Bilinear Matrix Inequalities and Robust Stability of Nonlinear Multi-Model MPC

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Abstract

A BMI-based approach to an on-line computationally efficient robust nonlinear MPC is proposed. Theoretical results and a simple example accompany the proposed method.

1 Introduction

Model predictive control (MPC) has been an active research area for close to two decades. The research has been driven by numerous successful applications of the technology [1], and during the last years a sound theoretical foundation has been established; [2], [3], and [4].

The issue of *robust* stability of MPC based control systems, however, is largely unsolved, at least for nonlinear MPC. Some results are available though. Works on robust MPC for linear systems include: [5] on constrained stable systems; [6] on unconstrained systems; [7] and [8] on constrained systems. Works on robust analysis of nonlinear MPC include: [9] and [10] on constrained continuous-time systems, and [11] on unconstrained discrete-time systems. Finally, works on robust synthesis, i.e. an uncertainty model is explicitly used when synthesizing the controller, of nonlinear MPC include: [12] on stable constrained discrete-time systems, [13] on input-affine constrained continuous-time systems, and [14] (based on [7]) on input-affine feedback linearizable constrained discrete-time systems.

In this paper we consider robust synthesis of MPC controllers for nonlinear constrained discrete-time systems. The approach applies to stable as well as unstable plants, it is not restricted to input-affine plants, and the on-line computational load is low (the off-line load, however, may be very high) as opposed to [13]. Furthermore, the approach does not have a min-max nature as have [13] and [14], this may give better performance when the nominal model used in the predictor is close to the real plant [12], thus its nature is close to the approach in [12] where nominal performance is optimized subject to robust stability.

Roughly speaking, our approach is based on ensuring that the constraint $||x_{k+1}||_P^2 < ||x_k||_P^2$ is satisfied for any possible plant within the model uncertainty class (to be defined later), where $x_k \in \mathbb{R}^n$ denotes the plant state at time-step k and $||x||_P^2 := x^T P x$ ($P = P^T > 0$). If this is the case, then certainly the origin is (robustly) asymptotically stable since $x \mapsto x^T P x$ constitutes a Liapunov function for the origin of the closed loop. This constraint is added to the other constraints on the predicted control inputs and states in the MPC associated optimization problem to be solved at each time-step k. The problem now, of course, is how to find a suitable P, and also how to find an initial feasible predicted control sequence at every time-step $k \geq 0$. To this end we have utilized results and ideas from LMI based control [15] and multi-model systems [16], [17].

In LMI-based control, which has gained a lot of interest in the last few years, control system analysis- and synthesis problems are formulated as convex optimization problems involving linear matrix inequalities (LMIs). The reason for this interest is the development of very efficient interior point algorithms for solving such problems [18]. Many interesting control problems, in particular robust control problems, can be solved within the LMI framework. There are, however, interesting control problems which are very hard or impossible to formulate within the LMI framework. Some of these problems can be formulated within the more general bilinear matrix inequalities (BMIs) framework [19]. BMI problems are much harder than LMI problems since they, in general, are nonconvex. The efficient algorithms developed for the LMI problems do, however, provide a constructive basis on which branch-and-bound algorithms for BMI problems can be developed, see [20], [21], [22], and [23].

In recent years much work has also been put into the development of nonlinear models which are composed of a set of local models [16], [17]. The local models are valid in different parts of some predefined operating set in which the operating point typically will be defined by some of the measured control inputs and/or systems outputs. Further, the nonlinear (global) model is formed as a convex combination of the local models. Other names for multi-model systems are operating regime based models and fuzzy models. In this work we focus on multi-model systems in which the local models are affine discrete-time state-space models, and we utilize this model structure to describe the model uncertainty class. This structure has at least three important advantages: *(i)* It is possible to utilize the affine structure of the local models for analysis and synthesis; *(ii)* the model class is rich in the sense that it approximates arbitrarily close a very large class of nonlinear systems; *(iii)* the model structure is transparent and there exist support tools for model identification.

Finally, we note that work on computationally procedures for stability analysis of certain classes of continuous-time nonlinear systems, in particular multi-model systems composed of affine local models, is lately reported in [24], and [25].

We utilize a piecewise affine state-feedback structure to formulate the robust constrained nonlinear stabilization problem as a BMI feasibility problem. If this problem is solvable, we get a quadratic Liapunov function and a piecewise affine state-feedback which can be used to provide an initial feasible control sequence at every time-step $k \geq 0$ provided the initial state is within a given level set of the Liapunov function.

The paper is organized as follows. Firstly, we present the considered multi-model uncertainty class. Then a BMI is found, which, if it is feasible, guarantees robust constrained stabilization of the origin of the uncertain system. After this, an approach to nonlinear constrained robust MPC is presented based on a solution to a BMI feasibility problem. Before the conclusion, an example is provided.

2 Model Uncertainty Class

The problem we investigate is to robustly stabilize the origin of a plant which can be described by a convex combination of affine discrete-time state-space systems. That is, the (nonlinear) plant is assumed to be given by

$$x_{k+1} = \sum_{j \in I_{N_m}} \omega_j(x_k, u_k, k)(A_j x_k + B_j u_k + c_j), \tag{1}$$

where $k \geq 0$, x_0 given, $x_k \in X_m \subset \mathbb{R}^n$, $u_k \in U_m \subset \mathbb{R}^m$, the local models (A_j, B_j, c_j)'s are triplets which elements have appropriate dimensions, N_m is the number of local models (subscript m indicates "model"), $I_{N_m} := \{1, \ldots, N_m\}$,

$$\omega_j : X_m \times U_m \times \mathbb{N} \to [0,1], \; \forall j \in I_{N_m},$$

and

$$\sum_{j \in I_{N_m}} \omega_j(x, u, k) = 1, \; \forall (x, u, k) \in X_m \times U_m \times \mathbb{N}. \tag{2}$$

X_m and U_m are assumed to be connected sets containing the origin.

Uncertainty is represented by allowing $\omega(\cdot,\cdot,\cdot) := (\omega_1(\cdot,\cdot,\cdot), \ldots, \omega_{N_m}(\cdot,\cdot,\cdot))$ to vary within a predefined set Ω. Next, this set will be defined.

The uncertainty description is based on knowledge of the state-space supports, X_j^S (superscript S indicates "support"), for the weights $\omega_j(\cdot,\cdot,\cdot)$, i.e., knowledge of the sets

$$X_j^S := \bigcup_{(u,k) \in U_m \times \mathbb{N}} \operatorname{supp} \omega_j(\cdot, u, k), \forall j \in I_{N_m}, \tag{3}$$

where, for a nonnegative real-valued function $a(\cdot)$, $\operatorname{supp} a(\cdot)$ returns the set on which $a(\cdot)$ is positive. By Eq. (2) the X_j^S's cover X_m.

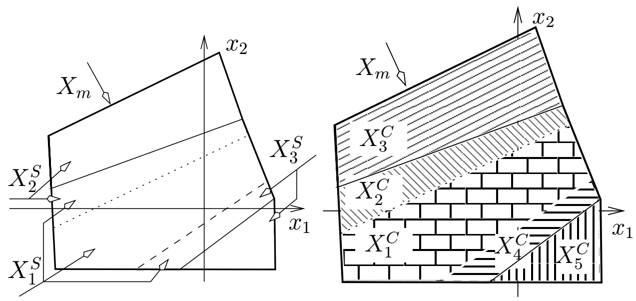

Figure 1: The state-space supports; X_1^S, X_2^S, and X_3^S for a multi-model system with three local models (left), and the associated 5 clusters (right).

An example of state-space supports for a 2-dimensional system is shown in the left part of Figure 1.

We notice that the projection on the state-space for all $u \in U_m$ in Eq. (3) implies that nonlinearities associated with the control input will be conservatively handled. It should be noted, however, that an arbitrary nonlinearity associated with the control input can be handled.

Associated with the state-space supports we define the following sets: for all $j \in I_{N_m}$

$$\Omega_j := \{\tilde{\omega} | \; \tilde{\omega} : X_m \times U_m \times \mathbb{N} \to [0,1] \text{ and } \tilde{\omega}(x,u,k) > 0 \text{ only when } x \in X_j^S\},$$

i.e. the set of all possible weights for local model number j. Now, let

$$\Omega := \{\omega = (\omega_1, \ldots, \omega_{N_m}) \in \Omega_1 \times \ldots \Omega_{N_m} | \sum_{j \in I_{N_m}} \omega_j(x,u,k) = 1, \; \forall (x,u,k) \in X_m \times U_m \times \mathbb{N}\},$$

i.e. the set of all valid convex combinations, and

$$f_\omega(x,u,k) := \sum_{j \in I_{N_m}} \omega_j(x,u,k)(A_j x + B_j u + c_j).$$

Finally

$$\mathcal{M} := \{f_\omega | \; \omega \in \Omega\}.$$

Thus, $\mathcal{M}$ denotes the assumed multi-model uncertainty class.

Local models with $c_j \neq 0$ are assumed not to have support in some neighborhood of the origin. This amounts to assuming that all the plants $f \in \mathcal{M}$, and in particular the real plant, satisfies $0 = f(0,0,k)$ for all $k \geq 0$, i.e. the equilibrium state and control input are assumed to be known.

With the state-space supports, X_j^S, we also associate a partitioning of the state space into a set of N_c clusters. A cluster, X_j^C, is a set on which the same local models have support on the whole set, and if it is extended, at least one of the local models will not have support on the extension. In Figure 1 (right part) the 5 clusters associated with the state-space supports given in Figure 1 (left part) are shown.

We will let X_m and U_m denote the state- and control constraints, respectively. In what follows the state- and control constraint sets could have been any connected subsets of, respectively, X_m and U_m containing the origin.

3 Robust Constrained Stabilization

3.1 Piecewise Affine State-Feedback

We finitely parameterize the state-feedback, $u(x)$, as a piecewise affine state-feedback. With the cluster containing the origin and the clusters which closure contains the origin, assumed (without loss of generality) to be the first N_c^o clusters, we associate a linear state feedback

$$u(x) = K_l x \text{ when } x \in X_l^C,\ l \in I_{N_c^o}. \tag{4a}$$

With all the other clusters we associate an affine state feedback, i.e. for $l \in \{N_c^o + 1, \ldots, N_c\}$

$$u(x) = K_l x + k_l \text{ when } x \in X_l^C. \tag{4b}$$

Remember that the clusters form a partition of X_m, so the above defined piecewise affine state-feedback is indeed well defined.

It should be noted that there is, in principle, no problem associating the piecewise state-feedback with a different partitioning of X_m than the one associated with the clusters. For reasons of clarity, however, we restrict the piecewise affine state-feedback to be associated with the clusters.

3.2 BMI for Robust Stabilization

In this subsection we present the essential results from [26] in a very condensed form (due to page constraints). This forms the basis for the forthcoming robust MPC result.

Theorem 1 [26] says that if a specified BMI, which basically describes conditions for the decrease, in the different clusters, of the Liapunov function $x \mapsto x^T P x$ along all possible closed-loop trajectories which can be generated by plants in $\mathcal{M}$ under the state-feedback (4), and conditions for that the level curve $\{x|x^T P x = \alpha,\ \alpha > 0\}$ is within X_m and outside the prespecified smallest acceptable region of attraction $\{x|x^T R_A x \leq 1, R_A > 0\}$, is feasible, then the origin is quadratically stable and has a region of attraction associated with $\{x|x^T R_A x \leq 1\}$ of at least $\{x|x^T P x \leq \alpha\}$. Quadratic stability [15] is a stability notion for uncertain systems which is natural to use when formulating LMIs or BMIs for robust stabilization. In this result we assume that $U_m = \mathbb{R}^m$, i.e. no control constraints, only state constraints X_m.

Theorem 2 [26] handles the case where both the states and controls are constrained. It can be shown that control constraints can be formulated as LMIs, or, less conservatively, as BMIs. Thus, by adding these LMIs or BMIs to the ones in Theorem 1, we get that feasibility of the resulting BMI guarantees that the origin is quadratically stable and has a region of attraction associated with $\{x|x^T R_A x \leq 1\}$ of at least $\{x|x^T P x \leq \alpha\}$ and that the control input constraints are satisfied on all closed-loop trajectories starting within $\{x|x^T P x \leq \alpha\}$.

As mentioned earlier, solving BMI problems is much harder than solving LMI problems. We use branch-and-bound algorithm 3 in [22] for solving the BMI feasibility problems associated with Theorems 1 and 2 in [26]. A similar approach for BMI problems can be found in [21]. In algorithm 3 the branching is done on a set of lower dimension, in our case much lower, than the total problem dimension, as opposed to [20] and [23] where the branching is done on a set with dimension equal to the total problem size. The number of so-called complicating variables gives the dimension of this lower dimensional set. The number of complicating variables is the smallest number of variables that need to be fixed to make the BMI an LMI. In our case the BMI structure arises due to one single BMI (when control constraints are represented as LMIs)

$$SP + PS \leq 2I \quad (\Leftarrow S \leq P^{-1})$$

where $0 < S = S^T, P = P^T \in \mathbb{R}^{n \times n}$. This gives $(n^2 + n)/2$ complicating variables which is much lower than the total problem size which can be maybe ten times the number of complicating variables. From this we also observe that the BMI problem reduces to an LMI problem if the Liapunov function is fixed, i.e. only the state-feedback (4) is computed.

4 Robust MPC

The results from the preceding sections will now be used to robustify MPC. This will be done by introducing the precomputed Liapunov function into the MPC optimization problem. Further, the precomputed feedback matrices (cf. (4)) will be used to compute an initial value for the optimization problem to be solved at each time-step.

First, an optimality criterion is defined on the prediction horizon N.

$$\phi(\pi_k, \chi_k; k, x_k, u_{k-1})$$

$$\pi_k := \{u_k, \ldots, u_{k+N-1}\}, \quad \chi_k := \{x_{k+1}, \ldots, x_{k+N}\},$$
$$\phi : \mathbb{R}^m \times \cdots \times \mathbb{R}^m \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty).$$

Second, the following optimization problem, to be found feasible or solved at each time step k, denoted $\mathcal{P}_{\text{MPC}}$ is specified.

$$\min_{\pi_k \in \Pi} \ \phi(\pi_k, \chi_k^{nom}; k, x_k, u_{k-1})$$

subject to

$$\|x_{k+1}^j\|_P^2 - \|x_k\|_P^2 \le -\|x_k\|_M^2,\ \forall j \in \{j | x_k \in X_j^S\},$$

and the soft constraint: $\chi_k^{nom} \in X \times \cdots \times X$.

P and M are given by Theorem 2 [26], cf. Section 3.2. $\Pi := U_m \times \cdots \times U_m$. x_{k+1}^j denotes the one-step ahead prediction using affine local model j, i.e. $x_{k+1}^j := A_j x_k + B_j u_k + c_j$. χ_k^{nom} denotes the prediction on the N-step ahead horizon using some model which typically, when restricted to $X_m \times U_m \times \mathbb{N}$, is within the model uncertainty class $\mathcal{M}$. This model is denoted the nominal model. The so-called soft constraint is defined by X. It should be noted that it is sensible that $X \subseteq X_m$, so as to "softly" force the predicted states to be within the state constraints. If the nominal model is within $\mathcal{M}$ on $X_m \times U_m \times \mathbb{N}$ and $X \subseteq \{x | x^T P x \le \alpha\}$ then the soft constraint can be satisfied, if wanted, for every $k \ge 0$. If the choice of X or the choice of nominal model makes the soft constraint infeasible at some k it can be dropped at that k while retaining the closed-loop plant state within X_m. This follows from the hard constraint.

The **solution procedure** for the MPC is defined as follows.

Step 1 At time-step k the initial choice for π_k in the iterative optimization algorithm is computed by the *precomputed* state-feedback (4), $\{K_l\}_{l=1}^{N_c}, \{k_l\}_{l=N_c^o+1}^{N_c}$, derived from Theorem 2 [26], using the nominal model for prediction.

Step 2 The iterative optimization algorithm for solving $\mathcal{P}_{\text{MPC}}$ is run until convergence or, alternatively, terminated earlier.

We may now formulate the following Theorem, based on Theorem 2 [26].

Theorem 1 (Nonlinear Robust MPC)
Assume that the system to be controlled is given by some element in $\mathcal{M}$. Then an MPC based on the solution procedure above renders the origin of the closed-loop quadratically stable with a region of attraction associated with $\{x | x^T R_A x \le 1\}$ of at least $\{x | x^T P x \le \alpha\} \subseteq X_m$, while satisfying the constraints on the control input. △

From the discussion above and the one to follow this Theorem should be clear, for a rigorous proof, see [26].

As mentioned above the soft constraint in $\mathcal{P}_{\text{MPC}}$ may or may not be satisfied. Importantly, it should be observed that the result does not depend on this.

The solution procedure utilizes an iterative optimization algorithm. Since the initial choice satisfies the hard constraint, consecutive iterations will also satisfy these constraints. The importance of the iterative search is to improve nominal performance within the frame of a robust stability guarantee. By this, the iterative algorithm may be terminated at an arbitrary iteration without affecting (quadratic) stability. This may e.g. be caused by limited computation time.

The optimization problem may be non-convex if the nominal prediction model is nonlinear, or if $\phi(\cdot,\cdot;k,x_k,u_{k-1})$ is non-convex. The latter is rarely encountered since $\phi(\cdot,\cdot;k,x_k,u_{k-1})$ typically is chosen as a norm-based function, and all such functions are convex. Again, this does not affect the stability result.

The proposed MPC algorithm may be interpreted as follows: *(i)* Robust stability is guaranteed by considering the 1-step ahead prediction of the whole model uncertainty class. This is possible since the set of possible 1-step ahead predictions is defined by a polytope where the vertices are given by the affine systems that are active at the present state x_k. It should be noted that this is, of course, less conservative than letting all the models have support X_m; *(ii)* Performance is taken care of by considering a nominal model on the entire prediction horizon. In practice the nominal model will be chosen as the most likely model. It may either be a linear or a nonlinear model.

5 Example

The example system, which is taken from [27], is depicted in Figure 2. It is a mechanical system consisting of a mass, a spring, and a damper. The nonlinear equations of motion have the following structure

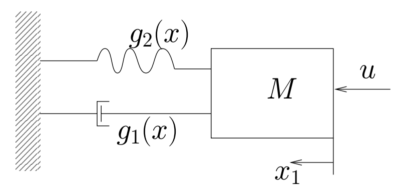

Figure 2: Example system.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{M}\left(-g_1(x_1,x_2) - g_2(x_1) + u\right)$$

where

$$g_1(x_1,x_2) = D(c_1 x_1 + c_2 x_2^3);\ \ g_2(x_1) = c_3 x_1 + c_4 x_1^3$$

and system knowledge permits us to limit the parameters to

$$M = 1 \qquad 0.9 \le D \le 1.2 \qquad 0 \le c_1 \le 0.02 \quad (4)$$
$$0.1 \le c_2 \le 0.2 \quad 0.01 \le c_3 \le 0.02 \quad 0.7 \le c_4 \le 1 \quad (5)$$

To facilitate the uncertainty modeling of this system we rewrite $\dot{x}_2$ as

$$\dot{x}_2 = f_1(x_1) + f_2(x_2) + \frac{u}{M}$$

$$f_1(x_1) = -\frac{1}{M}\left((Dc_1 + c_3)x_1 + c_4 x_1^3\right);\ f_2(x_2) = -\frac{Dc_2}{M}x_2$$

Next, we develop the model class $\mathcal{M}$, which provides an outer approximation to the set of possible systems defined by the uncertainty (4)-(5).

This is done by first finding the upper and lower bounds for each of $f_1(\cdot)$ and $f_2(\cdot)$ on $[-1.5, 1.5]$. Piecewise affine functions are then chosen, which tightly approximate the lower bounds from below, and the upper bounds from above. The upper and lower bounds for $f_1(\cdot)$ and $f_2(\cdot)$ as well as their tight outer approximations are shown in Figure 3 to the upper left and upper right, respectively. The bends of the piecewise upper and lower bounds for both functions are at ± 0.8667. Both the upper and lower outer approximations for each function consists of four pieces, this gives rise to $4 \times 4 \times 2 = 32$ "local" planes covering $f(x_1, x_2) := f(x_1) + f(x_2)$ on $[-1.5, 1.5] \times [-1.5, 1.5] =: X_m$ as shown at the bottom in Figure 3.

For each of the 32 "local" planes it is straightforward to find an associated discrete-time local model (A_j, B_j, c_j). We have used a sample-period of 0.2 and forward Euler in all our simulations.

The supports X_j^S are given by 8×2 rectangles and 8×2 squares, as can be seen in Figure 3. $\mathcal{M}$ is now defined by the supports and the local models. There are $4 \times 4 = 16$ clusters associated with the 32 supports. Two local models have support in each cluster, X_j^C (an upper and a lower bound for $f(\cdot, \cdot)$).

In this example we let $U_m := [-0.5, 0.5]$, while the smallest acceptable region of attraction is chosen to be $\{x | x^T x \leq 1\}$, i.e. the closed unit ball. We are now in a position to formulate a BMI for constrained robust stabilization of the origin. Solving the BMI feasibility problem, which in this case involves solving 5 LMI problems or three iterations in the branch-and-bound algorithm, results in a robust controller giving a closed loop trajectory as depicted by the dashed trajectory in the left part of Figure 4. The accompanying control input sequence is depicted by the dashed line in the right part of Figure 4. We observe that both the relation $\{x | x^T x \leq 1\} \subseteq \{x | x^T P x \leq \alpha\} \subseteq X_m$, and the control constraints are satisfied. In the simulations, D and c_4 have sinusoidal variations between their upper and lower bounds, the other parameters are fixed as follows: $M = 1$, $c_1 = 0.02$, $c_2 = 0.15$, and $c_3 = 0.015$ while $x_0 = (-0.6325, -0.9518)^T$.

Now, suppose that instead of merely robustly stabilize the plant while satisfying state and control constraints, we, in addition, want to optimize the performance as defined by the following cost function.

$$\tilde{\phi}(\{u_k\}_{k=0}^{\infty}; x_0, u_{-1}) = \sum_{k=0}^{\infty} (x_{2,k+1} - x_{2,k})^2 + x_{1,k+1}^2 + (u_k - u_{k-1})^2$$

where $x_{k+1} = \tilde{f}(x_k, u_k, k)$ is the state of the real plant at time $k+1$, while $x_{i,k}$ denotes the i'th element of the real plant's state vector at time k. This cost function express natural objectives as minimize actuator wear and mass acceleration while keeping the position at zero. Minimizing $\tilde{\phi}(\cdot; x_0, u_{-1})$ is of course impossible since the real plant, $\tilde{f}(\cdot, \cdot, \cdot)$, is not known. Moreover, there is an infinite number of decision variables and constraints. We propose to approach this problem using the robust MPC developed in Section 4. As a cost function we propose

$$\phi(\pi_k, \chi_k^{nom}; x_k, u_{k-1}) = \sum_{i=1}^{10} (x_{2,k+i}^{nom} - x_{2,k+i-1}^{nom})^2 + {x_{1,k+i}^{nom}}^2 + (u_{k+i-1} - u_{k+i-2})^2$$

Using the solution procedure proposed in Section 4 with $X = \mathbb{R}^n$, $u_{-1} = 0$, and a nominal model defined by: $M = 1$, $D = 1.1$ $c_1 = 0.02$, $c_2 = 0.15$, $c_3 = 0.015$, $c_4 = 0.7$, we get the closed loop trajectory as shown by the solid trajectory in the left part of Figure 4. The accompanying control input sequence is shown by the solid line in the right part of Figure 4. If we compare $\tilde{\phi}(\{u_k\}_{k=0}^{\infty}; x_0, u_{-1})$ for the two simulations, we get 10.61 for the piecewise affine state-feedback versus 4.79 for the robust MPC, that is, a 55 % cost reduction. Further, we note that the control constraints are reached in the MPC case as opposed to the case when the piecewise affine state-feedback is used. This is because the control constraints are somewhat conservatively handled when transforming them into LMIs or BMIs. Thus, such conservatism can be removed while retaining the robust stabilization property when using the MPC approach.

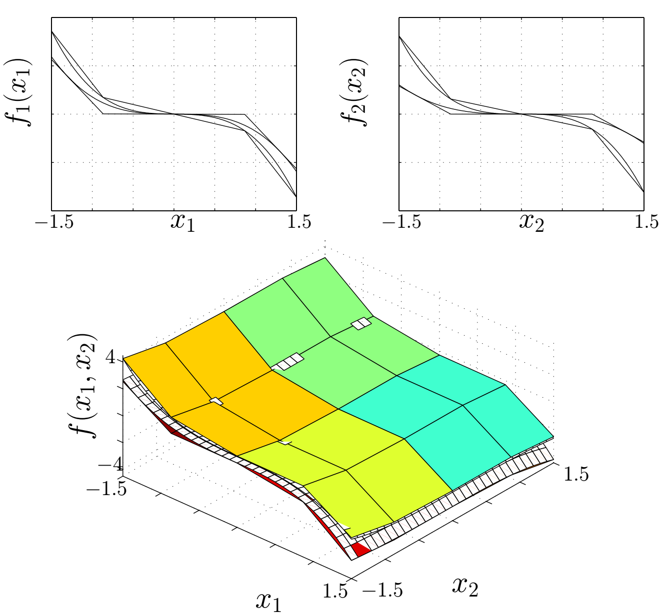

Figure 3: Graphs illustrating the development of the nonlinear multi-model uncertainty class for the example.

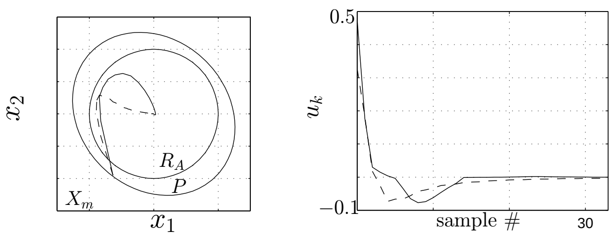

Figure 4: Left: Phase-plane plot of closed-loop trajectories using piecewise affine state-feedback (dashed) and robust MPC (solid). Right: Accompanying control input sequences.

6 Conclusion

An approach to nonlinear constrained robust MPC is presented. The method utilizes structural model uncertainty and guarantees robust stability.

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