Some Relations between Stability and Smoothness in Discrete-time Dynamic Models

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Abstract

Many discrete-time dynamic models of current interest are based on functions that, while generally continuous, are nonsmooth; examples include specific multimodels, hinging hyperplane models, and hybrid systems. We consider two models for which we can vary the smoothness and examine its influence on qualitative behavior. In the smooth regime, both models exhibit asymptotic stability for sufficiently small amplitude inputs; in the nonsmooth regime, the simpler model is shown to be BIBO stable but not asymptotically stable, and in both models nonlinear effects become more pronounced as the input amplitude decreases, in marked contrast to the behavior of smooth (i.e., linearizable) systems. Further, in the case of the simpler model the general character of this behavior in the nonsmooth regime cannot be changed with linear proportional feedback.

1 Introduction

Many of the nonlinear discrete-time dynamic models considered in the control literature map past inputs u_{k-j} and outputs y_{k-j} into the current output y_k via smooth multivariable maps. Examples include polynomial NARMAX models [3], various special cases (e.g., bilinear models [12] and polynomial Hammerstein models [6]), neural network models [11], and radial basis function models [4]. There, Taylor-series arguments imply that linear terms dominate for sufficiently small input amplitudes. Conversely, these arguments do not apply to models based on nonsmooth functions, as seen in Fig. 1, which shows four step responses for Model B introduced in Sec. 3 of this paper. These step responses are approximately

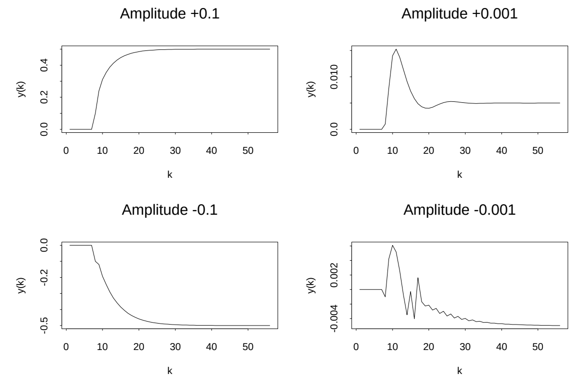

Figure 1: Model B step responses ($n = 2$, $a_1 = 0.8$,$a_2 = 0.0$, $b = 0.19$, $c = 0.5$)

symmetric with respect to sign reversal for amplitude 0.1 (behavior consistent with and characteristic of linearity), but become both much more complex and highly asymmetric for amplitude 0.001, clearly demonstrating the nonlinearity of this model. Also, note that this general behavior is qualitatively similar to certain friction phenomena, which are often described by continuous-time models that are also based on nonsmooth nonlinearities [1].

This paper examines the qualitative behavior of two simple discrete-time models based on the functions $\psi_c(x) = |x|^c$ and $\phi_c(v,w) = |v|^c - |vw|^{c/2}$, whose smoothness depends on the exponent c; this dependence allows us to investigate the influence of smoothness on qualitative model behavior. Motivation for this investigation lies in the increasing popularity of nonsmooth models in the identification and control literature, including hybrid systems [2], hinging hyperplane models [5], and some multimodels [7, 8]. For example, the following model exhibits the qualitative behavior seen in a simple chemical reactor [9]:

$$y_k = \alpha |y_{k-1}| + \beta u_{k-1}, \tag{1}$$

and has the multimodel representation:

$$y_k = \begin{cases} a y_{k-1} + b u_{k-1}, & y_{k-1} \geq 0 \\ -a y_{k-1} + b u_{k-1}, & y_{k-1} \leq 0. \end{cases}$$

Here, we analyze two models: Model A may be viewed as an extension of Eq. (1) and admits a complete analysis, part of which is described in Sec. 2. Model B is more complex and a partial analysis is given in Sec. 3. Finally, Sec. 4 presents a brief discussion of these results and outlines some possible extensions.

2 Analysis of Model A

Model A is defined by:

$$y_k = ay_{k-1} + b|y_{k-1}|^c + u_{k-1}.$$

For $b = 0$, this model is linear and stable if and only if $|a| < 1$; here, we assume $b \neq 0$ and view $|b|$ as a measure of the strength of the nonlinearity. Similarly, we view c as a smoothness index since the function $\psi_c(x) = |x|^c$ is singular if $c < 0$, $\psi_c \in C_0$ for $0 \leq c \leq 1$, and $\psi_c \in C^1$ for $c > 1$.

In discussing the stability of Model A, it is necessary to distinguish between the stability of equilibrium solutions and stability with respect to bounded inputs. First, note that Model A generally exhibits multiple equilibrium solutions, a point most easily seen in the autonomous case $u_k \equiv u_s = 0$; these solutions must satisfy

$$\begin{aligned} y_s - ay_s - b|y_s|^c &= 0 \\ \Leftrightarrow \left[1 - a - b|y_s|^{c-1} \operatorname{sign} y_s\right] y_s &= 0. \end{aligned}$$

Clearly, $y_s = 0$ is one solution, but if $|a| < 1$, $b \neq 0$, and $c \neq 1$, we also have the second solution:

$$y_s^+ = \left|\frac{1-a}{b}\right|^{1/(c-1)} \operatorname{sign} b.$$

In what follows, we consider both the stability of the equilibrium solution $y_s = 0$ for the autonomous case and stability with respect to bounded inputs. Due to space limitations, the results presented here are necessarily somewhat incomplete, but a more detailed treatment is in preparation. Lemma 1 below establishes sufficient conditions for *α-exponential stability* of the autonomous solution $y_s = 0$ for specified $0 < \alpha < 1$, meaning:

$$|y_k| \leq \alpha^k |y_0| \quad \text{for all } k > 0.$$

Lemma 1:

> For $c > 1$ and $|a| < \alpha < 1$, the autonomous response $y_s = 0$ of Model A is α-exponentially stable if
>
> $$|y_0| \leq \Delta \equiv \left(\frac{\alpha - |a|}{|b|}\right)^{1/(c-1)} < |y_s^+|.$$

Proof by induction:

Suppose $|y_{k-1}| \leq \Delta$, then:

$$\begin{aligned} |y_k| &\leq |a| \cdot |y_{k-1}| + |b| \cdot |y_{k-1}|^c \\ &= \left[|a| + |b| \cdot |y_{k-1}|^{c-1}\right] \cdot |y_{k-1}| \\ &\leq \left[|a| + |b|\Delta^{c-1}\right] \cdot |y_{k-1}| \\ &\leq \alpha |y_{k-1}| < \Delta. \end{aligned}$$

Hence, if $|y_0| \leq \Delta$, then $|y_k| \leq \alpha^k |y_0|$ for all $k \geq 0$.

□

Similar reasoning leads to bounds on the input amplitude $|u_k|$ that guarantee boundedness of the response $|y_k|$ for all k. It is important to note that these input bounds depend on the system parameters a, b, and c for $b \neq 0$ and $c > 1$. Consequently, these results are strictly weaker than *bounded-input, bounded-output* (BIBO) stability, which means that given *any* input bound $|u_k| \leq M$, there exists a corresponding bound $|y_k| \leq N$ on the response. In fact, for $c > 1$, Model A is not BIBO stable, since it may be shown that the response to an impulse input of sufficient magnitude exhibits unbounded exponential growth; again, proofs will be given elsewhere. For $0 < c \leq 1$, the following lemmas establish sufficient conditions for BIBO stability and illustrate the important differences in qualitative behavior that occur for $c = 1$ and $0 < c < 1$. The case $c = 1$ is interesting since Model A then belongs to the class of *positive-homogeneous* models [10, ch. 3]: if $u_k \to \lambda u_k$ for any $\lambda > 0$, then $y_k \to \lambda y_k$. Consequently, the qualitative behavior of this system cannot depend on the *magnitude* of the input, although it can depend on the sign; for example, taking $a = 1/2$ and $b = -1$ yields a model with stable responses to positive steps but unstable responses to negative steps.

Lemma 2:

> For $c = 1$, Model A is BIBO stable if both conditions $|a + b| < 1$ and $|a - b| < 1$ hold.

Proof:

For simplicity, we assume $y_0 = 0$ here, but the result extends easily to any finite initial condition. Let $\alpha = \max\{|a + b|, |a - b|\}$ and note that

$$\begin{aligned} |y_k| &\leq |a + b \operatorname{sign} y_{k-1}| \cdot |y_{k-1}| + |u_{k-1}| \\ &\leq \alpha |y_{k-1}| + |u_{k-1}|. \end{aligned}$$

Next, suppose $|u_{k-1}| \leq M$ for all k and proceed by induction:

$$|y_{k-1}| \leq \frac{\alpha M}{1-\alpha} \;\Rightarrow\; |y_k| \;\leq\; \frac{\alpha M}{1-\alpha} + M \;=\; \frac{M}{1-\alpha}.$$

Since $|y_0| = 0 < M/(1-\alpha)$, the bound holds for all $k > 0$.

□

Lemma 3:

For $c = 1$, the unique autonomous response $y_s = 0$ is α-exponentially stable for all initial conditions if $|a \pm b| \leq \alpha < 1$.

Proof:

For all y_{k-1}, we have

$$|y_k| \;\leq\; |a + b \,\text{sign}\, y_{k-1}| \cdot |y_{k-1}| \;\leq\; \alpha |y_{k-1}|.$$

Hence, it follows that $|y_k| \leq \alpha^k |y_0|$ for all y_0.

□

Lemma 4:

For $0 < c < 1$, Model A is BIBO stable if $|a| < 1$.

Proof by induction:

We wish to show that if $|u_k| \leq \gamma$ for all k and $|y_{k-1}| \leq \Gamma$, then $|y_k| \leq \Gamma$. By the triangle inequality:

$$|y_k| \leq |a|\Gamma + |b|\Gamma^c + \gamma.$$

The desired result follows if $\Gamma > 0$ can be found such that

$$\begin{aligned} |a|\Gamma + |b|\Gamma^c + \gamma \quad &\leq \quad \Gamma \\ \Leftrightarrow \left[\Gamma^{c-1} + \frac{\gamma}{|b|}\Gamma^{-1}\right] \quad &\leq \quad \frac{1-|a|}{|b|}. \end{aligned} \tag{2}$$

For $0 < c < 1$, the term in brackets decreases monotonically to zero as $\Gamma \to \infty$ so Γ may be chosen large enough to satisfy (2). Hence, if $|y_0| \leq \Gamma$ and $|u_k| \leq \gamma$ for all k, then $|y_k| \leq \Gamma$ for all k.

□

Lemma 5:

For $|a| < 1$ and $0 < c < 1$, the autonomous solution $y_s = 0$ for Model A is *not* asymptotically stable.

Proof:

The autonomous response of Model A satisfies:

$$\begin{aligned} |y_k| \quad &= \quad \left|a + b y_{k-1}^{c-1} \,\text{sign}\, y_{k-1}\right| \cdot |y_{k-1}| \\ &\geq \quad [|b| \cdot |y_{k-1}|^{c-1} - |a|] \cdot |y_{k-1}|, \end{aligned}$$

and is therefore unstable if the term in brackets is larger than 1. Since $c < 1$, this condition holds if:

$$|y_{k-1}| < \left(\frac{1+|a|}{|b|}\right)^{1/(c-1)} \;=\; \left(\frac{|b|}{1+|a|}\right)^{1/(1-c)},$$

establishing a region of local instability around the autonomous solution $y_s = 0$.

□

It is interesting to consider the control implications of these results. In particular, the linear feedback law $u_k = -\kappa y_k$ changes the parameter a to $a - \kappa$ but does not change b or c. Hence, for $c > 1$, proportional feedback could enlarge the stability region and for $0 < c < 1$ the system remains BIBO stable if $|a - \kappa| < 1$, but it is not asymptotically stable for any κ.

3 Analysis of Model B

Model B is a nonlinear system with the following n-dimensional state-space representation:

$$\begin{aligned} \mathbf{x}_{k+1} \quad &= \quad \mathbf{A}\mathbf{x}_k + \mathbf{B}\phi_c(\mathbf{x}_k) + \mathbf{D}^T u_k \\ \phi_c(\mathbf{x}_k) \quad &= \quad |x_1|^c - |x_1 x_2|^{c/2}. \end{aligned} \tag{3}$$

The function $\phi_c(\mathbf{x})$ is a multivariate generalization of the nonlinearity $\psi_c(x)$ on which Model A is based. Analogous to Model A, the magnitude of $||\mathbf{B}||$ may be viewed as a measure of the strength of the nonlinearity and c may be viewed as a smoothness index. In particular, ϕ_c is discontinuous for $c < 0$, $\phi_c \in C^0$ for $0 < c \leq 2$, and $\phi_c \in C^1$ for $c > 2$. The behavior of this function is illustrated in Figs. 2 and 3 for $c = 0.2$ and $c = 5$. The gradient of ϕ_c is particularly large close to the axes $x_1 = 0$ and $x_2 = 0$ for $c = 0.2$, but this behavior is not seen for $c > 2$.

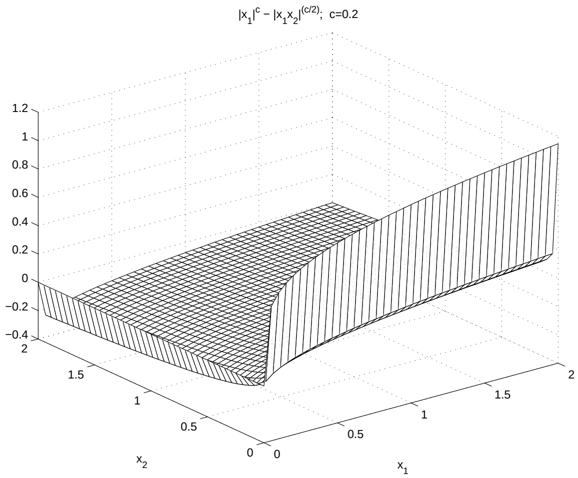

Figure 2: Plot of $\phi_c(x)$ for $c = 0.2$

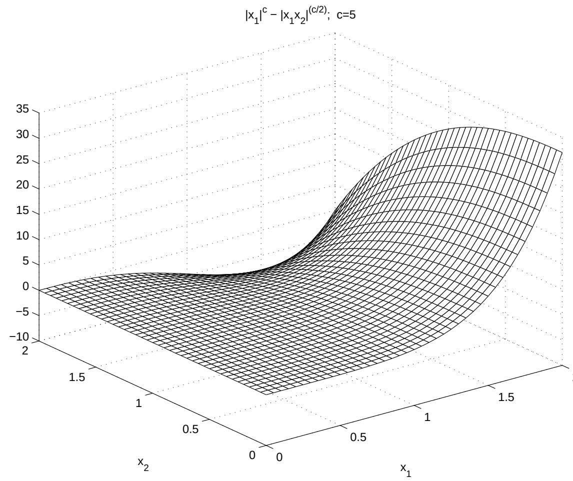

Figure 3: Plot of $\phi_c(x)$ for $c = 5.0$

3.1 Stability for $c > 1$

Lemma 6 below establishes the same results for Model B that Lemma 1 establishes for Model A. The essential basis for this result is the following inequality satisfied by the function $\phi_c(\mathbf{x}_k)$. First, note the following upper bound for $c > 0$:

$$\begin{aligned} \phi_c(\mathbf{x}_k) &\leq |x_1|^c \\ &\leq [(x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}]^c \;=\; ||\mathbf{x}_k||_2^c, \end{aligned}$$

and the corresponding lower bound:

$$\begin{aligned} \phi_c(\mathbf{x}_k) &\geq -|x_1 x_2|^{c/2} \\ &\geq -\{[\max\{|x_1|, |x_2|\}]^2\}^{c/2} \\ &\geq -\{x_1^2 + x_2^2\}^{c/2} \;\geq\; -||\mathbf{x}_k||_2^c. \end{aligned}$$

Combining these results, we have $|\phi_c(\mathbf{x}_k)| \leq ||\mathbf{x}_k||_2^c$ for all $c > 0$.

Lemma 6

> Consider the autonomous response of the system (3) for $c > 1$. This response is α-exponentially stable if $||\mathbf{A}|| < \alpha < 1$ and the initial condition $\mathbf{x}_0$ lies in the closed n-ball:
>
> $$||\mathbf{x}_0||_2 \leq \Delta \;\equiv\; \left(\frac{\alpha - ||\mathbf{A}||}{||\mathbf{B}||}\right)^{1/(c-1)}.$$

Proof by induction:

Suppose $||\mathbf{x}_k||_2 \leq \Delta$, then

$$\begin{aligned} ||\mathbf{x}_{k+1}||_2 &\leq ||\mathbf{A}|| \cdot ||\mathbf{x}_k||_2 + ||\mathbf{B}|| \cdot |\phi_c(\mathbf{x}_k)| \\ &\leq ||\mathbf{A}|| \cdot ||\mathbf{x}_k||_2 + ||\mathbf{B}|| \cdot ||\mathbf{x}_k||_2^c \\ &= [||\mathbf{A}|| + ||\mathbf{B}|| \cdot ||\mathbf{x}_k||_2^{c-1}] \cdot ||\mathbf{x}_k||_2 \\ &\leq [||\mathbf{A}|| + ||\mathbf{B}||\Delta^{c-1}] \cdot ||\mathbf{x}_k||_2 \\ &\leq \alpha ||\mathbf{x}_k||_2 \;<\; \Delta. \end{aligned}$$

Hence, if $||\mathbf{x}_0||_2 \leq \Delta$, then $||\mathbf{x}_k||_2 \leq \alpha^k ||\mathbf{x}_0||_2$ for all $k > 0$.

□

Note that the stability radius Δ established by this lemma depends on the stability margin of the linear system via $||\mathbf{A}||$, the strength of the nonlinearity $||\mathbf{B}||$, and the smoothness of the nonlinearity, determined by c. In particular, as $||\mathbf{B}|| \to 0$, we recover the usual linear stability result for all initial conditions $\mathbf{x}_0$. Further, if $||\mathbf{B}|| < \alpha - ||\mathbf{A}||$, it follows that $\Delta > 1$ but $\Delta \searrow 1$ as $c \to \infty$. Conversely, for $||\mathbf{B}|| > \alpha - ||\mathbf{A}||$, the opposite behavior is observed: $\Delta < 1$ and $\Delta \nearrow 1$ as $c \to \infty$. Finally, for $||\mathbf{B}|| = \alpha - ||\mathbf{A}||$, it follows that $\Delta = 1$ and Lemma 6 establishes the α-exponential stability of the closed unit ball, independent of c.

3.2 Stability for $0 < c \leq 1$

Using similar arguments, results analogous to Lemmas 2, 3, and 4 for Model A may also be developed for

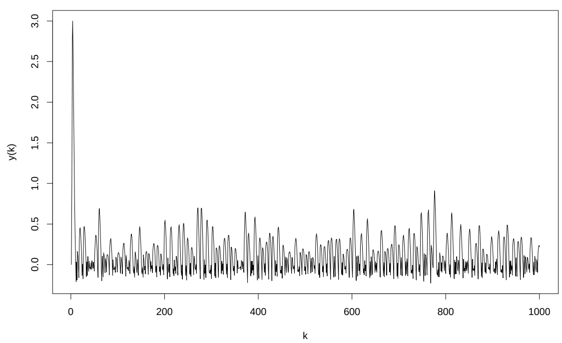

Figure 4: Unit impulse response, Model B

Model B and these will be reported elsewhere, including exponential stability results for the autonomous response when $c = 1$, BIBO stability results for $c = 1$, and BIBO stability results for $0 < c < 1$. Conversely, it is not obvious how to extend Lemma 5 to Model B, since that would require a nontrivial lower bound on $|\phi_c(\mathbf{x}_k)|$. However, simulation evidence suggests that Model B exhibits similar qualitative behavior in this nonsmooth regime. In particular, Fig. 4 shows the unit impulse response for a simple special case of Model B; although this response is bounded, it does not appear to be asymptotically stable.

4 Discussion and extensions

Motivated by the observation that a number of interesting discrete-time dynamic models are based on nonsmooth functions, this paper has examined the behavior of two simple models based on such functions, illustrating that nonsmoothness can have important and surprising behavioral consequences. Specific examples include the non-asymptotic BIBO stability demonstrated here for Model A and the increase of the effective nonlinearity with *decreasing* input amplitude shown in Fig. 1 for Model B. In addition, it was noted that the essential stability behavior of Model A cannot be changed with linear proportional feedback. A more detailed analysis of these two models is in preparation and we also intend to investigate extensions of these observations to other, more general models based on variable smoothness nonlinearities like the functions ψ_c and ϕ_c considered here.

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