which leaves $\rho_x(1,t)$ and $\iota_x(1,t)$ for control input. The controller presented in Section 3 was of Dirichlet type. In the collocated case, we need Neumann type actuation, which can easily be derived from transformation (15)–(16) by setting homogeneous Neumann boundary conditions at $x = 1$ for the target system. The resulting controller is

$$\begin{aligned} u_R(t) &= \int_0^1 \left[k_x(1,y)\rho(y,t) + k_{c,x}(1,y)\iota(y,t)\right]dy + k(1,1)\rho(1,t) + k_c(1,1)\iota(1,t), && (78) \\ u_I(t) &= \int_0^1 \left[-k_{c,x}(1,y)\rho(y,t) + k_x(1,y)\iota(y,t)\right]dy - k_c(1,1)\rho(1,t) + k(1,1)\iota(1,t), && (79) \end{aligned}$$

where k and k_c, are the kernels found in from Section 3. The boundary conditions in (11) must be replaced by

$$\rho_x(1,t) = u_R(t), \ \iota_x(1,t) = u_I(t). \tag{80}$$

Consider the observer

$$\begin{aligned} \hat\rho_t = {} & a_R\hat\rho_{xx} + b_R(x)\hat\rho - a_I\hat\iota_{xx} - b_I(x)\hat\iota \\ & + p_1(x)\left(\rho(1,t) - \hat\rho(1,t)\right) + p_{c,1}(x)\left(\iota(1,t) - \hat\iota(1,t)\right), \end{aligned} \tag{81}$$

$$\begin{aligned} \hat\iota_t = {} & a_I\hat\rho_{xx} + b_I(x)\hat\rho + a_R\hat\iota_{xx} + b_R(x)\hat\iota \\ & - p_{c,1}(x)\left(\rho(1,t) - \hat\rho(1,t)\right) + p_1(x)\left(\iota(1,t) - \hat\iota(1,t)\right), \end{aligned} \tag{82}$$

for $x \in (0,1)$, with boundary conditions

$$\hat\rho(0,t) = 0, \tag{83}$$

$$\hat\iota(0,t) = 0, \tag{84}$$

$$\begin{aligned} \hat\rho_x(1,t) &= p_0\left(\rho(1,t) - \hat\rho(1,t)\right) + p_{c,0}\left(\iota(1,t) - \hat\iota(1,t)\right) + u_R(t), && (85) \\ \hat\iota_x(1,t) &= -p_{c,0}\left(\rho(1,t) - \hat\rho(1,t)\right) + p_0\left(\iota(1,t) - \hat\iota(1,t)\right) + u_I(t). && (86) \end{aligned}$$

The observer error is governed by

$$\tilde{\rho}_t = a_R\tilde{\rho}_{xx} + b_R(x)\tilde{\rho} - a_I\tilde{\imath}_{xx} - b_I(x)\tilde{\imath} - p_1(x)\tilde{\rho}(1,t) - p_{c,1}(x)\tilde{\imath}(1,t), \tag{87}$$

$$\tilde{\imath}_t = a_I\tilde{\rho}_{xx} + b_I(x)\tilde{\rho} + a_R\tilde{\imath}_{xx} + b_R(x)\tilde{\imath} + p_{c,1}(x)\tilde{\rho}(1,t) - p_1(x)\tilde{\imath}(1,t), \tag{88}$$

for $x \in (0,1)$, with boundary conditions

$$\tilde{\rho}(0,t) = 0, \tag{89}$$

$$\tilde{\imath}(0,t) = 0, \tag{90}$$

$$\begin{aligned}\tilde{\rho}_x(1,t) &= -p_0\tilde{\rho}(1,t) - p_{c,0}\tilde{\imath}(1,t), &(91)\\ \tilde{\imath}_x(1,t) &= p_{c,0}\tilde{\rho}(1,t) - p_0\tilde{\imath}(1,t). &(92)\end{aligned}$$

As in the previous section we now seek a transformation

$$\begin{aligned}\tilde{\rho}(x,t) &= \tilde{\sigma}(x,t) - \int_x^1 [p(x,y)\tilde{\sigma}(y,t) + p_c(x,y)\tilde{\kappa}(y,t)]\,dy, &(93)\\ \tilde{\imath}(x,t) &= \tilde{\kappa}(x,t) - \int_x^1 [-p_c(x,y)\tilde{\sigma}(y,t) + p(x,y)\tilde{\kappa}(y,t)]\,dy, &(94)\end{aligned}$$

that transforms system (87)–(92) into the exponentially stable system

$$\begin{aligned}\tilde{\sigma}_t &= a_R\tilde{\sigma}_{xx} + f_R(x)\tilde{\sigma} - a_I\tilde{\kappa}_{xx} - f_I(x)\tilde{\kappa}, &(95)\\ \tilde{\kappa}_t &= a_I\tilde{\sigma}_{xx} + f_I(x)\tilde{\sigma} + a_R\tilde{\kappa}_{xx} + f_R(x)\tilde{\kappa}, &(96)\end{aligned}$$

for $x \in (0,1)$, with boundary conditions

$$\begin{aligned}\tilde{\sigma}(0,t) &= 0,\ \tilde{\kappa}(0,t) = 0, &(97)\\ \tilde{\sigma}_x(1,t) &= 0,\ \tilde{\kappa}_x(1,t) = 0. &(98)\end{aligned}$$

When the transformation is found, the observer gains are given by

$$\begin{aligned}p_1(x) &= -a_R p_y(x,1) - a_I p_{c,y}(x,1), &(99)\\ p_{c,1}(x) &= a_I p_y(x,1) - a_R p_{c,y}(x,1), &(100)\\ p_0 &= -p(1,1),\ p_{c,0} = -p_c(1,1). &(101)\end{aligned}$$

Subtracting (87)–(92) from (95)–(98), and using (93)–(94), we obtain

$$p_{xx} = p_{yy} - \bar{\beta}(x,y)\,p - \bar{\beta}_c(x,y)\,p_c, \tag{102}$$

$$p_{c,xx} = p_{c,yy} + \bar{\beta}_c(x,y)\,p - \bar{\beta}(x,y)\,p_c, \tag{103}$$

with boundary conditions

$$p(x,x) = -\frac{1}{2}\int_0^x \bar{\beta}(\gamma,\gamma)\,d\gamma, \tag{104}$$

$$p_c(x,x) = \frac{1}{2}\int_0^x \bar{\beta}_c(\gamma,\gamma)\,d\gamma, \tag{105}$$

$$p(0,y) = p_c(0,y) = 0, \tag{106}$$

where

$$\bar{\beta}(x,y) = \left[a_R\left(b_R(x) - f_R(y)\right) + a_I\left(b_I(x) - f_I(y)\right)\right] / \left(a_R^2 + a_I^2\right), \tag{107}$$

$$\bar{\beta}_c(x,y) = \left[a_R\left(b_I(x) - f_I(y)\right) - a_I\left(b_R(x) - f_R(y)\right)\right] / \left(a_R^2 + a_I^2\right). \tag{108}$$

Setting

$$\breve{x} = y,\ \breve{y} = x, \tag{109}$$

$$\breve{p}(\breve{x},\breve{y}) = p(x,y), \tag{110}$$

$$\breve{p}_c(\breve{x},\breve{y}) = p_c(x,y), \tag{111}$$

we get

$$\breve{p}_{\breve{x}\breve{x}} = \breve{p}_{\breve{y}\breve{y}} + \bar{\beta}(\breve{y},\breve{x})\,\breve{p} + \bar{\beta}_c(\breve{y},\breve{x})\,\breve{p}_c, \tag{112}$$

$$\breve{p}_{c,\breve{x}\breve{x}} = \breve{p}_{c,\breve{y}\breve{y}} - \bar{\beta}_c(\breve{y},\breve{x})\,\breve{p} + \bar{\beta}(\breve{y},\breve{x})\,\breve{p}_c \tag{113}$$

with boundary conditions

$$\breve{p}(\breve{x},\breve{x}) = -\frac{1}{2}\int_0^{\breve{x}} \bar{\beta}(\gamma,\gamma)\,d\gamma, \tag{114}$$

$$\breve{p}_c(\breve{x},\breve{x}) = \frac{1}{2}\int_0^{\breve{x}} \bar{\beta}_c(\gamma,\gamma)\,d\gamma, \tag{115}$$

$$\breve{p}(\breve{x},0) = \breve{p}_c(\breve{x},0) = 0. \tag{116}$$

Finally, noticing that $\bar{\beta}(\breve{y},\breve{x})=\beta(\breve{x},\breve{y})$ and $\bar{\beta}_c(\breve{y},\breve{x})=\beta_c(\breve{x},\breve{y})$, we obtain

$$\begin{aligned}
\breve{p}_{\breve{x}\breve{x}} &= \breve{p}_{\breve{y}\breve{y}}+\beta(\breve{x},\breve{y})\,\breve{p}+\beta_c(\breve{x},\breve{y})\,\breve{p}_c, && (117)\\
\breve{p}_{c,\breve{x}\breve{x}} &= \breve{p}_{c,\breve{y}\breve{y}}-\beta_c(\breve{x},\breve{y})\,\breve{p}+\beta(\breve{x},\breve{y})\,\breve{p}_c, && (118)
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
\breve{p}(\breve{x},\breve{x}) &= -\frac{1}{2}\int_0^{\breve{x}}\beta(\gamma,\gamma)\,d\gamma, && (119)\\
\breve{p}_c(\breve{x},\breve{x}) &= \frac{1}{2}\int_0^{\breve{x}}\beta_c(\gamma,\gamma)\,d\gamma, && (120)\\
\breve{p}(\breve{x},0) &= \breve{p}_c(\breve{x},0)=0. && (121)
\end{aligned}$$

From (99)–(100), we have

$$\begin{aligned}
\breve{p}_1(\breve{y}) &= -a_R\breve{p}_{\breve{x}}(1,\breve{y})-a_I\breve{p}_{c,\breve{x}}(1,\breve{y}), && (122)\\
\breve{p}_{c,1}(\breve{y}) &= a_I\breve{p}_{\breve{x}}(1,\breve{y})-a_R\breve{p}_{c,\breve{x}}(1,\breve{y}). && (123)
\end{aligned}$$

Since equation (117)–(121) is identical with equation (24)–(29), it follows that the observer gains can be obtained from the feedback gains as

$$\begin{aligned}
p_1(x) &= -a_Rk_x(1,x)-a_Ik_{c,x}(1,x), && (124)\\
p_{c,1}(x) &= a_Ik_x(1,x)-a_Rk_{c,x}(1,x), && (125)
\end{aligned}$$

and we get the following result.

Theorem 4 *Suppose that f_R and f_I satisfy (41), and that k,k_c is a solution of (24)–(29). Then for any intial data $(\tilde{\rho}_0,\tilde{\iota}_0)\in H_1(0,1)$, the system (87)–(92) with output injection gains given by (124)–(125) and (101) has a unique classical solution $(\tilde{\rho},\tilde{\iota})\in C^{2,1}((0,1)\times(0,\infty))$ and is exponentially stable at the origin in the $L_2(0,1)$ and $H_1(0,1)$ norms.*

We now formulate the solution to the ouput-feedback problem in the collocated case.

Theorem 5 *Suppose that f_R and f_I satisfy (41), and let k, k_c be the solution of (24)–(29). Then for any intial data $(\rho_0, \iota_0), (\hat{\rho}_0, \hat{\iota}_0) \in H_1(0,1)$, the system (8)–(10) with the controller*

$$\rho_x(1,t) = \int_0^1 [k_x(1,y)\hat{\rho}(y,t) + k_{c,x}(1,y)\hat{\iota}(y,t)]\,dy + k(1,1)\rho(1,t) + k_c(1,1)\iota(1,t), \tag{126}$$

$$\iota_x(1,t) = \int_0^1 [-k_{c,x}(1,y)\hat{\rho}(y,t) + k_x(1,y)\hat{\iota}(y,t)]\,dy - k_c(1,1)\rho(1,t) + k(1,1)\iota(1,t), \tag{127}$$

and the observer

$$\hat{\rho}_t = a_R\hat{\rho}_{xx} + b_R(x)\hat{\rho} - a_I\hat{\iota}_{xx} - b_I(x)\hat{\iota} + p_1(x)(\rho(1,t) - \hat{\rho}(1,t)) + p_{c,1}(x)(\iota(1,t) - \hat{\iota}(1,t)), \tag{128}$$

$$\hat{\iota}_t = a_I\hat{\rho}_{xx} + b_I(x)\hat{\rho} + a_R\hat{\iota}_{xx} + b_R(x)\hat{\iota} - p_{c,1}(x)(\rho(1,t) - \hat{\rho}(1,t)) + p_1(x)(\iota(1,t) - \hat{\iota}(1,t)), \tag{129}$$

$$\begin{aligned}
\hat{\rho}(0,t) &= 0, \\
\hat{\iota}(0,t) &= 0, \\
\hat{\rho}_x(1,t) &= p_0(\rho(1,t) - \hat{\rho}(1,t)) + p_{c,0}(\iota(1,t) - \hat{\iota}(1,t)) \\
&\quad + \int_0^1 [k_x(1,y)\hat{\rho}(y,t) + k_{c,x}(1,y)\hat{\iota}(y,t)]\,dy + k(1,1)\rho(1,t) + k_c(1,1)\iota(1,t), \\
\hat{\iota}_x(1,t) &= -p_{c,0}(\rho(1,t) - \hat{\rho}(1,t)) + p_0(\iota(1,t) - \hat{\iota}(1,t)) \\
&\quad + \int_0^1 [-k_{c,x}(1,y)\hat{\rho}(y,t) + k_x(1,y)\hat{\iota}(y,t)]\,dy - k_c(1,1)\rho(1,t) + k(1,1)\iota(1,t),
\end{aligned}$$

have unique classical solutions $(\rho, \iota), (\hat{\rho}, \hat{\iota}) \in C^{2,1}((0,1) \times (0,\infty))$ and are exponentially stable at the origin in the $L_2(0,1)$ and $H_1(0,1)$ norms.

Figure 7: Output injection gains,$p_1(x)$ and $p_{c,1}(x)$.

Figure 8: Closed-loop observer error.

Figure 9: Closed-loop plant response.

6 Simulations with Nonlinear Model

The observer designs in Sections 4 and 5 are linear designs, ignoring the last term in (1). In this section, we explore the performance of the observer (in the collocated case) in simulations

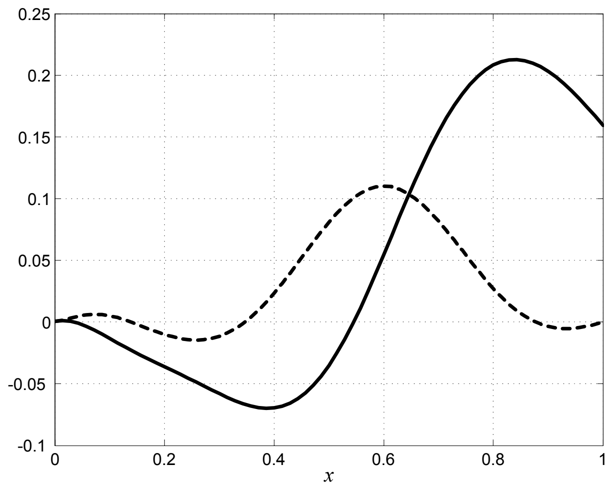

Figure 7: Observer gains.

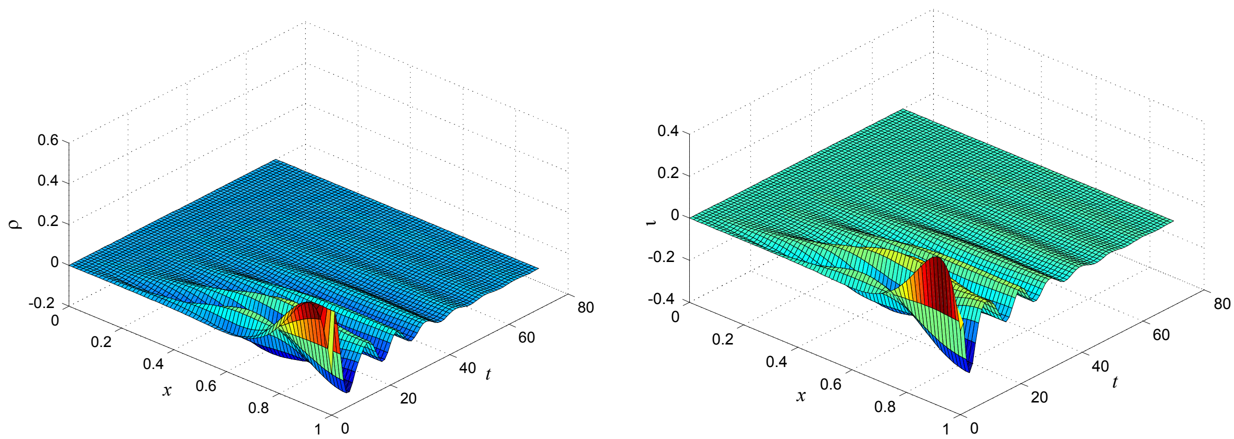

Figure 8: Observer error.

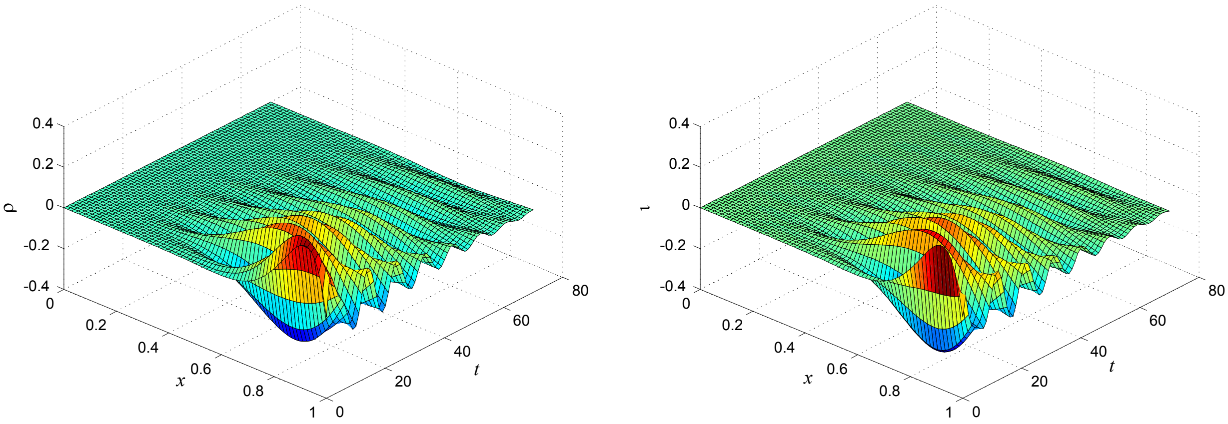

Figure 9: Plant response.

of the full, nonlinear model of vortex shedding. Including the nonlinear term, equation (8)–(9) becomes

$$\rho_t = a_R \rho_{xx} + \left(b_R(x) + c_R(x)\left(\rho^2+\iota^2\right)\right)\rho - a_I \iota_{xx} - \left(b_I(x) + c_I(x)\left(\rho^2+\iota^2\right)\right)\iota, \tag{130}$$

$$\iota_t = a_I \rho_{xx} + \left(b_I(x) + c_I(x)\left(\rho^2+\iota^2\right)\right)\rho + a_R \iota_{xx} + \left(b_R(x) + c_R(x)\left(\rho^2+\iota^2\right)\right)\iota, \tag{131}$$

where

$$c_R(x) = \Re(a_4)\exp\left(-\Re\left(\frac{1}{a_1}\int_{x_d}^{(1-x_d)x+x_d} a_2(\tau)\,d\tau\right)\right), \tag{132}$$

$$c_I(x) = \Im(a_4)\exp\left(-\Re\left(\frac{1}{a_1}\int_{x_d}^{(1-x_d)x+x_d} a_2(\tau)\,d\tau\right)\right). \tag{133}$$

Figure 10: Open-loop plant response for the nonlinear system.

Figure 11: Open-loop observer error for linear observer.

Figure 12: Open-loop observer error for nonlinear observer.

Figure 13: Closed-loop plant response for the nonlinear system, using the linear observer.

Figure 14: Closed-loop plant response for the nonlinear system, using the nonlinear observer consisting of a copy of (130)–(131) with the output injections designed for the linear case.

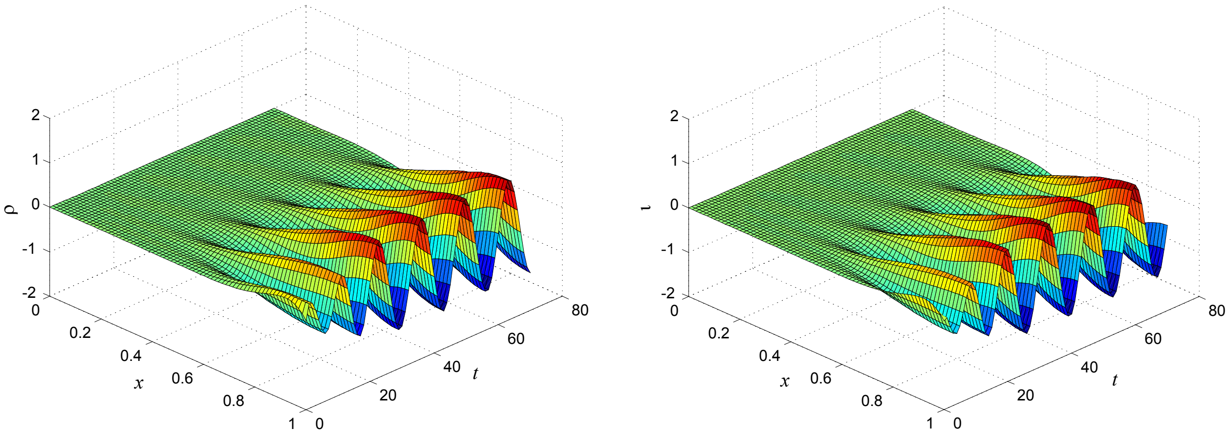

Figure 10: Open loop simulation of nonlinear system.

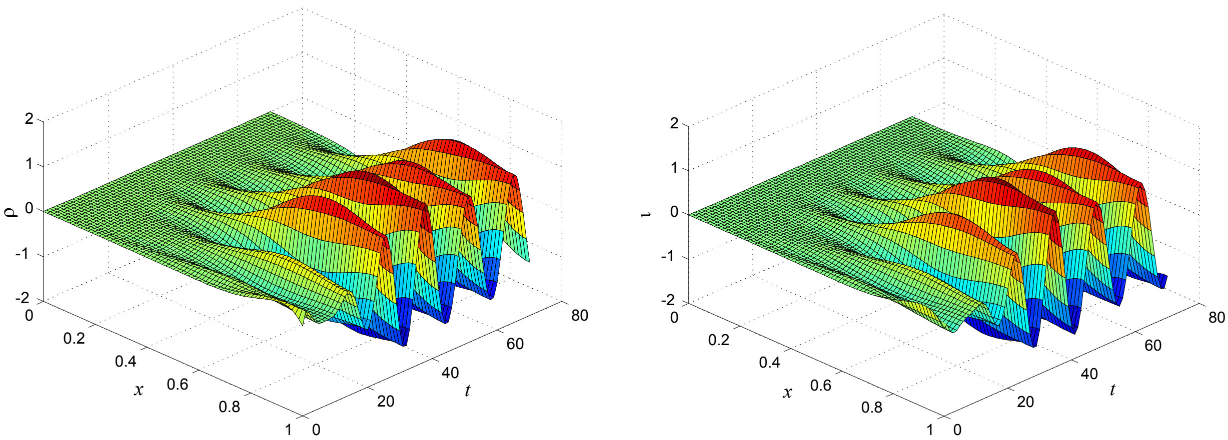

Figure 11: Observer error for linear observer.

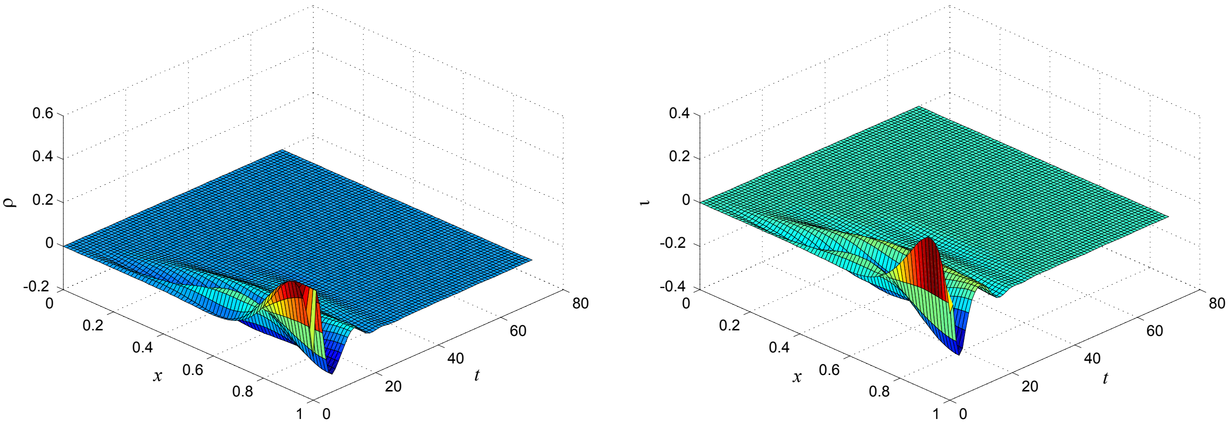

Figure 12: Observer error for nonlinear observer.

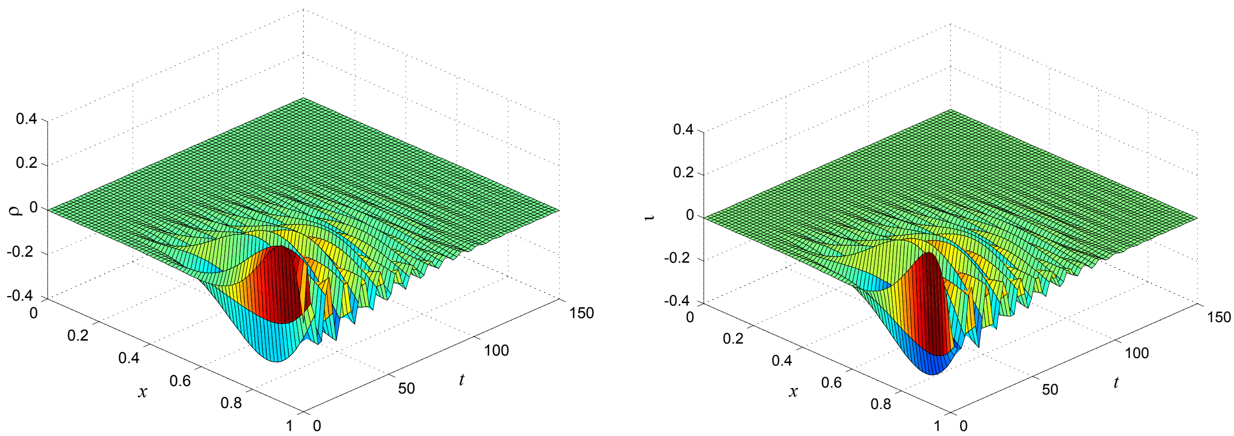

Figure 13: Closed-loop plant response for linear observer.

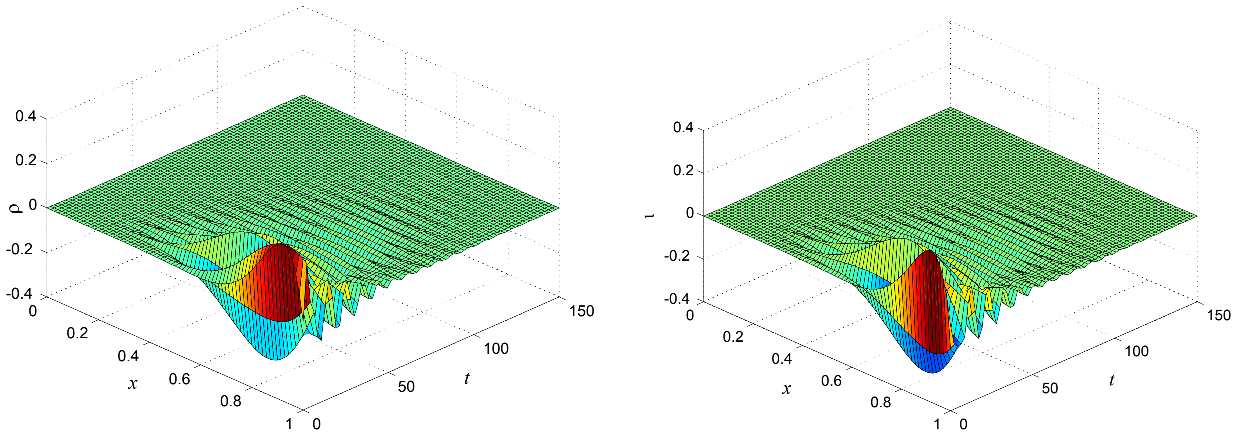

Figure 14: Closed-loop plant response for nonlinear observer.

7 Explicit Construction

Whereas the previous sections dealt with output feedback stabilization of the model of vortex shedding (2)–(4), we depart from that application in this section, and construct explicit formulas for the state feedback kernels in the special case of constant coefficients. We therefore assume that $b_R(x)$ and $b_I(x)$ in (8)–(9) are constants, which we denote b_R and b_I, respectively, and select the target system (17)–(18) by setting $f_R = -c$, for some non-negative constant c, and

$$f_I = b_I - \frac{a_I}{a_R}(b_R + c). \tag{134}$$

The stability criterion (41) is clearly satisfied. Furthermore, we have from (30)–(31) and (38) that

$$b = \frac{1}{a_R}(b_R + c), \tag{135}$$

$$b_c = 0. \tag{136}$$

Formulas (34)–(37) become

$$G_0(\xi, \eta) = -\frac{b}{4}(\xi - \eta), \tag{137}$$

$$G_{c,0}(\xi, \eta) = 0, \tag{138}$$

$$G_{n+1}(\xi,\eta) = \frac{b}{4}\int_\eta^\xi \int_0^\eta G_n(\tau,s)\,dsd\tau, \tag{139}$$
$$G_{c,n+1}(\xi,\eta) = \frac{b}{4}\int_\eta^\xi \int_0^\eta G_{c,n}(\tau,s)\,dsd\tau. \tag{140}$$

It is clear from (138) and (140), that

$$G_{c,n}(\xi,\eta) \equiv 0,\ n = 0,1,2,... \tag{141}$$

so it follows that

$$k_c(x,y) \equiv 0. \tag{142}$$

The problem of finding G is now identical to the problem solved in [4, Section 6] where it was found that

$$k(x,y) = -by\frac{I_1\left(\sqrt{b(x^2-y^2)}\right)}{\sqrt{b(x^2-y^2)}}, \tag{143}$$

where I_1 is the modified Bessel function of the first kind and of order one. In view of these calculations and Theorem 5, we obtain the following result.

Theorem 6 *Let $c \geq 0$. Then for any intial data $(\rho_0, \iota_0), (\hat{\rho}_0, \hat{\iota}_0) \in H_1(0,1)$, the system (8)–(10) with the controller*

$$\rho_x(1,t) = \int_0^1 k_2(y)\hat{\rho}(y,t)\,dy - \frac{1}{2a_R}(b_R+c)\rho(1,t), \tag{144}$$
$$\iota_x(1,t) = \int_0^1 k_2(y)\hat{\iota}(y,t)\,dy - \frac{1}{2a_R}(b_R+c)\iota(1,t), \tag{145}$$

and the observer

$$\hat{\rho}_t = a_R\hat{\rho}_{xx} + b_R\hat{\rho} - a_I\hat{\iota}_{xx} - b_I\hat{\iota} - k_2(x)\left[a_R(\rho(1,t)-\hat{\rho}(1,t)) - a_I(\iota(1,t)-\hat{\iota}(1,t))\right], \tag{146}$$

$$\hat{\iota}_t = a_I\hat{\rho}_{xx} + b_I\hat{\rho} + a_R\hat{\iota}_{xx} + b_R\hat{\iota} - k_2(x)\left[a_I(\rho(1,t)-\hat{\rho}(1,t)) + a_R(\iota(1,t)-\hat{\iota}(1,t))\right], \tag{147}$$

$$\hat{\rho}(0,t) = 0, \tag{148}$$
$$\hat{\iota}(0,t) = 0, \tag{149}$$
$$\hat{\rho}_x(1,t) = \frac{1}{2a_R}(b_R+c)(\rho(1,t)-\hat{\rho}(1,t)) + \int_0^1 k_2(y)\hat{\rho}(y,t)\,dy - \frac{1}{2a_R}(b_R+c)\rho(1,t), \tag{150}$$
$$\hat{\iota}_x(1,t) = \frac{1}{2a_R}(b_R+c)(\iota(1,t)-\hat{\iota}(1,t)) + \int_0^1 k_2(y)\hat{\iota}(y,t)\,dy - \frac{1}{2a_R}(b_R+c)\iota(1,t) \tag{151}$$

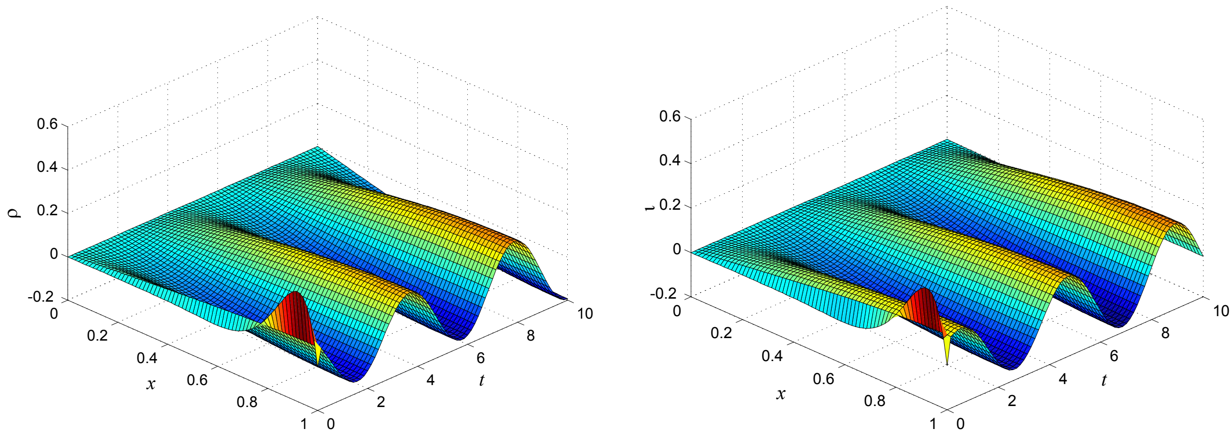

Figure 15: Open-loop plant response with $a_R = 1$, $a_I = 0$, $b_R = 2.5$, and $b_I = 1.5$.

where

$$k_2(x) = \frac{1}{a_R}(b_R + c)\, x \frac{I_2\left(\sqrt{\frac{1}{a_R}(b_R + c)(1 - x^2)}\right)}{1 - x^2}, \tag{152}$$

have unique classical solutions $(\rho, \iota), (\hat{\rho}, \hat{\iota}) \in C^{2,1}((0,1) \times (0,\infty))$ *and are exponentially stable at the origin in the* $L_2(0,1)$ *and* $H_1(0,1)$ *norms.*

In (152), I_2 is the modified Bessel function of the first kind, and of order two.

Figure 15: Open-loop plant response for the system with constant coefficients.

Figure 16: Open-loop observer error for explicit observer.

Figure 17: Closed-loop plant response for explicit controller/observer.

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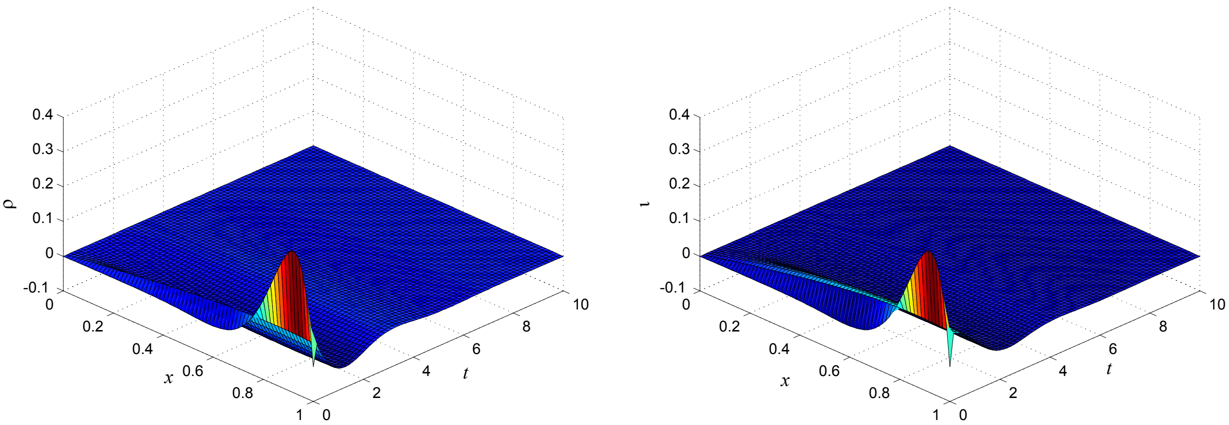

Figure 16: Open-loop observer error.

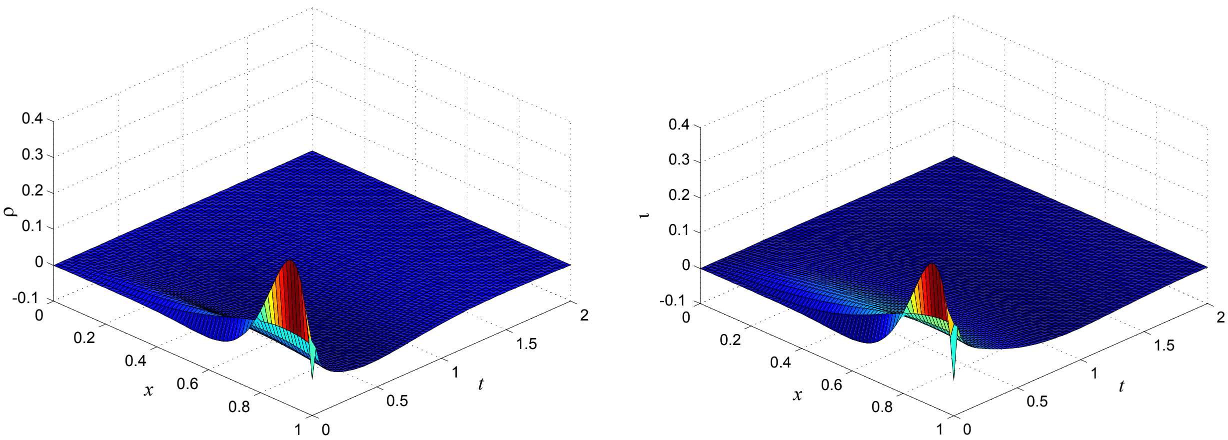

Figure 17: Closed-loop plant response.

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